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The Existence and Uniqueness of Positive Solutions for a Singular Nonlinear Three-Point Boundary Value Problems

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Abstract

Using the method of lower and upper solutions, we study the following singular nonlinear three-point boundary value problems:
$\begin{cases} -x''(t)+K(t)x^{-q}(t)=\lambda x^{p}(t), & t\in(0,1), \\ x(0)=0, & x(1)=ax(\eta), \end{cases}$, where $K\in C[0,1]$, $0<a<1$, $0<\eta<1$ and λ is a positive parameter and present the existence, uniqueness, and the dependency on parameters of the positive solutions under various assumptions. Our result improves those in the previous literatures.

Keywords

Three-Point Boundary Value Problem, Positive Solution, Lower and Upper Solutions, Eigenvalue and Eigenfunction

1. Introduction and Main Results

In this paper, we consider the three-point boundary value problem

$$\begin{cases} -x''(t)+K(t)x^{-q}(t)=\lambda x^{p}(t), & t\in(0,1), \\ x(0)=0, \quad x(1)=ax(\eta), \end{cases} \tag{1.1}$$

where $K\in C[0,1]$, $0<a<1$, $0<\eta<1$, and λ is a positive parameter.

The m-point boundary value problem for linear second-order ordinary differential equations was initiated by Ilin and Moiseev [1] [2]. Since then, there are many results on the existence of general nonlinear multi-point boundary value problems, see [3] [4] [5] [6] and their references. For examples, in [6], Rynne studied the \$m\$-point boundary value problem

$$\begin{cases} -u'' = f(u), \quad on(0,1), \quad u \in R \times X, \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

where $m \geq 3$, $\eta_i \in (0,1)$, $\alpha_i > 0$ with $\sum_{i=1}^{m-2} \alpha_i < 1$ and presented the existence of the sign changing solutions by Rabinowitz bifurcation theorem. Especially, Rynne ([7]) discussed the three-point boundary value problem

$$\begin{cases} -u'' = f(u) + h, \quad on(0,1), \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases}$$

and showed the solvability and non-solvability results from either the half-eigenvalue or the Fucik spectrum approach. As we known, the method of upper and lower solutions is very important for the study of the boundary value problems, see [8]-[18]. Therefore, establishing the method of upper and lower solutions for three-point boundary value problems is necessary and important.

In [19], when *f* is nondecreasing on *x*, Du and Zhao got the methods of upper and lower solutions of

$$\begin{cases} -x''(t) = f(t, x(t)), \quad t \in (0,1), \\ x(0) = ax(\eta), \quad x(1) = 0, \end{cases}$$

and used iterative techniques to study the existence of positive solutions. And in [3] when *f* is decreasing on *u*, Du and Zhao considered the existence and uniqueness of positive solutions of the problem

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u(1) = 0 \end{cases}$$

by constructing lower and upper solutions. Wei ([15]) constructed the method of upper and lower solutions for three-point boundary value problems and gave the sufficient and necessary conditions for the existence of positive solutions of the problem

$$\begin{cases} -x''(t) = f(t, x(t)), \quad t \in (0,1), \\ x(0) = ax(\eta), \quad x(1) = 0. \end{cases}$$

On the other hand, singular boundary problems arise in the contexts of chemical heterogeneous catalysts, non-Newtonian fluids and also the theory of heat conduction in electrically conducting materials, see [20]-[25] for a detailed discussion. An interesting result comes from [25], in which, using method of upper and lower solutions, Shi and Yao discussed the following problem

$$\begin{cases} -\Delta u + K(x) u^{-q} = \lambda u^p, \quad x \in \Omega, \\ u(x) > 0, \quad \forall x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $K \in C^{2,\beta}(\bar{\Omega})$, $p, q \in (0,1)$ and λ is a positive parameter. Under various appropriate assumptions on $K(x)$, Shi and Yao obtained the existence and uniqueness of classical solutions.

Motivated by above works, under various appropriate assumptions on *p*, *q* and $K(t)$, we will obtain the existence and uniqueness of positive solution of problem (1.1) for λ in different circumstances. In our proof, the upper and lower solutions theorem (see [16]) plays an important role in the paper.

Define

$$K^* = \max_{t\in[0,1]} K(t), K_* = \min_{t\in[0,1]} K(t).$$

The main results of this paper are stated in the following theorems.

***Theorem* 1.1.** *When* $K_* > 0$,

1) If $0 < p,q < 1$*, there exists* $\bar{\lambda} > 0$ *such that the problem* (1.1) *has at least one C[0,1] positive solution* $x_\lambda(t)$ *for* $\lambda > \bar{\lambda}$.

2) For $\lambda > \bar{\lambda}$, (1.1) *has a maximal solution* $\bar{x}_\lambda(t)$ *and* $\bar{x}_\lambda(t)$ *is increasing with respect to* λ.

Theorem 1.2. *When* $K^* < 0$,

1) If $0 < p < 1, 0 < q$, (1.1) *has at least one* $C[0,1]$ *positive solution for all* $\lambda > 0$.

2) If $0 < p,q < 1$, (1.1) *has an unique* $C^1[0,1]$ *positive solution* $x_\lambda(t)$ *for all* $\lambda > 0$.

3) $x_\lambda(t)$ *in* (2) *is increasing with respect to* λ.

Theorem 1.3. *When* $K_* < 0 < K^*$,

1) *If* $0 < p,q < 1$*, there exists a* $\lambda_* > 0$ *such that the problem* (1.1) *has at least one* $C[0,1]$ *posit- ive solution* $x_\lambda(t)$ *for* $\lambda > \lambda_*$.

2) *For* $\lambda > \lambda_*$, $x_\lambda(t)$ *in* (1) *is increasing with respect to* λ.

Remark 1.1: Note $K(t) > 0$ in Theorem (1.1). This is different from the conditions in [3] [15] [19] because $K(t) < 0$ in these references.

Remark 1.2: The unique result in Theorem 1.2 is different from that in [3] because we remove the monotonicity of nonlinearity *f* in *x*.

Remark 1.3: Note $K(t)$ is sigh-changing in Theorem 1.3. This is different from the conditions in [3] [15] [19] because $K(t) < 0$ in these references and is different from conditions in [1] [2] [4] [5] [6] [7] [26] because *f* is continuous at $x = 0$ in these references.

This paper is organised as follows. Some preliminary lemmas are stated and proved in Section 2. And Section 3 is devoted to prove the results.

2. Preliminaries

In this section, we first consider the following problem

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in (0,1), \\ x(0) = 0, x(\eta) = ax(1), \end{cases} \tag{2.1}$$

where $\eta \in (0,1)$, $0 < a < 1$ and $f \in [0,1]\times\mathbb{R}\times\mathbb{R}$.

Let $C^1[0,1] = \{x : [0,1] \to \mathbb{R} \mid x(t)$ is differential continuous on $[0,1]\}$ with norm

$$\|x\| = \max\{|x|_\infty, |x'|_\infty\},$$

where $|x'|_\infty = \max_{t\in[0,1]} |x(t)|$. Obviously, $C^1[0,1]$ is a Banach space. Now we give the definitions of lower and upper solutions for problem (2.1).

Definition 2.1. A function $\alpha(t)$ is called a lower solution to the problem (2.1), if $\alpha(t) \in C[0,1] \cap C^2(0,1)$ and satisfies

$$\begin{cases} -\alpha''(t) \le f(t,\alpha(t),\alpha'(t)), & t\in(0,1), \\ \alpha(0) \le 0, \quad \alpha(1) \le a\alpha(\eta). \end{cases} \tag{2.2}$$

Upper solution is defined by reversing the above inequality signs in problem (2.2).

If there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ to problem (2.1) such that $\alpha(t) \le \beta(t)$, then $(\alpha(t), \beta(t))$ is called a couple of upper and lower solutions of problem (2.1).

Set $D_\alpha^\beta = \{(t,x) \in (0,1)\times\mathbb{R}^+, \alpha(t) \le x \le \beta(t), t\in(0,1)\}$.

We list a lemma for the eigenvalues and eigenfunctions for the following linear problem

$$\begin{cases} -x''(t) = \lambda x(t), & t\in(0,1), \\ x(0) = 0, \quad x(1) = ax(\eta). \end{cases} \tag{2.3}$$

***Lemma* 2.1.** (*see* [6]) *The spectrum* $\sigma(L)$ *of problem* (*2.3*) *consists of a strictly increasing sequence ofeigenvalues* $\lambda_k > 0$, $k = 1,2,\cdots$, *with eigenfuctions* $\phi_k = \sin(\lambda_k^{\frac{1}{2}} t)$. *In addition,*

1) $\lim_{k\to+\infty} \lambda_k = +\infty$;

2) $\phi_k(t)$ *has exact* $k-1$ *simple zeros in* $(0,1)$, $k = 2,3,\cdots$ *and* ϕ_1 *is strictly positive on* $(0,1)$.

***Lemma* 2.2.** *Suppose that* $h \in L^1(0,1)$. *Then, for each* $\lambda > 0$, *the problem*

$$\begin{cases} -x''(t) + \lambda x = h(t), & t\in(0,1), \\ x(0) = 0, x(\eta) = \alpha x(1) \end{cases} \tag{2.4}$$

has an unique solution in $C[0,1]$.

Proof. Assume that $v_1(t)$ and $v_2(t)$ satisfies that

$$\begin{cases} -x''(t) + \lambda x = h(t), & t\in(0,1), \\ x(0) = 0, x'(0) = 1 \end{cases}$$

and

$$\begin{cases} -x''(t) + \lambda x = h(t), & t\in(0,1), \\ x(1) = 0, x'(1) = -1 \end{cases}$$

respectively. Define

$$G(t,s) = \frac{1}{\omega}\begin{cases} v_2(t)v_1(s), & 0\le s\le t\le 1, \\ v_1(t)v_1(s), & 0\le t\le s\le 1, \end{cases}$$

and

$$x(t) = \int_0^1 G(t,s)h(s)ds + \frac{e_1(t)}{e_1(1) - \alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds, \quad s\in[0,1].$$

Then

$$
\begin{aligned}
-x''(t)+\lambda x(t) &= -\frac{1}{\omega}[\int_0^t v_2(t)v_1(s)h(s)ds+\int_t^1 v_1(t)v_2(s)h(s)ds]'' \\
&\quad -\frac{e_1''(t)}{e_1(1)-\alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds+\lambda x(t) \\
&= -\frac{1}{\omega}[v_2'(t)v_1(t)-v_1'(t)v_2(t)]h(t) \\
&\quad -\frac{1}{\omega}[\lambda\int_0^t v_2(t)v_1(s)h(s)ds+\lambda\int_t^1 v_1(t)v_2(s)h(s)ds] \\
&\quad -\frac{\lambda e_1(t)}{e_1(1)-\alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds+\lambda x(t) \\
&= h(t)-\lambda\frac{1}{\omega}[\int_0^t v_2(t)v_1(s)h(s)ds+\int_t^1 v_1(t)v_2(s)h(s)ds] \\
&\quad -\lambda\frac{e_1(t)}{e_1(1)-\alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds+\lambda x(t) \\
&= h(t), \qquad t\in(0,1)
\end{aligned}
$$

and

$$
\begin{aligned}
x(1)-\alpha x(\eta) &= \int_0^1 G(1,s)h(s)ds+\frac{e_1(1)}{e_1(1)-\alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds \\
&\quad -\alpha[\int_0^1 G(\eta,s)h(s)ds+\frac{e_1(\eta)}{e_1(1)-\alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds] \\
&= \frac{e_1(1)}{e_1(1)-\alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds \\
&\quad -\alpha[\int_0^1 G(\eta,s)h(s)ds+\frac{e_1(\eta)}{e_1(1)-\alpha e_1(\eta)}\alpha\int_0^1 G(\eta,s)h(s)ds] \\
&= 0.
\end{aligned}
$$

Hence, $x(t)$ is a C[0,1] solution to problem(2.4). Since $\lambda>0$, Lemma 2.1 guarantees that problem (2.4) has an unique C[0,1] solution. The proof is complete. □

***Theorem* 2.1.** *Let* α *and* $\beta\in C([0,1])\cap C^1(0,1)$ *be lower and upper solutions of* (2.1) *such that* $\alpha\le\beta$. *Let* $\bar\psi\in L^1[0,1]$ *and* $\bar\phi:\mathbb{R}^+\to\mathbb{R}_0^+$ *be a continuous function that satisfies*

$$\int_0^\infty \frac{1}{\bar\phi(s)}ds=+\infty. \tag{2.5}$$

Suppose $f:D_\alpha^\beta\times\mathbb{R}\to\mathbb{R}$ *is an* L^1*-Carathéodory-function such that*

$$|f(t,x,v)|\le\bar\psi(t)\bar\phi(|v|),\ \forall(t,x)\in D_\alpha^\beta,\ v\in\mathbb{R}. \tag{2.6}$$

Then the problem (2.1) *has at least one solution* $x\in C^1[0,1]$ *such that for all* $t\in[0,1]$,

$$\alpha(t)\le x(t)\le\beta(t).$$

Proof. The proof proceeds in five steps.

Step 1. We consider a new modified problem. From (2.5), there is an $R>0$ be large enough so that

$$\int_0^R \frac{1}{\bar\phi(s)}ds>\|\psi\|_1. \tag{2.7}$$

And (2.6) guarantees that there is an $\bar{N}(t)$ with $\bar{N} \in L^1[0,1]$ such that

$$|f(t,x,v)| \le \bar{N}(t),\ \forall (t,x) \in D_\alpha^\beta,\ |v| \le R. \tag{2.8}$$

Define then

$$\chi(t,x) = \begin{cases} \alpha(t), & if \quad x < \alpha(t), \\ x, & if \quad \alpha(t) \le x \le \beta(t), \\ \beta(t), & if \quad x > \beta(t) \end{cases} \tag{2.9}$$

and

$$g(t,x,v) = \max\{\min\{f(t,\chi(t,x),v), \bar{N}(t)\}, -\bar{N}(t)\}. \tag{2.10}$$

Choose a $\lambda > 0$ and consider the new boundary value problem

$$\begin{cases} -x''(t) + \lambda x = g(t,x(t),x'(t)) + \lambda\chi(t,x(t)), & t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta), \end{cases} \tag{2.11}$$

where $0 < a < 1$, $0 < \eta < 1$.

Step 2. We discuss the existence of a $C^1[0,1]$ solution of (2.11).

Now Lemma 2.2 guarantees that for each $h \in L^1[0,1]$, the linear problem

$$\begin{cases} -x''(t) + \lambda x = h, & t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta) \end{cases}$$

has an unique $C[0,1]$ solution

$$v(t) = \int_0^1 G(t,s)h(s)ds + \frac{e_1(t)}{e_1(1) - ae_1(\eta)} a\int_0^1 G(\eta,s)h(s)ds,\ s \in [0,1].$$

For $x \in C^1[0,1]$, define

$$(Fx)(t) = g(t,x(t),x'(t)) + \lambda\chi(t,x(t)),\ t \in [0,1]$$

and

$$(Tx)(t) = \int_0^1 G(t,s)(Fx)(s)ds + \frac{e_1(t)}{e_1(1) - ae_1(\eta)} a\int_0^1 G(\eta,s)(Fx)(s)ds,\ s \in [0,1].$$

From (2.9) and (2.10), we have

$|g(t,x(t),x'(t)) + \lambda\chi(t,x(t))| \le \bar{N}(t) + \lambda \max\{\sup_{t\in[0,1]}|\alpha(t)|, \sup_{t\in[0,1]}|\beta(t)|\}$, which implies that the functions belonging to $\{(Tx)(t) : x \in C^1[0,1]\}$ and $\{(Tx)'(t) : x \in C^1[0,1]\}$ are bounded and equicontinuous. The Arzela-Ascoli Theorem guarantees that $TC^1[0,1]$ is relatively compact. The proof of the continuity of T is standard. Using the Schauder's fixed point theorem, we assert that T has at least one fixed point $x \in C^1[0,1]$.

Step 3. The solution x of (2.11) is such that $\alpha(t) \le x(t) \le \beta(t)$.

We prove that $x(t) \le \beta(t)$ for $t \in [0,1]$ only. In fact, suppose that there exist a $t_0 \in [0,1)$ such that $x(t_0) > \beta(t_0)$. Since $x(0) = 0 \le \beta(0)$, $t_0 > 0$. Let $w(t) = x(t) - \beta(t)$, $t \in [0,1]$. Then $w(0) \le 0$ and $w(t_0) > 0$.

Let $t^* = \sup\{t|\ w(s) > 0, s \in [t_0,t]\}$, $t_* = \inf\{t|\ w(s) > 0, s \in [t,t_0]\}$.

It is obvious that $w(t) > 0$ for all $t \in (t_*,t^*)$, $w(t_*) = 0$ and $w(t^*) \ge 0$. If $w(t^*) = 0$, then there exists a $t' \in (t_*,t^*)$ such that $w(t') = \max_{t\in[t_*,t^*]} w(t)$. If

$w(t^*) > 0$, obviously $t^* = 1$ and $w(1) = x(1) - \beta(1) > 0$. Since $w(\eta) = x(\eta) - \beta(\eta) = \frac{1}{a}(x(1) - \beta(1)) = \frac{1}{a} w(1) > w(1)$, there exists $t' \in (t_*, t^*)$ such that $w(t') = \max_{t \in [t_*, t^*]} w(t)$ also. Hence, $w'(t') = 0$ (*i.e.*, $\beta'(t') = x'(t')$) and $-w''(t') \geq 0$. On the other hand, since

$$\begin{aligned}
-w''(t') &= \beta''(t') - x''(t') \\
&\leq -f(t', \beta(t'), \beta(t')) + g(t', x(t'), x'(t)) + \lambda\chi(t', x(t')) - \lambda x(t') \\
&= -f(t', \beta(t'), \beta'(t')) + \max\{\min\{f(t', \beta(t'), \beta'(t)), \bar{N}(t)\}, -\bar{N}(t)\} \\
&\quad + \lambda\beta(t') - \lambda x(t') \\
&= -f(t', \beta(t'), \beta'(t')) + f(t', \beta(t'), \beta'(t')) + \lambda\beta(t') - \lambda x(t') \\
&= \lambda(\beta(t') - x(t')) < 0.
\end{aligned}$$

This is a contradiction.

A similar argument holds to prove $x(t) \leq \beta(t)$ for all $t \in [0,1]$.

Hence, from (2.10), one know that x satisfies that

$$\begin{cases} -x''(t) = g(t, x(t), x'(t)) = \max\{\min\{f(t, x(t), x'(t)), \bar{N}(t)\}, & t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta). \end{cases} \tag{2.12}$$

Step 4. The solution x of (2.11) is such that $|x'|_\infty \leq R$.

On the contrary, suppose that there is a $t' \in (0,1)$ such that $|x'(t')| > R$. Without loss of generality, we assume that $x'(t') > R$. Since $x(0) = 0$ and $x(1) = ax(\eta)$ with $0 < a < 1$, there is a $t_0 \in (0,1)$ such that $x'(t_0) = 0$. Without loss of generality, we assume that $x'(t) > 0$ for all (t', t_0). Observe that, for all $(t, x) \in D_\alpha^\beta$, $v \in \mathbb{R}$,

$$\max\{\min\{f(t, x, v), \bar{N}(t)\}, -\bar{N}(t)\} \leq \bar{\psi}(t)\bar{\phi}(|v|).$$

Then, from (2.12), one has

$$\begin{aligned}
\int_0^R \frac{1}{\bar{\phi}(s)} ds &= \left| \int_{x'(t_0)}^{x'(t')} \frac{1}{\bar{\phi}(s)} ds \right| = \left| \int_{t'}^{t_0} \frac{1}{\bar{\phi}(x'(t))} dx'(t) \right| \\
&= \left| \int_{t'}^{t_0} \frac{x''(t)}{\bar{\phi}(x'(t))} dt \right| = \left| \int_{t'}^{t_0} \frac{g(t, x(t), x'(t))}{\bar{\phi}(x'(t))} dt \right| \\
&= \int_{t'}^{t_0} \frac{\bar{\psi}(t)\bar{\phi}(x'(t))}{\bar{\phi}(x'(t))} dt = \int_{t'}^{t_0} \bar{\psi}(t) dt = \|\bar{\psi}\|_1.
\end{aligned}$$

This contradicts to (2.7).

Hence $|f(t, x(t), x'(t))| \leq \bar{N}(t)$, which together with $u \in [\alpha, \beta]$ guarantees that

$$g(t, x(t), x'(t)) = f(t, x(t), x'(t)), \quad \forall t \in (0,1).$$

Step 5. We claim that $x(t)$ satisfies (2.1).

Since $|x'|_\infty \leq R$ and $\alpha(t) \leq x(t) \leq \beta(t)$, by (2.8), (2.10) and (2.12), we have

$$\begin{cases} -x''(t) = \max\{\min\{f(t, x(t), x'(t)), \bar{N}(t)\} = f(t, x(t), x'(t)), & t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta), \end{cases}$$

that is, $x(t)$ is a $C^1[0,1]$ solution of (2.1). The proof is complete. □

Now we consider the following problem

$$\begin{cases} -x''(t) = f(t, x(t)), & t \in (0,1), \\ x(0) = 0, x(\eta) = ax(1), \end{cases} \tag{2.13}$$

where $\eta\in(0,1)$, $0<a<1$ and $f\in[0,1]\times\mathbb{R}\times\mathbb{R}$.

Now we give the definitions of lower and upper solutions for problem (2.13).

Definition 2.2. (see [16]) A function $\alpha(t)$ is called a lower solution to the problem (2.13), if $\alpha(t)\in C[0,1]\cap C^2(0,1)$ and satisfies

$$\begin{cases}-\alpha''(t)\le f(t,\alpha(t)), & t\in(0,1),\\ \alpha(0)\le 0, \quad \alpha(1)\le a\alpha(\eta).\end{cases} \tag{2.14}$$

Upper solution is defined by reversing the above inequality signs in problem (2.14).

By Theorem 2.1, we have following result.

Corollary 2.1. Suppose that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of problem (2.1) such that $\alpha(t)\le\beta(t)$, $t\in[0,1]$ and there exists $F\in L^1[0,1]$ such that $|f(t,x)|\le F(t)$ for all $(t,x)\in D_\alpha^\beta$. Then the problem (2.13) has at least one C[0,1] solution $x(t)$ satisfies $\alpha(t)\le x(t)\le\beta(t)$, $t\in[0,1]$.

Remark 2.1: This result can be found in [15]. So our theorem improves the works in the previous literature.

Lemma 2.3. Suppose that $f:(0,1)\times[0,+\infty)\to\mathbb{R}$ is a continuous functions such that $s^{-1}f(t,s)$ is strictly decreasing for $s>0$ at each $t\in(0,1)$. Let $w,v\in C[0,1]\cap C^2(0,1)$ satisfies:

1) $w''+f(t,w)\le 0\le v''+f(t,v)$, $t\in(0,1)$;

2) $w,v>0$, $t\in(0,1)$ and $w(0)\ge v(0)$, $w(1)\ge aw(\eta)$, $v(1)\le av(\eta)$;

3) $v''\in L^1[0,1]$.

Then $w(t)\ge v(t)$, $t\in[0,1]$.

Proof. By $v''\in L^1(0,1)$, we know that $v'(0+)$ and $v'(1-)$ exist and then $v\in C^1[0,1]$.

Suppose conversely $v(t)\not\le w(t)$ on [0,1]. We may assume without loss of generality that there exists $t_0\in(0,1)$ such that $v(t_0)-w(t_0)=\max_{0\le t\le 1}(v(t)-w(t))>0$. Let

$$t_*=\inf\{t_1\,|\,0\le t_1<t_0, v(t)>w(t), t\in(t_1,t_0)\},$$

$$t^*=\sup\{t_2\,|\,t_0\le t_2<1, v(t)>w(t), t\in(t_0,t_2)\}.$$

It's obvious that $0\le t_*<t^*\le 1$ and $v(t_*)=w(t_*)$, $v'(t_*+)\ge D^+w(t_*+)$, where D^+ denote Dini derivatives.

For $t^*\le 1$, there are three cases.

1) $t^*<1$. Then $v(t^*)=w(t^*)$, $v'(t^*)\le w'(t^*)$, $v(t)>w(t)$ for all $t\in(t_*,t^*)$.

2) $t^*=1$ and $v(t^*)=w(t^*)$, $v'(t^*-)\le D_-w(t^*-)$, $v(t)>w(t)$ for all $t\in(t_*,t^*)$, where D_- denotes Dini derivatives.

3) $t^*=1$ and $v(t^*)>w(t^*)$, $v(t)>w(t)$ for all $t\in(t_*,t^*]$. Since $v(1)-w(1)\le a(v(\eta)-w(\eta))<v(\eta)-w(\eta)$, then there is $t'\in[\eta,1]$ such that

$$v(t')-w(t')>0,\ (v(t')-w(t'))'<0.$$

Combining above (1), (2) and (3), there is a $t'>t_*$ such that

$$v(t_*)=w(t_*), v'(t_*+)\ge D^+w(t_*+), v(t')\ge w(t'), v'(t'-)\le D_-w(t'-),$$

and

$$v(t) > w(t), \forall t \in (t_*, t').$$

Let $y(t) = v'(t)w(t) - w'(t)v(t)$, $t \in (t_*, t')$. Then we have

$$\liminf_{t \to t_* +} y(t) \geq 0 \geq \limsup_{t \to t'-} y(t). \tag{2.15}$$

On the other hand,

$$\begin{aligned} y'(t) &= w(t)v''(t) - w''(t)v(t) \\ &= -w(t)f(t, v(t) + v(t)f(t, w(t)) \\ &= w(t)v(t)\left(\frac{f(t, w(t))}{w(t)} - \frac{f(t, v(t))}{v(t)}\right) \\ &\geq 0 \end{aligned}$$

for $t \in (t_*, t')$ and $y'(t) \not\equiv 0$ on (α, β). This implies $y(t') > y(t_*)$. This contradicts (2.15), so $v(t) \leq w(t)$. The proof is complete. □

By analogous methods in [19], we establish the following maximal theorem, which can be used in the proof of the uniqueness of positive solutions.

Lemma 2.4. (maximal theorem) Suppose that $0 < \eta < 1$, and $F = \{x \in C[0,1] \cap C^2(0,1), x(1) - ax(\eta) \geq 0, x(0) \geq 0\}$, if $x(t) \in F$ such that $-x''(t) \geq 0$ for $t \in (0,1)$, then $x(t) \geq 0$ for $t \in [0,1]$.

3. Proofs of Main Theorems

In this section, we'll always assume that $f(t, x) = \lambda x^p - K(t)x^{-q}$.

(A) The proof of Theorem 1.1.

Proof.

1) We consider the problem

$$\begin{cases} -x''(t) + K(t)x^{-q}(t) = \lambda x^p(t), & t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta), \end{cases} \tag{3.1.1}$$

where $0 < q, p < 1$, $K \in C[0,1]$, $K_* > 0$, $0 < a < 1$, $0 < \eta < 1$ and λ is a positive parameter.

In [19], when $f(t, x)$ is increasing in *x*, the problem

$$\begin{cases} -x''(t) = f(t, x), & t \in (0,1), \\ x(0) = ax(\eta), \quad x(1) = 0 \end{cases}$$

has an unique $C^1[0,1]$ positive solution. From that, suppose that $x_*(t)$ is an unique $C^1[0,1]$ positive solution of the problem

$$\begin{cases} -x''(t) = x^p(t), & t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta), \end{cases} \tag{3.1.2}$$

where $0 < a < 1$, $0 < \eta < 1$.

Set $\beta(t) = \lambda^{\frac{1}{1-p}} x_*(t)$. Then

$$\begin{aligned} -\beta''(t) + K(t)\beta^{-q}(t) &= \lambda^{\frac{1}{1-p}} x_*(t) + K(t)\lambda^{\frac{-q}{1-p}} x_*^{-q}(t) \\ &> \lambda^{\frac{1}{1-p}} x_*(t) + K_*\lambda^{\frac{-q}{1-p}} x_*^{-q}(t) \\ &> \lambda^{\frac{1}{1-p}} x_*^p(t), \end{aligned}$$

$$\lambda\beta^p(t)=\lambda^{\frac{1}{1-p}}x_*^p(t).$$

Thus $-\beta''(t)+K(t)\beta^{-q}(t)>\lambda\beta^p(t)$. Combining it with (3.1.2) we obtain

$$\begin{cases}-\beta''(t)+K(t)\beta^{-q}(t)>\lambda\beta^p(t), & t\in(0,1),\\ \beta(0)=0, \quad \beta(1)=a\beta(\eta).\end{cases}$$

Consequently, $\beta(t)$ is a upper solution of (3.1.1).

Set $\alpha(t)=M\varphi_1^{1+q}$, where M is a positive constant and φ_1 is the first eigenfunction. Then

$$-\alpha''(t)+K(t)\alpha^{-q}(t)=-\frac{2M}{1+q}\varphi_1^{\frac{1-q}{1+q}}(t)\varphi_1''(t)+\frac{K(t)}{M^q\varphi_1^{\frac{2q}{1+q}}}-\frac{2(1-q)M|\varphi_1'|^2}{(1+q)^2\varphi_1^{\frac{2q}{1+q}}}$$

$$=\frac{2\lambda M}{1+q}\varphi_1^{\frac{2}{1+q}}+\frac{K(t)}{M^q}\varphi_1^{\frac{2q}{1+q}}-\frac{2(1-q)M|\varphi_1'|^2}{(1+q)^2\varphi_1^{\frac{2q}{1+q}}}$$

$$<\frac{2\lambda M}{\varphi_1^{\frac{2}{1+q}}}+\frac{K^*}{M^q}\varphi_1^{\frac{2q}{1+q}}-\frac{2(1-q)M|\varphi_1'|^2}{(1+q)^2\varphi_1^{\frac{2q}{1+q}}}.$$

By Lemma 2.1 we have $\varphi_1(t)=\sin(\sqrt{\lambda_1}t)$, $\varphi_1(t)=\sqrt{\lambda_1}\cos(\sqrt{\lambda_1}t)$. Thus there exists $\delta_0>0$ and $b\in(0,1)$ such that

$$|\varphi_1'(t)|=|\sqrt{\lambda_1}\cos(\sqrt{\lambda_1}t)|>\delta_0, \quad t\in[0,b),$$

$$|\varphi_1(t)|=|\sin(\sqrt{\lambda_1}t)|>\delta_0, \quad t\in[b,1].$$

a) On $[0,b)$, choosing $M\ge M_1=[\frac{(1+q)^2K^*}{2(1-q)\delta_0^2}]^{\frac{1}{1+q}}$, then we have

$$\frac{K^*}{M^q\varphi_1^{\frac{2q}{1+q}}}\le\frac{\lambda_1 M}{1+q}\varphi_1^{\frac{2}{1+q}}.$$

b) On $[b,1]$, choosing $M\ge M_2=[\frac{(1+q)^2K^*}{2(1-q)\delta_0^2}]^{\frac{1}{1+q}}$, then we have

$$\frac{K^*}{M^q\varphi_1^{\frac{2q}{1+q}}}\le\frac{\lambda_1 M}{1+q}\varphi_1^{\frac{2}{1+q}}.$$

Fixing $M=\max\{M_1,M_2\}$, then

$$-\alpha''(t)+K(t)\alpha^{-q}(t)\le\frac{3\lambda_1 M}{1+q}\varphi_1^{\frac{2}{1+q}}$$

and

$$\lambda\alpha^p(t)=\lambda M^p\varphi_1^{\frac{2q}{1+q}}.$$

Set $\lambda_0=\frac{3M^{1-q}}{1+q}|\varphi_1|_\infty^{\frac{2-2p}{1+q}}$. Then we have

$$\frac{3M\lambda_1}{1+q}\varphi_1^{\frac{2}{1+q}} < \lambda M^p \varphi_1^{\frac{2p}{1+q}},\ \forall \lambda > \lambda_0.$$

Hence, $-\alpha''(t) + K(t)\alpha^{-q}(t) < \lambda\alpha^p(t)$, $\forall \lambda > \lambda_0$.

It follows from Lemma (2.1) that

$$\alpha(0) = M\varphi_1^{\frac{2}{1+q}}(0) = 0$$

and

$$\alpha(1) = M\varphi_1^{\frac{2}{1+q}}(1) = M[a\varphi_1(\eta)]^{\frac{2}{1+q}} = Ma^{\frac{2}{1+q}}\varphi_1^{\frac{2}{1+q}}(\eta) < aM\varphi_1^{\frac{2}{1+q}}(\eta) = a\alpha(\eta).$$

Set $\lambda_2 = (M|\frac{\varphi_1}{x_*}|_\infty |\varphi_1|_\infty^{\frac{1-q}{1+q}})^{1-p}$. Then $\alpha(t) = M\varphi_1^{\frac{2}{1+q}}(t) \le \lambda^{\frac{1}{1-p}} x_*(t) = \beta(t)$ for all $\lambda > \lambda_2$. Thus we choose $\bar{\lambda} = \max\{\lambda_0, \lambda_2\}$ and $\lambda > \bar{\lambda}$, then $(\alpha(t), \beta(t))$ is a couple of upper and lower solutions of (3.1.1).

We choose $F(t) = \lambda\beta^p + K^*\beta^{-q}$, then $|f(t,x)| \le F(t)$ for all $(t,x) \in D_\alpha^\beta$. It's easy to see that $F(t) \in L^1[0,1]$. From Corollary 2.1, the problem (3.1.1) has at least one $C[0,1]$ positive solution $x(t)$ satisfying $\alpha(t) \le x(t) \le \beta(t)$ for $\lambda > \bar{\lambda}$.

2) (Existence of the maximal solution) We observe the problem

$$\begin{cases} -x''(t) = \lambda x^p(t), & t \in (0,1), \\ x(0) = 0, & x(1) = ax(\eta). \end{cases} \tag{3.1.3}$$

From [19], we note the unique solution of (3.1.3) is $w_\lambda(t)$ for any $\lambda > 0$. In (1) we obtained the solution $x_\lambda(t)$ of (3.1.1) then we have

$$w_\lambda''(t) + \lambda w_\lambda^p(t) = 0 < x_\lambda''(t) + \lambda x_\lambda^p(t)$$

and $x^{-1}f(t,x) = \lambda x_\lambda^{p-1}(t)$ is decreasing in *x*. Noting that $x_\lambda(t) \in L^1[0,1]$ by (1). From Lemma 2.3, we have $x_\lambda(t) \le w_\lambda(t)$.

Let $\Omega_j = [\frac{1}{i_0 + j}, 1)$, $j = 1, 2, \cdots$ and $w_j(t)$ be the solution of

$$\begin{cases} -x''(t) + K(t)w_{j-1}^{-q}(t) = \lambda w_{j-1}^p(t), & t \in \Omega_j, \\ x(t) = w_{j-1}(t),\ t \in [0, \frac{1}{i_0 + j}), \\ x(1) = ax(\eta) \end{cases} \tag{3.1.4}$$

for $j = 1, 2, \cdots$, with $w_0(t) = w_\lambda(t)$ defined in (3.1.3). Let $x_\lambda(t)$ be a solution of (3.1.1).

In (3.1.4), letting $j = 1$ we have

$$\begin{cases} -w_1''(t) + K(t)w_\lambda^{-q}(t) = \lambda w_\lambda^p(t), & t \in \Omega_1, \\ w_1(t) = w_\lambda(t),\ t \in [0, \frac{1}{i_0 + j}), \\ w_1(1) = aw_1(\eta). \end{cases} \tag{3.1.5}$$

Combining (3.1.5) with (3.1.3) we have $w_1''(t) - w_\lambda''(t) \ge 0$ for $t \in \Omega_1$. By maximum principle, we have $w_1(t) \le w_0(t) = w_\lambda(t)$. Similarly, we can obtain that $w_{j+1}(t) \le w_j(t) \le w_\lambda(t)$.

Furthermore, we observe problem (3.1.1)

$$\begin{cases} -x''(t)+K(t)x^{-q}(t)=\lambda x^{p}(t), & t\in(0,1), \\ x(0)=0, \quad x(1)=ax(\eta). \end{cases}$$

Combining it with (3.1.5) we have

$$-w_1''(t)+x_\lambda''(t)+K(t)(w_\lambda^{-q}(t)-x_\lambda^{-q}(t))=\lambda(w_\lambda^{p}(t)-x_\lambda^{p}(t))\geq 0,$$

thus $x_\lambda''(t)-w_1''(t)\geq 0$ for $t\in\Omega_1$. It's easy to verify that $x_\lambda(t)\leq w_1(t)$ for $t\in[0,1]$ by maximum principle. By similar method we can obtain $x_\lambda(t)\leq w_{j+1}(t)\leq w_j(t)\leq w_\lambda(t)$ for $t\in[0,1]$.

Furthermore, we have $\{w_j(t)\}_{j\in N}$ is bounded from below by $x_\lambda(t)$.

Because $w_j(t)$ is a solution to (3.1.3),

$$\begin{aligned} -w_j''(t) &= \lambda w_{j-1}^{p}(t)-K(t)w_{j-1}^{-q}(t) \\ &\leq \lambda w_{j-1}^{p}(t)-K_* w_{j-1}^{-q}(t) \\ &\leq [\lambda w_{j-1}^{p+q}(t)-K_*]w_{j-1}^{-q}(t) \\ &\leq [\lambda w_{j-1}^{p+q}(t)-K_*]w_{j}^{-q}(t). \end{aligned}$$

Suppose that $t_0\in(0,1)$, $w_j(t_0)=\max_{0\leq t\leq 1} w_j(t)$, then $w_j'(t_0)=0$ and $w_j(t)$ is increasing on (t,t_0). By integration of $-w_j''(t)$ from t to t_0, we have

$$\int_t^{t_0} -w_j''(s)ds \leq \int_t^{t_0}[\lambda w_{j-1}^{p+q}(s)-K_*]w_j^{-q}(s)ds\ .$$

So $w_j'(t)w_j^{q}(t)\leq \lambda w_{j-1}^{p+q}(t_0)-K_*$. Similarly, by integration of $-w_j''(t)$ from t_0 to t, we can obtain $|w_j'(t)w_j(t)|\leq \lambda w_{j-1}^{p+q}(t_0)-K_*$. For giving $t_1,t_2\in[0,1]$, we have

$$\int_{t_1}^{t_2} w_j'(s)w_j^{q}(s)ds \leq \int_{t_1}^{t_2}|w_j'(s)w_j^{q}(s)|\,ds \leq \int_{t_1}^{t_2}[\lambda w_{j-1}^{p+q}(t_0)-K_*]ds.$$

We can find K large such that $|\lambda w_{j-1}^{p+q}(t_0)-K_*|<K$. Then

$$\int_{t_1}^{t_2} w_j'(s)w_j^{q}(s)ds \leq K|t_2-t_1|,\ |w_j^{q+1}(t_2)-w_j^{q+1}(t_1)|\leq K|t_2-t_1|. \quad (3.1.4)$$

We define an operator $I(w)=w^{q+1}$, then $I^{-1}(w)=w^{\frac{1}{q+1}}$. It follows from (3.1.4) that $\{I(w_j(t))\}_{j\in N}$ is a uniformly bounded and equicontinuous functions in [0,1]. Obviously, I^{-1} is uniformly continuous in a bounded and closed domain Ω, *i.e.*, for all $\varepsilon>0$, there exists a $\delta>0$ such that when w_1, $w_2\in\Omega$, $|w_1-w_2|<\delta$, we have $|I^{-1}(w_1)-I^{-1}(w_2)|<\varepsilon$. Since $0<w_j(t)<w_0(t)$, there exists a $M>0$ such that $w_j(t)\in(0,M]$. From (3.1.4), for the above $\delta>0$, there exists $\delta'>0$ such that when $|t_1-t_2|<\delta'$, we have $|w_j^{q+1}(t_2)-w_j^{q+1}(t_1)|<\delta$.

Therefore, for all $\varepsilon>0$, there exists $\delta'>0$ such that when $|t_1-t_2|<\delta'$, we have

$$|w_j(t_2)-w_j(t_1)|=|I^{-1}(w_j^{q+1}(t_2))-I^{-1}(w_j^{q+1}(t_1))|<\varepsilon\ .$$

Thus $\{w_j(t)\}_{j\in N}$ is equicontinuous. Using Arzela-Ascoli theorem, there exists a subsequence $\{w_{j_k}(t)\}_{j_k\in\{i\}}$ such that $\lim_{j_k\to+\infty} w_{j_k}(t)=\overline{x}_\lambda(t)$. Without loss of generality, we assume that

$$\lim_{j\to+\infty} w_j(t) = \overline{x}_\lambda(t), \quad t\in[0,1]. \tag{3.1.5}$$

In the following, we shall show that $\overline{x}_\lambda(t)$ is a C[0,1] positive solution of (3.1.1).

Fixing $t\in(0,1)(t\neq\frac{1}{2})$, then $w_j(t)$ can be stated

$$w_j(t) = w_j(\frac{1}{2}) + w_j{}'(\frac{1}{2})(t-\frac{1}{2}) + \int_{\frac{1}{2}}^{t}(s-t)[K(s)w_{j-1}^{-q}(s) - \lambda w_{j-1}^{p}(s)]ds. \tag{3.1.6}$$

Fixing $j\in N$, by Lagrange mean value theorem, there exists $t_n\in(\frac{1}{2},1)$ such that $x_\lambda(1) - w_j(\frac{1}{2}) \le w_j(1) - w_j(\frac{1}{2}) = w_{j'}(t_n)(1-\frac{1}{2}) < w_0(1)$.

So there exists $M_1>0$ such that $|w_{j'}(t_n)| < 2M_1$. Since $\{w_j(t)\}_{j\in N}$ is bounded in [0,1], we may assume that $m<w_j(t)<M_2$, $t\in[\frac{1}{2},t_n]$,

$$\begin{aligned}|\int_{\frac{1}{2}}^{t_n} -w_j''(s)ds| &= |\int_{\frac{1}{2}}^{t_n}[\lambda w_{j-1}^{p}(s) - K(s)w_{j-1}^{-q}(s)]ds| \\ &\le |\int_{\frac{1}{2}}^{t_n}[\lambda w_{j-1}^{p}(s) - K_* w_{j-1}^{-q}(s)]ds| \\ &\le \lambda M^p - K_* m^{-q}.\end{aligned}$$

Thus

$$|w_j{}'(\frac{1}{2})| - |w_j{}'(t_n)| \le |w_j{}'(\frac{1}{2}) - w_j{}'(t_n)| \le \lambda M_2^p - K^* m^{-q}$$

i.e.,

$$|w_j{}'(\frac{1}{2})| \le 2M_1 + \lambda M_2^p - K^* m^{-q}.$$

Thus both $\{w_j{}'(\frac{1}{2})\}_{j\in N}$ and $\{w_j(\frac{1}{2})\}_{j\in N}$ are bounded. Then they all have a convergence subsequence. Without loss of generality, we note the subsequences are $\{w_j(\frac{1}{2})\}_{j\in N}$ and $\{w_j{}'(\frac{1}{2})\}_{j\in N}$. And fixing $j\in N$, we assume $\lim_{j\to\infty} w_j{}'(\frac{1}{2}) = r_0$.

In equation (3.1.6), letting $j\to\infty$ we have

$$\overline{x}_\lambda(t) = \overline{x}_\lambda(\frac{1}{2}) + r_0(t-\frac{1}{2}) + \int_{\frac{1}{2}}^{t}(s-t)[K(s)\overline{x}_\lambda^{-q}(s) - \lambda\overline{x}_\lambda^{p}(s)]ds$$

for $t\in(0,1)$, *i.e.*, $-\overline{x}_\lambda{}''(t) + K(t)\overline{x}_\lambda^{-q}(t) = \lambda\overline{x}_\lambda^{p}(t)$. Therefore $\overline{x}_\lambda(t)$ is a C[0,1] positive solution of (3.1.1). Therefore $\overline{x}_\lambda(t)$ is the maximal solution of (3.1.1).

Next we shall verify the dependence on λ of maximal solution $\overline{x}_\lambda(t)$.

Let $H = \{\mu>0$: (3.1.1) has a C[0,1] positive solution with $\lambda=\mu\}$.

Obviously, by (1), $H\neq\varnothing$. Let $\lambda_1\in H$. and $\overline{x}_\lambda(t)$ be the corresponding maximal solution of (3.1.1) for $\lambda=\lambda_1$. Then for any $\lambda_2>\lambda_1>\overline{\lambda}$, $\overline{x}_{\lambda_1}{}''(t) + \lambda_1\overline{x}_{\lambda_1}^{p}(t) \ge 0, t\in(0,1)$. By Lemma (2.3), $\overline{x}_{\lambda_1}(t) \le w_{\lambda_2}(t)$ in [0,1]. Just replacing $x_\lambda(t)$ by $\overline{x}_{\lambda_1}(t)$ in above proof. We can easily find that

$$\begin{cases}-\overline{x}_{\lambda_1}''(t) + K(t)\overline{x}_{\lambda_1}^{-q}(t) = \lambda_1\overline{x}_{\lambda_1}^{p} \le \lambda_2\overline{x}_{\lambda_1}^{p}, & t\in(0,1), \\ -w_{\lambda_2}''(t) + K(t)w_{\lambda_2}^{-q}(t) \ge \lambda_2 w_{\lambda_2}^{p}(t).\end{cases}$$

Combining it with boundary conditions, we can obtain that $(\overline{x}_{\lambda_1}(t), w_{\lambda_2}(t))$ is a couple of lower and upper solutions of (3.1.1) for $\lambda = \lambda_2 > \lambda_1$. One can be prove that there is a solution $x_{\lambda_2}(t)$ of (3.1.1) with $\lambda = \lambda_2$ such that

$$\overline{x}_{\lambda_1}(t) \le x_{\lambda_2}(t) \le w_{\lambda_2}(t).$$

Therefore $\lambda_2 \in H$. Moreover, by (ii), for any $\lambda_2 > \lambda_1 \ge \overline{\lambda}$, $\overline{x}_{\lambda_2}(t) \ge \overline{x}_{\lambda_1}(t)$.

This completes the proof of Theorem 1.1. □

(B) The proof of Theorem 1.2.

Proof. 1) We consider the problem

$$\begin{cases} -x''(t) + K(t)x^{-q}(t) = \lambda x^p(t), & t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta), \end{cases} \tag{3.2.1}$$

where $q > 0, 0 < p < 1$, $K(t) \in C[0,1]$, $K^* < 0$, $0 < a < 1$, $0 < \eta < 1$ and λ is a positive parameter.

Now we consider an approximate problem of (3.2.1) as follows

$$\begin{cases} -x''(t) + K(t)x^{-q}(t) = \lambda x^p(t), & t \in (0,1), \\ x(0) = \dfrac{1}{n}, \quad x(1) = ax(\eta) + \dfrac{1}{n}, \end{cases} \tag{3.2.2}$$

where $0 < a < 1$, $0 < \eta < 1$, $n \ge 1$.

Let ε very small. We'll verify that $\alpha_n(t) = \varepsilon\varphi_1(t) + \frac{1}{n}$ is a lower solution of (3.2.2). Indeed, when n is big enough, we can obtain that $\varepsilon\varphi_1(t) + \frac{1}{n}$ is close to 0. Since $\lambda_1 \in (0, \sqrt{\frac{\pi}{2}})$ (see [6]), we can deduce

$$\begin{aligned} & -\alpha_n''(t) + K(t)\alpha_n^{-q}(t) - \lambda\alpha_n^p(t) \\ & = \lambda_1\varepsilon\varphi_1(t) + K(t)(\varepsilon\varphi_1(t) + \frac{1}{n})^{-q} - \lambda(\varepsilon\varphi_1(t) + \frac{1}{n})^p \\ & < \lambda_1\varepsilon\varphi_1(t) - \lambda(\varepsilon\varphi_1(t) + \frac{1}{n})^p \\ & < \varepsilon\varphi_1(t)[\lambda_1 - \lambda(\varepsilon\varphi_1(t) + \frac{1}{n})^{p-1}] \\ & < 0, \end{aligned}$$

$$\alpha_n(0) - \frac{1}{n} = \varepsilon\varphi_1(0) = 0$$

and $\alpha_n(1) - [a\alpha_n(\eta) + \frac{1}{n}] = \varepsilon a\varphi_1(\eta) + \frac{1}{n} - a\varepsilon\varphi_1(\eta) - \frac{a}{n} - \frac{1}{n} < 0$, which imply that $\alpha_n(t)$ is a lower solutions of (3.2.2).

In the following, we'll construct an upper solution of (3.2.2). Let

$$\beta(t) = -Mt^2 + (M + aM)t + M,$$

where M is big enough for $M > \{(2\lambda)^{\frac{1}{1-p}}, \frac{1}{n(1-a)}\}$. We can obtain

$$\begin{aligned}-\beta''(t)+K(t)\beta^{-q}(t)&=2M+K(t)[-Mt^2+(M+aM)t+M]^{-q}\\&>2M+K_*M^{-q}\\&>M,\end{aligned}$$

$$\begin{aligned}\lambda\beta^{p}(t)&=\lambda[-Mt^2+(M+aM)t+M]^{p}\\&<\lambda[\frac{M(1+a)^2}{4}+M]^{p}\\&<\lambda(2M)^{p},\end{aligned}$$

$$-\beta''(t)+K(t)\beta^{-q}(t)\geq\lambda\beta^{p}(t),$$

$$\begin{aligned}\beta(1)-(a\beta(\eta)+\frac{1}{n})&=(a+1)M-a[-M\eta^2+(M+aM)\eta+M]-\frac{1}{n}\\&>(a+1)M-2aM-\frac{1}{n}\\&=M-aM-\frac{1}{n}\\&>0\end{aligned}$$

and $\beta(0)-\frac{1}{n}=M-\frac{1}{n}>0$. It's easy to see that $\beta(t)$ is na upper solution of (3.2.2).

Choosing $F_n(t)=\lambda\beta^{p}-K_*\alpha_n^{-q}$, then $|f(t,x)|\leq F_n(t)$, for all $(t,x)\in D_{\alpha_n}^{\beta}$. It's easy to verify that $F_n(t)\in L^1[0,1]$. Because that ε is small and *n* is big enough, $\alpha_n(t)\leq\beta(t)$. From Corollary 2.1, $(\alpha_n(t),\beta(t))$ is a couple of upper and lower solutions of (3.2.2). And for all $n\in N$, (3.2.2) has at least one C[0,1] positive solution $x_n(t)$ such that $\alpha_n(t)\leq x_n(t)\leq\beta(t)$.

In the following, we shall obtain a result as follows, there exists a subsequence $\{x_{n_k}(t)\}$ and $x(t)$ such that $\lim_{n_k\to\infty}x_{n_k}(t)=x(t)$.

Since $\beta(t)\in C[0,1]\cap C^2(0,1)$, $\beta(t)$ is bounded. Therefore $\{x_n(t)\}_{n\in N}$ is a uniformly bounded sequence of functions in [0,1]. Because $x_n(t)$ is a C[0,1] positive solution of (3.2.2), $x_n(t)$ satisfies

$$\begin{aligned}-x_n''(t)&=\lambda x_n^{p}(t)-K(t)x_n^{-q}(t)\\&\leq\lambda x_n^{p}(t)-K_*x_n^{-q}(t)\\&\leq[\lambda x_n^{p+q}(t)-K_*]x_n^{-q}(t).\end{aligned}$$

Suppose that $t_0\in(0,1)$, $x_n(t_0)=\max_{0\leq t\leq1}x_n(t)$, then $x_n'(t_0)=0$ and $x_n(t)$ is increasing on (t,t_0). By integration of $-x_n''(t)$ from *t* to t_0, we have

$$\int_t^{t_0}-x_n''(s)ds\leq\int_t^{t_0}[\lambda x_n^{p+q}(s)-K_*]x_n^{-q}(s)ds.$$

So $x_n'(t)\leq\frac{1}{x_n^{q}(t)}[\lambda x_n^{p+q}(t_0)-K_*]$. We can find a $K>0$ such that $x_n'(t)x_n^{q}(t)\leq K$. And by integration of $-x_n(t)$ from t_0 to *t*, we have

$$\int_t^{t_0}-x_n''(s)ds\leq\int_t^{t_0}[\lambda x_n^{p+q}(s)-K_*]x_n^{-q}(s)ds.$$

So $-x_n'(t)\leq\frac{1}{x_n^{q}(t)}[\lambda x_n^{p+q}(t_0)-K_*]$. For above *K*, we have $|-x_n'(t)x_n^{q}(t)|\leq K$, *i.e.*, $|x_n'(t)x_n^q(t)|\le K$.

For giving $t_1,t_2\in[0,1]$, we have

$$\int_{t_1}^{t_2} x_n'(s)x_n^q(s)ds \le \int_{t_1}^{t_2}|x_n'(s)x_n^q(s)|ds \le \int_{t_1}^{t_2} Kds.$$

Then $\int_{t_1}^{t_2} x_n'(s)x_n^q(s)ds \le K|t_2-t_1|$. The above inequality can be rewritten as

$$\left|\int_{x_n(t_1)}^{x_n(t_2)} x_n^q(s)dx_n(s)\right|\le K|t_2-t_1|,\quad |x_n^{q+1}(t_2)-x_n^{q+1}(t_1)|\le K|t_2-t_1|. \tag{3.2.3}$$

We now define an operator $I(x)=x^{q+1}$, then $I^{-1}(x)=x^{\frac{1}{q+1}}$. It follows from (3.2.3) that $\{I(x_n(t))\}_{n\in N}$ is a uniformly bounded and equicontinuous functions in [0,1]. Obviously, I^{-1} is uniformly continuous in a bounded and closed domain Ω, *i.e.*, for all $\varepsilon>0$, there exists a $\delta>0$ such that $|I^{-1}(x_1)-I^{-1}(x_2)|<\varepsilon$ for $|x_1-x_2|<\delta$, $x_1,x_2\in\Omega$. Since $0<x_n(t)<\beta(t)$, there exists a $M>0$ such that $x_n(t)\in(0,M]$. From (3.2.3), for the above $\delta>0$, there exists $\delta'>0$ such that $|x_n^{q+1}(t_2)-x_n^{q+1}(t_1)|<\delta$ for $|t_1-t_2|<\delta'$.

Therefore, for all $\varepsilon>0$, there exists $\delta'>0$ such that

$$|x_n(t_2)-x_n(t_1)|=|I^{-1}(x_n^{q+1}(t_2))-I^{-1}(x_n^{q+1}(t_1))|<\varepsilon$$

for $|t_1-t_2|<\delta'$. Consequently, $\{x_n(t)\}_{n\in N}$ is equicontinuous. Using Arzela-Ascoli theorem, there exists a subsequence $\{x_{n_k}(t)\}$ such that $\lim_{n_k\to+\infty} x_{n_k}(t)=x(t)$. Without loss of generality, we assume that

$$\lim_{n\to+\infty} x_n(t)=x(t),\quad t\in[0,1]. \tag{3.2.4}$$

In the following, we shall show that $x(t)$ is a C[0,1] positive solution of (3.2.1). Fixing $t\in(0,1)(t\ne\frac{1}{2})$, $x_n(t)$ can be stated

$$x_n(t)=x_n(\frac{1}{2})+x_n'(\frac{1}{2})(t-\frac{1}{2})+\int_{\frac{1}{2}}^{t}(s-t)[K(s)x_n^{-q}(s)-\lambda x_n^p(s)]ds. \tag{3.2.5}$$

Fixing $n\in N$, by Lagrange mean value theorem, there exists $t_n\in(\frac{1}{2},1)$ such that $\alpha_n(1)-x_n(\frac{1}{2})\le x_n(1)-x_n(\frac{1}{2})=x_n'(t_n)(1-\frac{1}{2})\le\beta(1)$.

So there exists $M_1>0$ such that $|x_n'(t_n)|\le 2M_1$. Since $\{x_n(t)\}_{n\in N}$ is bounded in [0,1], we may assume that $m\le x_n(t)\le M_2$, $t\in[\frac{1}{2},t_n]$.

$$\left|\int_{\frac{1}{2}}^{t_n} -x_n''(s)ds\right|=\left|\int_{\frac{1}{2}}^{t_n}[\lambda x_n^p(s)-K(s)x_n^{-q}(s)]ds\right|.$$

We can obtain

$$|-x_n'(t_n)+x_n'(\frac{1}{2})|\le\lambda M_2^p-K_*M_2^{-q} \text{ and } |x_n'(\frac{1}{2})|\le 2M_1+\lambda M_2^p-K_*M_2^{-q}.$$

Therefore both $\{x_n(\frac{1}{2})\}_{n\in N}$ and $\{x_n'(\frac{1}{2})\}_{n\in N}$ are bounded. They all have a convergence subsequence. Without loss of generality, we note the subsequences are $\{x_n(\frac{1}{2})\}_{n\in N}$ and $\{x_n'(\frac{1}{2})\}_{n\in N}$. And fixing $n\in N$, we assume $\lim_{n\to\infty} x_n'(\frac{1}{2})=r_0$.

From (3.2.5), letting $n\to\infty$, we obtain

$$x(t)=x(\frac{1}{2})+r_0(t-\frac{1}{2})+\int_{\frac{1}{2}}^{t}(s-t)[K(s)x^{-q}(s)-\lambda x^{p}(s)]ds.$$

By derivation twice of $x(t)$, we have

$$-x''(t)+K(t)x^{-q}(t)=\lambda x^{p}(t).$$

Combining it with (3.2.4), we can obtain that $x(t)$ is a C[0,1] positive solution of (3.2.1).

2) We study the uniqueness of $C^1[0,1]$ positive solution of problem (3.2.1).

Let $F(t)=\lambda\beta^{p}-K_*(\varepsilon\varphi_1)^{-q}$. Obviously, when $0<q<1$, $F(t)$ is integrable over (0,1). Since $|x''(t)|\le F(t)$, $x(t)$ is absolutely integrable over (0,1). Then both $x'(0+)$ and $x'(1-)$ exist, *i.e.*, $x(t)\in C^1[0,1]$.

Suppose conversely that $x_1(t)$, $x_2(t)$ are two $C^1[0,1]$ positive solutions of the problem (3.2.1), $x_1(t)\not\equiv x_2(t)$ on [0,1]. We may assume without loss of generality that there exists $t^*\in(0,1)$ such that $x_2(t^*)-x_1(t^*)=\max_{0\le t\le 1}(x_2(t)-x_1(t))>0$. Let

$$\alpha=\inf\{t_1\mid 0\le t_1<t^*,x_2(t)>x_1(t),t\in(t_1,t^*)\},$$

$$\beta=\sup\{t_2\mid t^*\le t_2<1,x_2(t)>x_1(t),t\in(t^*,t_2)\}.$$

It's obvious that $0\le\alpha<\beta\le 1$ and

$$x_1(\alpha)=x_2(\alpha),x_1'(\alpha)\le x_2'(\alpha),x_1(\beta)\le x_2(\beta),$$
$$x_1'(\beta+)\ge x_2'(\beta+),x_1(t)<x_2(t),t\in(\alpha,\beta).$$

Let $y(t)=x_1(t)x_2'(t)-x_2(t)x_1'(t),t\in(\alpha,\beta)$. Then we have

$$\liminf_{t\to\alpha+}y(t)\ge 0\ge\limsup_{t\to\beta+}y(t). \tag{3.2.6}$$

On the other hand,

$$\begin{aligned}y'(t)&=x_1x_2''-x_2x_1''\\&=x_1(Kx_2^{-q}-\lambda x_2^{p})+x_2(\lambda x_1^{p}-Kx_1^{-q})\\&=Kx_1x_2^{-q}-\lambda x_1x_2^{p}+\lambda x_1^{p}x_2-Kx_1^{-q}x_2\\&=Kx_1x_2(x_2^{-q-1}-x_1^{-q-1})+\lambda x_1x_2(x_1^{p-1}-x_2^{p-1})\\&\ge 0\end{aligned}$$

for $t\in(\alpha,\beta)$ and $y'(t)\not\equiv 0$ on (α,β). This implies $y(\beta-)>y(\alpha+)$, contradicts (3.2.6), so $x_1(t)\equiv x_2(t)$. Thus the $C^1[0,1]$ positive solution of (3.2.6) is unique.

3) We assume that $0<\lambda_1<\lambda_2$ and $x_{\lambda_1}(t)$, $x_{\lambda_2}(t)$ are the corresponding unique $C^1[0,1]$ positive solutions to (3.2.1). Obviously, $x_{\lambda_1}''(t)\in L^1[0,1]$. In (3.2.1), $f(t,x)=\lambda x^{p}(t)-K(t)x^{-q}(t)$ is continuous.

Since $p,q\in(0,1)$, $K_*<0$, it's easy to see that $x^{-1}f(t,x)=\lambda x^{p-1}(t)-K(t)x^{-q-1}(t)$ is decreasing for $x>0$ at each $t\in[0,1]$.

$$x_{\lambda_2}''(t)-K(t)x_{\lambda_2}^{-q}(t)+\lambda_2x_{\lambda_2}^{p}(t)=0<x_{\lambda_1}''(t)-K(t)x_{\lambda_1}^{-q}(t)+\lambda_2x_{\lambda_1}^{p}(t)$$

for $t\in(0,1)$, $x_{\lambda_2}(0)\ge x_{\lambda_1}(0)$, $x_{\lambda_2}(1)\ge ax_{\lambda_2}(\eta)$ and $x_{\lambda_1}(1)\ge ax_{\lambda_1}(\eta)$. Therefore, by Lemma 2.3,

$$x_{\lambda_1(t)} \le x_{\lambda_2}(t), \quad t \in [0,1].$$

So $x(t)$ is increasing with respect to λ.

This completes the proof of Theorem 1.2. □

(C) The proof of Theorem 1.3.

Proof.

1) We consider the problem

$$\begin{cases} -x''(t) + K(t)x^{-q}(t) = \lambda x^p(t), \quad t \in (0,1), \\ x(0) = 0, \quad x(1) = ax(\eta), \end{cases} \tag{3.3.1}$$

where $0 < p, q < 1$, $K(t) \in C[0,1]$, $K_* < 0 < K^*$, $0 < a < 1$, $0 < \eta < 1$ and λ is a positive parameter.

Since $K^* > 0 > K_*$, then by Theorem 1.1, there exists a $\lambda^* > 0$, such that for $\lambda > \lambda^*$, the problem

$$\begin{cases} -v''(t) + K^* v^{-q}(t) = \lambda v^p(t), \quad t \in (0,1), \\ v(0) = 0, \quad v(1) = av(\eta) \end{cases}$$

has a maximal solution $v_\lambda(t)$. Let $v_k(t) = v_\lambda(t) + \frac{1}{k}$. We observe that

$$\begin{aligned} -v_k''(t) + K(t)v_k^{-q}(t) &= -v_\lambda''(t) + K(t)(v_\lambda + \frac{1}{k})^{-q} \\ &= \lambda v_\lambda^p(t) - K^* v_\lambda^{-q}(t) + K(v_\lambda + \frac{1}{k})^{-q} \\ &< \lambda v_\lambda^p(t) + K^*(v_\lambda + \frac{1}{k})^{-q} - K^* v_\lambda^{-q}(t) \\ &< \lambda v_\lambda^p(t). \end{aligned}$$

$$\lambda v_k^p(t) = \lambda(v_\lambda + \frac{1}{k}) > \lambda v_\lambda^p(t),$$

$$v_k(0) = v_\lambda(0) + \frac{1}{k} = \frac{1}{k} \quad \text{and} \quad v_k(1) = v_\lambda(1) + \frac{1}{k} = av_\lambda(\eta) + \frac{1}{k} \le av_k(\eta) + \frac{1}{k}.$$

Consequently, $v_k(t) = v_\lambda(t) + \frac{1}{k}$ is a lower solution of $P_k(\lambda)$:

$$\begin{cases} -x''(t) + K(t)x^{-q}(t) = \lambda x^p(t), \quad t \in (0,1), \\ x(0) = \frac{1}{k}, \quad x(1) = ax(\eta) + \frac{1}{k}. \end{cases}$$

On the other hand, the problem

$$\begin{cases} -w''(t) + K_* w^{-q}(t) = \lambda w^p(t), \quad t \in (0,1), \\ w(0) = \frac{1}{k}, \quad w(1) = aw(\eta) \end{cases}$$

has a solution $w_k(t)$ for any $k \in N$. Then

$$\begin{aligned} -w_k''(t) + K(t)w_k^{-q}(t) &= \lambda w_k^p(t) - K_* w_k^{-q}(t) + K(t)w_k^{-q}(t) \\ &> \lambda w_k^p(t) - K_*(t)w_k^{-q}(t) + K_* w_k^{-q}(t), \\ &= \lambda w_k^p(t). \end{aligned}$$

So we have

$$\begin{cases} -w_k''(t) + K(t)w_k^{-q}(t) > \lambda w_k^p(t), & t \in (0,1) \\ w_k(0) = \dfrac{1}{k}, \\ w_k(1) = aw_k(\eta). \end{cases}$$

Therefore, $w_k(t)$ is an upper solution of $P_k(\lambda)$. Since $w_k''(t) + \lambda w_k^p(t) \le 0 \le v_k''(t) + \lambda v_k^p(t)$, $w(0) \ge v(0)$, $w(1) \ge aw(\eta)$, $v(1) \le av(\eta)$, $v''(t) \in L^1[0,1]$ and $x^{-1} f(t,x) = \lambda x^{p-1}(t)$ is decreasing in *x*, by Lemma 2.3,

$$v_k(t) \le w_k(t), \quad t \in [0,1].$$

Obviously, there exists a minimal solution $x_\lambda^{(1)}(t)$ of $P_1(\lambda)$, satisfying $v_1(t) \le x_\lambda^{(1)}(t) \le w_1(t)$. Similarly, taking $x_\lambda^{(1)}(t)$ and $v_2(t)$ as a couple of lower and upper solutions for $P_2(\lambda)$, we conclude that there exists a minimal solution $x_\lambda^{(2)}(t)$ of $P_2(\lambda)$ such that

$$v_2(t) \le x_\lambda^{(2)}(t) \le x_\lambda^{(1)}(t).$$

Repeating the above arguments, we obtain a sequence $\{x_\lambda^{(k)}(t)\}_{k\in N}$ which is decreasing in *k*. Therefore, similar to the proof of Theorem 1.2 (1), we obtain a solution $x_\lambda(t) = \lim_{k\to\infty} x_\lambda^{(k)}(t)$, and $v_\lambda(t) \le x_\lambda(t) \le w_1(t)$.

2) (Dependence on λ) Let $\lambda^* < \lambda_1 < \lambda_2$, $x_{\lambda_1}(t)$ and $x_{\lambda_2}(t)$ be the corresponding solutions of (3.3.1) for $\lambda = \lambda_1$ and λ_2 which we obtained in (1). We observe that

$$\begin{cases} -(x_{\lambda_2}^{(k)}(t))'' + K_*(x_{\lambda_2}^{(k)}(t))^{-q} = \lambda_2 (x_{\lambda_2}^{(k)}(t))^p \ge \lambda_1 (x_{\lambda_2}^{(k)}(t))^p, & t \in (0,1), \\ x_{\lambda_2}^{(k)}(0) = 0, \quad x_{\lambda_2}^{(k)}(1) = ax_{\lambda_2}^{(k)}(\eta), \end{cases}$$

$x_{\lambda_2}^{(k)}(t)$ is an upper solution of $P_k(\lambda_1)$, and

$$x_{\lambda_2}^{(k)}(t) \ge v_{\lambda_2}(t) + \frac{1}{k} \ge v_{\lambda_1}(t) + \frac{1}{k}, \ t \in [0,1].$$

Therefore $x_{\lambda_2}^{(k)}(t) \ge x_{\lambda_1}^{(k)}(t)$, since $x_{\lambda_1}^{(k)}(t)$ is a minimal solution of $P_k(\lambda_1)$ which satisfies $x_{\lambda_1}^{(k)}(t) \ge v_{\lambda_1}(t) + \dfrac{1}{k}$. Therefore we must have $x_{\lambda_1}(t) \le x_{\lambda_2}(t)$.

Thus Theorem 1.3 is true. □

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Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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