Invariance of Weighted Bajraktarević Mean with Respect to the Beckenbach-Gini means

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Abstract
Under some conditions on the functions $\varphi$ and $\psi$ defined on $I$, the weighted Bajraktarević mean is given by

$$B^{\varphi,\psi}_{\lambda,\mu}(x,y) = \left(\frac{\varphi}{\psi}\right)^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)ight) + \left(\mu\psi(x) + (1-\mu)\psi(y)\right), \quad x, y \in I,$$

where $\lambda, \mu \in [0,1]$. In this paper, we study the invariance of the weighted Bajraktarević mean with respect to Beckenbach-Gini means.

Keywords
Weighted Bajraktarević Mean, Beckenbach-Gini Mean, Invariance Equation, Functional Equation

1. Introduction
Let $I \subset \mathbb{R}$ be an open interval. A two-variable function $M : I^2 \to I$ is called a mean on the interval $I$ if

$$\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}, \quad x, y \in I$$

holds. If for all $x, y \in I, x \neq y$, these inequalities are strict, $M$ is called strict. Obviously, if $M$ is a mean, then $M$ is reflexive, i.e., $M(x,x) = x$ for all $x \in I$.

A quasi-arithmetic mean, generated by the function $\varphi$, is defined by

$$M(x,y) = \mathcal{A}_\varphi(x,y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right), \quad x, y \in I,$$

for a continuous, strictly monotone function $\varphi : I \to \mathbb{R}$.

A more general mean is the class of the weighted quasi-arithmetic means, which is defined by

$$M(x,y) = \mathcal{A}_{\varphi,\lambda}(x,y) := \varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right), \quad x, y \in I,$$
where $\varphi : I \rightarrow \mathbb{R}$ is a continuous strictly monotone function, and the constant $\lambda \in (0,1)$.

A Lagrangian mean is defined by

$$M(x,y) = \mathcal{L}_\varphi(x,y) = \begin{cases} \varphi^{-1}\left(\frac{1}{y-x}\int_x^y \varphi(t)\,dt\right), & \text{if } x \neq y, \quad x, y \in I, \\ x, & \text{if } x = y, \end{cases}$$

where $\varphi : I \rightarrow \mathbb{R}$ is a continuous strictly monotone function.

Given the continuous functions $\varphi, \psi : I \rightarrow \mathbb{R}$ satisfy $\psi(x) \neq 0$ for $x \in I$ and $\frac{\varphi}{\psi}$ is one-to-one, the Bajraktarević mean of generators $\varphi$ and $\psi$ [1] is defined by

$$M(x,y) = B^{[\varphi,\psi]} := \left(\varphi^{(x)}/\psi^{(y)}\right)^{-1}\left(\varphi(x) + \varphi(y)\right) / \left(\psi(x) + \psi(y)\right), \quad x, y \in I. \quad (1.1)$$

$B^{[\varphi,\psi]}$ is a strict mean, and it is a generalization of quasi-arithmetic mean. Note that if $\varphi(x) = x, x \in I$, we have

$$B^{[\varphi,\psi]} = B^{[\varphi]} := \frac{x\psi(x) + y\psi(y)}{\psi(x) + \psi(y)}, \quad x, y \in I, \quad (1.2)$$

where the mean $B^{[\varphi]}$ is called Beckenbach-Gini mean of a generator $\psi$ [2].

Quotient mean $Q^{[\varphi,\psi]} : I^2 \rightarrow \mathbb{R}$ is defined by

$$Q^{[\varphi,\psi]}(x,y) := \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\varphi(x)}{\psi(x)} + \frac{\varphi(y)}{\psi(y)}\right), \quad x, y \in I, \quad (1.3)$$

where the functions $\varphi$ and $\psi$ are continuous, positive, and of different type of strict monotonicity in $I$ [3]. For $I = (0,\infty), \varphi(x) = x, \psi(x) = \frac{1}{x}$, we have

$$Q^{[\varphi,\psi]}(x,y) = \sqrt{xy} = G, \text{ where } G \text{ is geometric mean.}$$

Now we define the weighted Bajraktarević mean as follows:

$$M(x,y) = B^{[\varphi,\psi]}_{\lambda,\mu} := \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\lambda \varphi(x) + (1-\lambda) \varphi(y)}{\mu \psi(x) + (1-\mu) \psi(y)}\right), \quad x, y \in I, \quad (1.4)$$

where $\lambda, \mu \in [0,1]$, $\varphi, \psi : I \rightarrow \mathbb{R}$ are continuous, positive, and of different type of strict monotonicity and $\frac{\varphi}{\psi}$ is one-to-one. Note that if $\lambda = \mu = \frac{1}{2}$, $B^{[\varphi,\psi]}_{1,1} = B^{[\varphi,\psi]}$.

If $\lambda = 1, \mu = 0$, the weighted Bajraktarević mean becomes quotient mean, that is $B^{[\varphi,\psi]}_{1,0} = Q^{[\varphi,\psi]}(x,y)$. Without any loss of generality, we can assume that $\varphi$ is strictly increasing and $\psi$ is strictly decreasing.

Let $M, N : I^2 \rightarrow I$ be means. A mean $K : I^2 \rightarrow I$ is called invariant with respect to the mean-type mappings $(M, N)$, shortly, $(M, N)$-invariant [4], if

$$K(M(x,y), N(x,y)) = K(x,y), \quad x, y \in I.$$
identity
\[ G(A(x, y), H(x, y)) = G(x, y), \quad x, y > 0, \]
where \( A, H, G \) denote the arithmetic, harmonic and geometric means, respectively.

The invariance of the arithmetic mean with respect to various quasi-arithmetic means has been extensively investigated. Firstly we came upon the work of Sutô [5] [6] presented in 1914, in which he gave analytic solutions for the invariance equation
\[ A_\phi(x, y) + A_\psi(x, y) = x + y, \quad x, y \in I. \quad (1.5) \]
Then Matkowski solved the above equation under assumptions that \( \phi(x) \) and \( \psi(x) \) are twice continuously differentiable [4]. These regularity assumptions were weaken step-by-step by Daróczy, Maksa and Páles in [7] [8]. Finally, without any regularity assumptions, the problem was solved by Daróczy and Páles in [9].

Also, the form of Equation (1.5) was generalized by many authors. Concretely, Burai considered the invariance of the arithmetic mean with respect to weighted quasi-arithmetic means in [10]. Daróczy, Hajdu, Jarczyk and Matkowski studied the invariance equation involving three weighted quasi-arithmetic means [11] [12] [13]. Matkowski solved the invariance equation involving the arithmetic mean in class of Lagrangian mean-type mappings [14]. In [15], Makó and Páles investigated the invariance of the arithmetic mean with respect to generalized quasi-arithmetic means. The invariance of the geometric mean in class of Lagrangian mean-type mappings has been studied by Głazowska and Matkowski in [16]. All pairs of Stolarsky’s means for which the geometric mean is invariant were determined in [17]. Zhang and Xu considered the invariance of the geometric mean with respect to generalized quasi-arithmetic means in [18] and some invariance of the quotient mean with respect to Makó-Páles means in [19]. Recently, Jarczyk provided a review on the invariance of means [20].

Matkowski studied the invariance of the quotient mean with respect to weighted quasi-arithmetic mean type mapping [3]. He also studied the invariance of the Bajraktarević means with respect to quasi-arithmetic means in [21] and the invariance of the Bajraktarević means with respect to the Beckenbach-Gini means in [22]. Motivated by the above mentioned works, in this paper, we study the invariance of the weighted Bajraktarević mean with respect to the Beckenbach-Gini means, i.e., solve the functional equation
\[ B_{\phi, \psi}^{[\alpha, \beta]}(x, y) + B_{\phi, \psi}^{[\alpha, \beta]}(x, y) = B_{\phi, \psi}^{[\alpha, \beta]}(x, y), \quad x, y \in I, \quad (1.6) \]
where \( I \subset \mathbb{R}, \ \phi, \psi : I \rightarrow (0, +\infty) \) are continuous functions and \( \phi \) is strictly increasing, \( \psi \) is strictly decreasing.

### 2. Main Result

**Lemma 1.** Let \( I \subset \mathbb{R} \) be an interval. Suppose that the function \( \phi : I \rightarrow (0, +\infty) \) is differentiable, then we have
\[ \frac{\partial B^{[\theta]}(x,x)}{\partial x} = \frac{1}{2}. \quad (2.1) \]

If the function \( \varphi : I \to (0, +\infty) \) is twice differentiable, then we have
\[
\frac{\partial^2 B^{[\theta]}(x,x)}{\partial x^2} = \frac{\varphi''(x)}{2\varphi(x)}. \quad (2.2)
\]

**Proof.** By the definition of \( B^{[\theta]} \), we have
\[
\frac{\partial B^{[\theta]}(x,y)}{\partial x} = \frac{\varphi^2(x) + \varphi(x)\varphi(y) + x\varphi'(x)\varphi(y) - y\varphi'(x)\varphi(y)}{(\varphi(x) + \varphi(y))^2},
\]
then let \( y = x \), we can get that
\[
\frac{\partial B^{[\theta]}(x,x)}{\partial x} = \frac{1}{2}.
\]

Also we have
\[
\frac{\partial^2 B^{[\theta]}(x,y)}{\partial x^2} = \frac{2\varphi'(x)\varphi(y) + x\varphi'(x)\varphi(y) - y\varphi'(x)\varphi(y)}{(\varphi(x) + \varphi(y))^2} - \frac{2\varphi'(x)(x\varphi'(x)\varphi(y) - y\varphi'(x)\varphi(y))}{(\varphi(x) + \varphi(y))^3},
\]
letting \( y = x \), we can get (2.2).

**Lemma 2.** Let \( I \subset \mathbb{R} \) be an interval and \( \lambda, \mu \in [0,1], \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2} \). Suppose that the functions \( \varphi, \psi : I \to (0, +\infty) \) is differentiable, \( \varphi \) strictly increasing, \( \psi \) strictly decreasing and \( \frac{\varphi}{\psi} \) is one-to-one. If \( B_{\lambda,\mu}^{[\psi,\varphi]} \) is invariant with respect to the mean-type mapping \( \left(B_{\psi}^{[\psi]}, B_{\varphi}^{[\varphi]}\right) \) i.e., the Equation (1.6) holds, then there exists a positive number \( c \) such that
\[
\psi(x) = c\varphi(x)^{1-2\lambda}, \quad x \in I. \quad (2.3)
\]

**Proof.** By the definition of the mean \( B_{\lambda,\mu}^{[\psi,\varphi]} \) and (1.6) we have
\[
\left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\lambda\varphi\left(B_{\psi}^{[\psi]}(x,y)\right) + (1-\lambda)\varphi\left(B_{\varphi}^{[\varphi]}(x,y)\right)}{\mu\psi\left(B_{\psi}^{[\psi]}(x,y)\right) + (1-\mu)\psi\left(B_{\varphi}^{[\varphi]}(x,y)\right)}\right)
\]
\[
= \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\lambda\varphi(x) + (1-\lambda)\varphi(y)}{\mu\psi(x) + (1-\mu)\psi(y)}\right), \quad x, y \in I,
\]
whence, for all \( x, y \in I \)
\[
\left(\lambda\varphi\left(B_{\psi}^{[\psi]}(x,y)\right) + (1-\lambda)\varphi\left(B_{\varphi}^{[\varphi]}(x,y)\right)\right)\left(\mu\psi(x) + (1-\mu)\psi(y)\right)
\]
\[
= \left(\mu\psi\left(B_{\psi}^{[\psi]}(x,y)\right) + (1-\mu)\psi\left(B_{\varphi}^{[\varphi]}(x,y)\right)\right)\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right) \quad (2.4)
\]

Differentiating the above equation with respect to \( x \), we get that
\[
\left(\lambda\varphi\left(B_{\psi}^{[\psi]}\right)\frac{\partial B_{\psi}^{[\psi]}}{\partial x} + (1-\lambda)\varphi'\left(B_{\varphi}^{[\varphi]}\right)\frac{\partial B_{\varphi}^{[\varphi]}}{\partial x}\right)\left(\mu\psi(x) + (1-\mu)\psi(y)\right)
\]
\[
+ \left(\lambda\varphi\left(B_{\psi}^{[\psi]}\right) + (1-\lambda)\varphi\left(B_{\varphi}^{[\varphi]}\right)\right)\mu\psi'(x)
\]
\[
\begin{align*}
&= \left( \mu \psi' \left( B^{[\nu]} \right) \frac{\partial B^{[\nu]}}{\partial x} + (1 - \mu) \psi' \left( B^{[\nu]} \right) \frac{\partial B^{[\nu]}}{\partial x} \right) \left( \lambda \phi(x) + (1 - \lambda) \psi(y) \right) \\
&+ \left( \mu \psi \left( B^{[\nu]} \right) + (1 - \mu) \psi \left( B^{[\nu]} \right) \right) \lambda \phi'(x)
\end{align*}
\]

Then, letting \( y = x \), since \( B^{[\nu]} (x, x) = B^{[\nu]} (x, x) = x \) and Lemma 1 we obtain

\[
\begin{align*}
\left( \frac{1}{2} - \lambda \right) \phi'(x) \psi(x) &= \left( \frac{1}{2} - \mu \right) \phi(x) \psi'(x), \quad x \in I,
\end{align*}
\]

that is,

\[
\frac{\psi'(x)}{\psi(x)} = \frac{1 - 2\lambda}{1 - 2\mu} \phi'(x).
\]

Thus we can get that (2.3) holds.

**Theorem 1.** Let \( I \subset \mathbb{R} \) be an interval and \( \lambda, \mu \in [0,1], \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2} \). Suppose that the functions \( \phi, \psi : I \to (0, +\infty) \) is twice differentiable, \( \phi \) strictly increasing, \( \psi \) strictly decreasing and \( \frac{\phi}{\psi} \) is one-to-one. Then if the weighted Bajraktarević mean \( B^{[\nu,\mu]}_{\lambda,\mu} \) is invariant with respect to the mean-type mapping \( \left( B^{[\nu]}, B^{[\nu]} \right) \), that is (1.6) holds, then there exist \( a, b, p, q \in \mathbb{R}, p, q \neq 0, a, b > 0 \), such that

\[
\phi(x) = ae^p, \quad \psi(x) = be^q, \quad x \in I;
\]

where \( q = \frac{1 - 2\lambda}{1 - 2\mu} p \).

**Proof.** Assume that \( B^{[\nu,\mu]}_{\lambda,\mu} \) is invariant with respect to the mean-type mapping \( \left( B^{[\nu]}, B^{[\nu]} \right) \). Then the equality (2.4) is satisfied. Differentiating two times (2.4) with respect to \( x \), we get

\[
\begin{align*}
&= \left( \mu \psi' \left( B^{[\nu]} \right) \frac{\partial B^{[\nu]}}{\partial x} + (1 - \mu) \psi' \left( B^{[\nu]} \right) \frac{\partial B^{[\nu]}}{\partial x} \right) \left( \lambda \phi(x) + (1 - \lambda) \psi(y) \right) \\
&+ \left( \mu \psi \left( B^{[\nu]} \right) + (1 - \mu) \psi \left( B^{[\nu]} \right) \right) \lambda \phi'(x)
\end{align*}
\]
Letting \( y = x \) and dividing \( \varphi(x)\psi(x) \), since Lemma 1, we get that
\[
\left( \frac{1}{4} - \lambda \right) \frac{\varphi^*(x)}{\varphi(x)} - \left( \frac{1}{4} - \mu \right) \frac{\psi^*(x)}{\psi(x)} + \left( \frac{1}{2} - \frac{3}{2}\lambda + \frac{1}{2}\mu \right) \frac{\varphi'(x)}{\varphi(x)} \frac{\psi'(x)}{\psi(x)}
+ \frac{\lambda}{2} \left( \frac{\varphi'(x)}{\varphi(x)} \right)^2 - \frac{1}{2} \left( \frac{\psi'(x)}{\psi(x)} \right)^2 = 0.
\]
(2.7)

From Formula (2.5), after simple calculations, we have
\[
\frac{\psi'(x)}{\psi(x)} = 1 - 2\lambda \frac{\varphi'(x)}{\varphi(x)} + \frac{1}{2} \left( 1 - 2\lambda \right) \frac{\varphi^*(x)}{\varphi(x)}
= \frac{1 - 2\lambda}{1 - 2\mu} \frac{\varphi^*(x)}{\varphi(x)} + \frac{1 - 2\lambda}{1 - 2\mu} \left( 1 - 2\mu \right) \frac{\varphi'(x)}{\varphi(x)} \left( \frac{\varphi'(x)}{\varphi(x)} \right)^2.
\]

Substituting them into Equation (2.7), we get that
\[
\frac{\varphi^*(x)}{\varphi(x)} - \left( \frac{\varphi'(x)}{\varphi(x)} \right)^2 = 0,
\]
that is
\[
\left( \frac{\varphi'(x)}{\varphi(x)} \right)^2 = 0.
\]

Solving this equation we obtain, for some \( a, p \in \mathbb{R}, \ p \neq 0, \ a > 0 \)
\[
\varphi(x) = ae^{px}.
\]
(2.8)

Since Lemma 2, we can get that \( \psi(x) = be^{qx} \) where \( q = \frac{1 - 2\lambda}{1 - 2\mu}, \ p \) and \( b = \frac{c}{a} > 0 \).

**Corollary 1.** Let \( I \subset \mathbb{R} \) be an interval and \( \lambda, \mu \in [0,1], \lambda \neq 0, \mu \neq \frac{1}{2}, \lambda + \mu = 1 \).
Suppose that the functions \( \varphi, \psi : I \to (0, +\infty) \) is twice differentiable, \( \varphi \) strictly increasing, \( \psi \) strictly decreasing and \( \frac{\varphi}{\psi} \) is one-to-one. Then the following conditions are equivalent:
1) \( B_{x,\mu}^{[\varphi,\psi]} \) is invariant with respect to the mean-type mapping \( (B^{[\varphi]}, B^{[\psi]}) \), i.e.,
\[
B_{x,\mu}^{[\varphi,\psi]} (B^{[\varphi]}, B^{[\psi]}) = B_{x,\mu}^{[\varphi,\psi]};
\]
2) there exist \( a, b, p \in \mathbb{R}, \ p \neq 0, \ a, b > 0 \), such that
\[
\varphi(x) = ae^{px}, \ \psi(x) = be^{-px}, \ x \in I ;
\]
3) there exist \( p \in \mathbb{R}, \ p \neq 0 \) such that
\[
B_{x,\mu}^{[\varphi,\psi]} (x, y) = \frac{x + y}{2}, \ B^{[\varphi]}(x, y) = \frac{xe^{px} + ye^{-py}}{e^{px} + e^{-py}}, \ B^{[\psi]}(x, y) = \frac{xe^{px} + ye^{-py}}{e^{px} + e^{-py}}
\]
for all \( x, y \in \mathbb{R} \).
Remark 1. For the case \((1-2\lambda)(1-2\mu) = 0\), since (2.5) and \(\varphi\) is strictly increasing, \(\psi\) is strictly decreasing, we have \(\lambda = \mu = \frac{1}{2}\). Then the Equation (2.7) becomes

\[
\frac{\varphi''(x)}{\varphi'(x)} - \left(\frac{\varphi'(x)}{\psi'(x)}\right)^2 = \frac{\psi''(x)}{\psi'(x)} - \left(\frac{\psi'(x)}{\varphi'(x)}\right)^2, \quad x, y \in I. \tag{2.9}
\]

Then assuming \(\varphi, \psi\) are three times differentiable, we can find the result for this case in [21].

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References


