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Decay Rates of the Compressible Hall-MHD Equations for Quantum Plasmas

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Abstract

In this paper, we consider the global existence and decay rates of strong solutions to the three-dimensional compressible quantum Hall-magneto-hydrodynamics equations. By combing the L^p-L^q estimates for the linearized equations and a standard energy method, the global existence and its convergence rates are obtained in various norms for the solution to the equilibrium state in the whole space when the initial perturbation of the stationary solution is small in some Sobolev norms. More precisely, the decay rates in time of the solution and its first order derivatives in L^2-norm are obtained when the L^1-norm of the perturbation is bounded.

Keywords

Compressible Hall-MHD Equations, Global Existence, Optimal Decay Rates, Energy Estimates

1. Introduction

In this paper, we consider the following compressible Hall-MHD equations for quantum plasmas in three dimensional whole space $\mathbb{R}^3$:

$$\begin{cases} \rho_t + div(\rho u) = 0, \\ (\rho u)_t + div(\rho u \otimes u) + \nabla P(\rho) - \dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \\ = B\cdot\nabla B + \mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u, \\ B_t + u\cdot\nabla B + \nabla\times\big((\nabla\times B)\times B\big) - \Delta B = B\cdot\nabla u, \quad \nabla\cdot B = 0, \end{cases} \tag{1.1}$$

for $(t,x)\in[0,+\infty)\times\mathbb{R}^3$ with the initial conditions:

$$(\rho,u,B)\big|_{t=0} = \big(\rho_0(x),u_0(x),B_0(x)\big), \quad x\in\mathbb{R}^3. \tag{1.2}$$

Here $\rho > 0$, $u = (u^1, u^2, u^3)$ and $B = (B^1, B^2, B^3)$ denote the density, the velocity and magnetic field, respectively. The pressure $P = P(\rho)$ is a smooth function with $P'(\rho) > 0$ for $\rho > 0$, μ and λ are referred to as the shear viscosity and the bulk viscosity coefficients of the fluid, which satisfy the usual condition

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

where $\hbar > 0$ is the Planck constant. The $\hbar^2$-term is referred to as the quantum potential or Bohm potential term [1], which is strongly nonlinearly degenerate and leads to the system non-diagonal and should be regarded as a consequence from dispersive properties of the quantum fluid.

The quantum terms date back to Wigner [2], where quantum corrections were considered for the thermodynamic equilibrium. The quantum correction to the stress tensor was proposed in [3] [4]. One may see Hass [5] for many physics backgrounds and mathematical derivation of many interesting models. Pu and Guo [6] established the global existence of strong solutions and the semiclassical limit for the full compressible quantum Navier-Stokes. Later, they [7] obtained the following decay rates

$$\left\|\nabla^k (n-1)(t)\right\|_{H^{N+2-k}} + \left\|\nabla^k u(t)\right\|_{H^{N+1-k}} + \left\|\nabla^k (T-1)(t)\right\|_{H^{N-k}} \leq C(1+t)^{-\frac{3+2k}{4}}.$$

with $k = 0,1$. Recently, Pu and Xu [8] showed the decay rates for smooth solutions of the magnetohydrodynamic model for quantum plasmas as follows:

$$\left\|\nabla^k (\rho-1, u, B)(t)\right\|_{L^2} + \left\|\hbar \nabla^{k+1} (\rho-1)(t)\right\|_{L^2} \leq C(1+t)^{-\frac{5}{4}},$$

where $k = 1,2,3,4$. The interested reader can refer to [9] [10] and references therein for more results of the quantum term.

Without the quantum effects, the above system (1.1) is usual compressible Hall-MHD equations, which represent the momentum conservation equation for the plasma fluid. Compared with the classical MHD equations, there exists the Hall term $\nabla \times ((\nabla \times B) \times B)$ in $(1.1)_3$, which makes Hall-MHD equations entirely different from MHD equations for understanding the problem of magnetic reconnection, due to the froze-field effect. Thus, we note that the Hall-MHD equations are useful in describing many phenomena such as magnetic reconnection in space plasmas, star formation, neutron stars and geo-dynamo (see [11] [12] [13] and references therein).

The compressible Hall-MHD equations have received some results in recent years. In particular, Fan *et al.* [14] proved the local existence of strong solutions with positive initial density and global small classical solutions with small initial perturbation belongs to $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. They also obtained optimal time decay rate for strong solutions as follows:

$$\left\|(\rho-1, u, B)(t)\right\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}.$$

Motivated by Fan *et al.*, Gao and Yao [15] improved the optimal time decay rates for higher order spatial derivatives of classical solutions under the condition that the initial data belongs to $H^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$. For the case of initial data belonging to some negative Sobolev space, Xu *et al.* showed the fast time decay rates for the higher-order spatial derivatives of solutions in [16]. Recently, they [17] established the unique global solvability and the optimal time decay rates of strong solutions in Besov spaces. On the other hand, there are also many works of incompressible Hall-MHD equations, see [18]-[26].

To our knowledge, so far there is no result on the large-time behaviors of the Cauchy problem (1.1)-(1.2). Therefore, the main purpose of this paper is to investigate global existence and decay rate in time of smooth solutions in H^4-framework. The decay rate of solutions towards the steady state has been an important problem in the PDE theory, which has been investigated extensively, see for instance [27]-[34] and the references therein. Compared with the general compressible H-MHD equations [14] [15] [35], the quantum term (higher order) appears in $(1.1)_2$, which leads to new difficulties in decay analysis than those results. The major method is to make a hypothesis (3.1) to cooperate with the special structure of (1.1). We first construct the global existence of strong solutions by the standard energy method under the condition that the initial data are close to the equilibrium state $(1,0,0)$ in H^4-norm. Furthermore, by assuming that the initial data in L^1-norm are finite additionally, we establish the optimal time decay rates of strong solutions by the method of spectral analysis and energy estimates. More precisely, we obtain the following time decay rates

$$\left\|\nabla(\rho-1,u,B)(t)\right\|_{H^3(\mathbb{R}^3)}+\left\|\hbar^2\nabla(\rho-1)\right\|_{H^3(\mathbb{R}^3)}\le C(1+t)^{-\frac{5}{4}},$$

for all $t\ge 0$.

Our main results of this paper are stated as the following theorem.

Theorem 1.1 *Assume that the initial condition* $(\rho_0-1,u_0,B_0)\in H^5(\mathbb{R}^3)\times H^4(\mathbb{R}^3)\times H^4(\mathbb{R}^3)$ *satisfies the constraints* (1.2), *there exists a constant* $\delta>0$ *such that if*

$$\left\|\rho_0-1\right\|_{H^5(\mathbb{R}^3)}+\left\|u_0\right\|_{H^4(\mathbb{R}^3)}+\left\|B_0\right\|_{H^4(\mathbb{R}^3)}\le\delta, \tag{1.3}$$

then there exists a unique global solution (ρ,u,B) of the Cauchy problem (1.1)-(1.2) satisfying

$$\begin{aligned}&\left\|(\rho-1,u,B)(t)\right\|_{H^4(\mathbb{R}^3)}^2+\left\|\hbar\nabla\rho(t)\right\|_{H^4(\mathbb{R}^3)}^2+\int_0^t\left\|\nabla(u,B,\hbar\rho)(\tau)\right\|_{H^4(\mathbb{R}^3)}^2\,\mathrm{d}\tau\\&\le C\left(\left\|\rho_0-1\right\|_{H^5(\mathbb{R}^3)}^2+\left\|u_0\right\|_{H^4(\mathbb{R}^3)}^2+\left\|B_0\right\|_{H^4(\mathbb{R}^3)}^2\right).\end{aligned} \tag{1.4}$$

Furthermore, if $(\rho_0-1,u_0,B_0)\in L^1(\mathbb{R}^3)$, the solution (ρ,u,B) enjoys the following decay properties

$$\left\|(\rho-1,u,B)(t)\right\|_{L^p(\mathbb{R}^3)}\le C_0(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)},\quad 2\le p\le 6, \tag{1.5}$$

$$\left\|(\rho-1,u,B)(t)\right\|_{L^\infty(\mathbb{R}^3)}\le C_0(1+t)^{-\frac{5}{4}}, \tag{1.6}$$

$$\left\|\nabla\left(\rho-1,u,B\right)(t)\right\|_{H^3\left(\mathbb{R}^3\right)}+\left\|\hbar\nabla\left(\rho-1\right)\right\|_{H^3\left(\mathbb{R}^3\right)}\le C_0\left(1+t\right)^{-\frac{5}{4}}, \tag{1.7}$$

$$\left\|\partial_t\left(\rho-1,u,B\right)(t)\right\|_{L^2\left(\mathbb{R}^3\right)}\le C_0\left(1+t\right)^{-\frac{5}{4}}, \tag{1.8}$$

for some positive constant C_0.

Notation. Throughout this paper, we denote the norms in Sobolev spaces $H^m\left(\mathbb{R}^3\right)$ and $W^{m,p}\left(\mathbb{R}^3\right)$ by $\|\cdot\|_{H^m}$ and $\|\cdot\|_{W^{m,p}}$ for $m\ge 0$ and $p\ge 1$ respectively. In particular, for $m=0$, we shall simply use $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^p}$. Moreover, $\nabla=\left(\partial_1,\partial_2,\partial_3\right)$, $\partial_i=\partial_{x_i}$ $(i=1,2,3)$ and for any integer $\ell\ge 0$, $\nabla^\ell f$ denotes all derivatives of order ℓ of the function *f*. In addition, *C* denotes the generic positive constant which may vary in different places and the integration domain $\mathbb{R}^3$ will be always omitted without any ambiguity. Finally, $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2\left(\mathbb{R}^3\right)$.

The rest of this paper is organized as follows. In Section 2 we reformulate the system (1.1)-(1.2) into a more convenient form. In Section 3, we make some crucial energy estimates for the solution that will play an essential role for us to construct the global existence of strong solutions. In Section 4, we use the energy estimates derived in Section 3 to build the global existence of the solution, which combine with the linear decay estimates imply Theorem 1.1. In Appendix, we list some useful inequalities.

2. Reformations

To make it more convenient to prove Theorem 1.1, in this section, we will reformulate the problem (1.1) and (1.2). More precisely, we set

$$n=\rho-1,\quad v=\frac{u}{\gamma},\quad B=B,$$

then the system (1.1) and (1.2) can be rewritten as

$$\begin{cases} n_t+\gamma\nabla\cdot v=F_1,\\ v_t+\gamma\nabla n-\dfrac{\hbar^2}{4\gamma}\nabla\Delta n-\mu\Delta v-\left(\lambda+\mu\right)\nabla\nabla\cdot v=F_2,\\ B_t-\Delta B=F_3,\quad \nabla\cdot B=0,\\ \left(n,v,B\right)\big|_{t=0}=\left(n_0,v_0,B_0\right)(x)\to(0,0,0),\quad \text{as } |x|\to\infty, \end{cases} \tag{2.1}$$

where $\gamma=\sqrt{P'(1)}$ and the source terms $\left(F_1,F_2,F_3\right)$ are

$$F_1=-\gamma\nabla\cdot(nv),$$

$$F_2=-\gamma v\cdot\nabla v-h(n)\nabla n+\frac{\hbar^2}{4\gamma}\left(\frac{|\nabla n|^2\nabla n}{(n+1)^3}-\frac{\nabla n\Delta n}{(n+1)^2}-\frac{\nabla n\cdot\nabla^2 n}{(n+1)^2}-g(n)\nabla\Delta n\right)$$
$$-g(n)\left(\mu\Delta v+(\lambda+\mu)\nabla\nabla\cdot v\right)+\frac{B\cdot\nabla B}{\gamma(n+1)},$$

$$F_2=-\gamma v\cdot\nabla B+\gamma B\cdot\nabla v-\nabla\times\left(\left(\nabla\times B\right)\times B\right).$$

We defined the two nonlinear function of *n* by

$$g(n)=\frac{n}{n+1},\quad h(n)=\frac{P'(n+1)}{\gamma(n+1)}-\gamma. \tag{2.2}$$

In the following, we will establish the global existence and time decay rates of the solution (n,v,B) to the stead state $(0,\vec{0},\vec{0})$. We first define the solution space of the initial value problem (2.1) by

$$\begin{aligned}X(0,T)=\Big\{&(n,v,B)\mid n,B\in C^0\big(0,T;H^4(\mathbb{R}^3)\big)\cap C^1\big(0,T;H^3(\mathbb{R}^3)\big),\\ &\hbar\nabla n\in C^0\big(0,T;H^4(\mathbb{R}^3)\big)\cap C^1\big(0,T;H^3(\mathbb{R}^3)\big),\\ &v\in C^0\big(0,T;H^4(\mathbb{R}^3)\big)\cap C^1\big(0,T;H^2(\mathbb{R}^3)\big)\Big\},\end{aligned}$$

and

$$N(0,T)^2=\sup_{0\le t\le T}\|(n,v,B)(t)\|_{H^4}^2+\sup_{0\le t\le T}\|\hbar\nabla n(t)\|_{H^4}^2+\int_0^T\|\nabla(n,v,B)(\tau)\|_{H^4}^2\,\mathrm{d}\tau,$$

for any $0\le T\le\infty$. By the standard continuity argument, the global existence of solutions to (2.1) will be obtained by combining the local existence result together with a priori estimates.

Proposition 2.1 (*Local existence*). *Assume that* $(n_0,v_0,B_0,\hbar\nabla n_0)\in H^4(\mathbb{R}^3)$ *and*

$$\inf_{x\in\mathbb{R}^3}\{n_0+1\}>0.$$

Then there exists a positive constant $T_0>0$ depending on $N(0,0)$ such that the initial value problem (2.1) has a unique solution $(n,v,B,\hbar\nabla n)\in X(0,T_0)$ satisfying $N(0,T_0)\le 2N(0,0)$ and

$$\inf_{x\in\mathbb{R}^3,0\le t\le T}\{n(x,t)+1\}>0.$$

Proposition 2.2 (*A priori estimate*). *Let* $(n_0,v_0,B_0,\hbar\nabla n_0)\in H^4(\mathbb{R}^3)$. *Suppose that the initial value problem* (2.1) *has a solution* $(n,v,B,\hbar\nabla n)\in X(0,T)$ *for some* $T>0$. *Then there exist a small constant* $\delta>0$ *and a constant* $\tilde{C}_1$, *which are independent of T, such that if*

$$\sup_{0\le t\le T}\|(n,v,B,\hbar\nabla n)(t)\|_{H^4}\le\delta,$$

then for any $t\in[0,T]$, *it holds that*

$$\begin{aligned}&\|(n,v,B)(t)\|_{H^4}^2+\|\hbar\nabla n(t)\|_{H^4}^2+\int_0^t\|\nabla(v,B,\hbar n)(\tau)\|_{H^4}^2\,\mathrm{d}\tau\\ &\le\tilde{C}_1\left(\|n_0-1\|_{H^5}^2+\|v_0\|_{H^4}^2+\|B_0\|_{H^4}^2\right).\end{aligned} \tag{2.3}$$

Furthermore, there is a constant C_1' *such that for any* $t\in[0,T]$, *the global solution* $(n,v,B,\hbar\nabla n)(x,t)$ *has the decay properties*

$$\|(n,v,B)(t)\|_{L^p}\le C_1'(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)},\quad 2\le p\le 6, \tag{2.4}$$

$$\|(n,v,B)(t)\|_{L^\infty}\le C_1'(1+t)^{-\frac{5}{4}}, \tag{2.5}$$

$$\|\nabla(n,v,B)(t)\|_{H^3}+\|\hbar\nabla n\|_{H^3}\le C_1'(1+t)^{-\frac{5}{4}}, \tag{2.6}$$

$$\left\|\partial_t(n,v,B)(t)\right\|_{L^2} \le C_1'(1+t)^{-\frac{5}{4}}. \tag{2.7}$$

The proof of Theorem 1.1 is followed from Proposition 2.1 and Proposition 2.2 by the standard iteration arguments. The proof of Proposition 2.1 is standard and thus omitted. Proposition 2.2 will be proved in Section 3 and Section 4.

3. Energy Estimates

In this section we will drive some a priori energy estimates for the solutions to the system (2.1). We assume a priori that for sufficiently small $\delta > 0$,

$$\left\|(n,v,B)(t)\right\|_{H^4}^2 + \left\|\hbar\nabla n(t)\right\|_{H^4}^2 \le \delta. \tag{3.1}$$

By (2.1) and Sobolev's inequality, we then obtain

$$\frac{1}{2} \le n+1 \le 2.$$

Therefore, for $C > 0$, we have

$$|g(n)|,|h(n)| \le C|n| \text{ and } \left|g^{(k)}(n)\right|,\left|h^{(k)}(n)\right| \le C, \quad \text{for any } k \ge 1. \tag{3.2}$$

In the first place, we will obtain the dissipation estimate for v.

Lemma 3.1 *Let* (n,v,B) *be a smooth solution to* (2.1), *then it holds that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|(n,v,B)\right\|_{L^2}^2 + \frac{\hbar^2}{4\gamma}\left\|\nabla n\right\|_{L^2}^2\right) + C\left\|\nabla v\right\|_{L^2}^2 + C\left\|\nabla B\right\|_{L^2}^2\\ &\le C\delta\left(\left\|\nabla n\right\|_{L^2}^2 + \hbar^2\left\|\nabla n\right\|_{L^2}^2\right).\end{aligned} \tag{3.3}$$

Proof. Multiplying $(2.1)_1$, $(2.1)_2$ and $(2.1)_3$ by n, v and B respectively, and then integrating them over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|(n,v,B)\right\|_{L^2}^2 + \mu\left\|\nabla v\right\|_{L^2}^2 + (\mu+\lambda)\left\|\nabla\cdot v\right\|_{L^2}^2 + \left\|\nabla B\right\|_{L^2}^2\\ &= \left\langle \frac{\hbar^2}{4\gamma}\nabla\Delta n, v\right\rangle + \langle F_1, n\rangle + \langle F_2, v\rangle + \langle F_3, B\rangle.\end{aligned} \tag{3.4}$$

We will estimate the three terms on the right-hand side.

Firstly, for the first term, by the continuity equation and integration by parts twice, we have

$$\begin{aligned}\left\langle \frac{\hbar^2}{4\gamma}\nabla\Delta n, v\right\rangle &= \left\langle \frac{\hbar^2}{4\gamma}\nabla n, \Delta v\right\rangle = \left\langle \frac{\hbar^2}{4\gamma}\nabla n, \frac{1}{\gamma}\nabla F_1 - \frac{1}{\gamma}\nabla n_t\right\rangle\\ &= -\left\langle \frac{\hbar^2}{4\gamma^2}\nabla n, \nabla n_t\right\rangle + \left\langle \frac{\hbar^2}{4\gamma}\Delta n, (\nabla n\cdot v + n\nabla\cdot v)\right\rangle\\ &\le -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\hbar^2}{4\gamma^2}\left\|\nabla n\right\|_{L^2}^2\right) + C\hbar^2\left\|\Delta n\right\|_{L^3}\left(\left\|\nabla n\right\|_{L^2}\left\|v\right\|_{L^6} + \left\|\nabla v\right\|_{L^2}\left\|n\right\|_{L^6}\right)\\ &\le -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\hbar^2}{4\gamma^2}\left\|\nabla n\right\|_{L^2}^2\right) + C\hbar^2\delta\left(\left\|\nabla n\right\|_{L^2}^2 + \left\|\nabla v\right\|_{L^2}^2\right).\end{aligned} \tag{3.5}$$

Secondly, for the second term, it follows from Lemma 5.1, the assumption (3.1), the Hölder inequality and the Young inequality that

$$\begin{aligned}\langle F_1, n\rangle &= -\langle \gamma n\nabla\cdot v, n\rangle - \langle \gamma v\cdot\nabla n, n\rangle \\ &\le \gamma\|n\|_{L^3}\|\nabla\cdot v\|_{L^2}\|n\|_{L^6} + \gamma\|v\|_{L^6}\|\nabla n\|_{L^2}\|n\|_{L^3} \\ &\le C\delta\left(\|\nabla n\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right).\end{aligned} \tag{3.6}$$

Next, for the third term, we have

$$\begin{aligned}\langle F_2, v\rangle &= -\langle \gamma v\cdot\nabla v, v\rangle - \langle h(n)\nabla n, v\rangle \\ &\quad + \left\langle \frac{\hbar^2}{4\gamma}\left(\frac{|\nabla n|^2\nabla n}{(n+1)^3} - \frac{\nabla n\Delta n}{(n+1)^2} - \frac{\nabla n\cdot\nabla^2 n}{(n+1)^2} - g(n)\nabla\Delta n\right), v\right\rangle \\ &\quad - \langle g(n)(\mu\Delta v + (\lambda+\mu)\nabla\nabla\cdot v), v\rangle + \left\langle \frac{B\cdot\nabla B}{\gamma(n+1)}, v\right\rangle \\ &:= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{3.7}$$

For the term I_1 and I_2, using (3.1), (3.2), Hölder's inequality, Young's inequality and Lemma 5.1, we obtain

$$I_1 + I_2 \le C\|v\|_{L^3}\|\nabla v\|_{L^2}\|v\|_{L^6} + C\|h(n)\|_{L^3}\|\nabla n\|_{L^2}\|v\|_{L^6} \le C\delta\left(\|\nabla n\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right).$$

For the term I_3, we have by Hölder's inequality, Lemma 5.1 and (3.1) that

$$\begin{aligned}I_3 &= \left\langle \frac{\hbar^2}{4\gamma}\left(\frac{|\nabla n|^2\nabla n}{(n+1)^3} - \frac{\nabla n\Delta n}{(n+1)^2} - \frac{\nabla n\cdot\nabla^2 n}{(n+1)^2}\right), v\right\rangle + \langle g'(n)\cdot v, \Delta n\rangle + \langle g(n)\nabla\cdot v, \Delta n\rangle \\ &\le C\hbar^2\left(\left\|\frac{\nabla n}{(n+1)^3}\right\|_{L^\infty}\|\nabla n\|_{L^2}\|\Delta n\|_{L^3}\|v\|_{L^6} + \left\|\frac{1}{(n+1)^2}\right\|_{L^\infty}\|\nabla n\|_{L^2}\|\Delta n\|_{L^3}\|v\|_{L^6}\right. \\ &\quad + \left\|\frac{\nabla n}{(n+1)^2}\right\|_{L^\infty}\|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^3}\|v\|_{L^6} \\ &\quad \left. + \|g'(n)\|_{L^2}\|v\|_{L^6}\|\Delta n\|_{L^3} + \|g(n)\|_{L^3}\|\Delta n\|_{L^6}\|\nabla\cdot v\|_{L^3}\right) \\ &\le C\delta\hbar^2\left(\|\nabla n\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right).\end{aligned}$$

Let $I_4 = I_{41} + I_{42}$. For the term I_{41}, by (3.1), (3.2), the Hölder inequality and integration by parts, we have

$$\begin{aligned}I_{41} &\le C\langle \nabla v, g(n)\nabla\cdot v\rangle + C\langle \nabla v, g'(n)\nabla n\cdot v\rangle \\ &\le C\|\nabla v\|_{L^2}^2\|g(n)\|_{L^\infty} + C\|\nabla v\|_{L^2}\|g'(n)\|_{L^\infty}\|\nabla n\|_{L^2}\|v\|_{L^\infty} \\ &\le C\delta\left(\|\nabla n\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right).\end{aligned}$$

In a similar way, we have

$$I_{42} \le C\delta\left(\|\nabla n\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right).$$

For the term I_5, we similarly obtain

$$I_5 \le C\delta\left(\|\nabla B\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right).$$

In light of the estimates $I_1 \sim I_5$, we can get

$$\langle F_2, v\rangle \le C\delta\left(\|\nabla n\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right). \tag{3.8}$$

Finally, for the last term, we have

$$\langle F_3, B\rangle = -\langle \gamma v\cdot\nabla B, B\rangle + \langle \gamma B\cdot\nabla v, B\rangle - \langle \nabla\times((\nabla\times B)\times B), B\rangle. \tag{3.9}$$

Similarly, we bound the first and second terms on the right hand side of (3.9) by

$$-\langle \gamma v\cdot\nabla B, B\rangle + \langle \gamma B\cdot\nabla v, B\rangle \le C\delta\left(\|\nabla B\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right). \tag{3.10}$$

For the last term on the right hand side of (3.9), by integration by part, we have

$$-\langle \nabla\times((\nabla\times B)\times B), B\rangle = \langle (\nabla\times B)\times B, \nabla\times B\rangle = 0. \tag{3.11}$$

Combined with (3.10) and (3.11), we get

$$\langle F_3, B\rangle \le C\delta\left(\|\nabla v\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right). \tag{3.12}$$

Substituting (3.5), (3.6), (3.8) and (3.12) yields into (3.4), by the smallness of δ, we get (3.3). □

In the following lemma, we derive the higher-order dissipative estimates.

Lemma 3.2 *Let* (n, v, B) *be a smooth solution to* (2.1)*, then*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla(n,v,B)\|_{H^3}^2 + \hbar^2\|\nabla^2 n\|_{H^3}^2\right) + C\|\nabla^2 v\|_{H^3}^2 + C\|\nabla^2 B\|_{H^3}^2 \\
&\le C\delta\left(\|\nabla n\|_{H^3}^2 + \hbar^2\|\nabla^2 n\|_{H^3}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right).
\end{aligned} \tag{3.13}$$

Proof. For $0 \le k \le 3$, applying ∇^{k+1} to $(2.1)_1$-$(2.1)_3$ and then taking L^2-inner product with $(\nabla^{k+1}n, \nabla^{k+1}v, \nabla^{k+1}B)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{k+1}(n,v,B)\|_{L^2}^2 + \mu\|\nabla^{k+2}v\|_{L^2}^2 + (\mu+\lambda)\|\nabla^{k+1}\nabla\cdot v\|_{L^2}^2 + \|\nabla^{k+2}B\|_{L^2}^2 \\
&= \left\langle \frac{\hbar^2}{4\gamma}\nabla^{k+1}\left(\frac{1}{n+1}\nabla\Delta n\right), \nabla^{k+1}v\right\rangle - \langle \gamma\nabla^{k+1}\nabla\cdot(n,v), \nabla^{k+1}n\rangle \\
&\quad - \langle \nabla^{k+1}(\gamma v\cdot\nabla v), \nabla^{k+1}v\rangle - \langle \nabla^{k+1}(h(n)\nabla n), \nabla^{k+1}v\rangle \\
&\quad + \left\langle \frac{\hbar^2}{4\gamma}\nabla^{k+1}\left(-\frac{\nabla n\cdot\nabla^2 n}{(n+1)^2} - \frac{\nabla n\Delta n}{(n+1)^2} + \frac{|\nabla n|^2\nabla n}{(n+1)^3}\right), \nabla^{k+1}v\right\rangle \\
&\quad - \langle \nabla^{k+1}(g(n)(\mu\Delta v + (\lambda+\mu)\nabla\nabla\cdot v)), \nabla^{k+1}v\rangle + \left\langle \nabla^{k+1}\left(\frac{B\cdot\nabla B}{\gamma(n+1)}\right), \nabla^{k+1}v\right\rangle \\
&\quad - \langle \nabla^{k+1}(\gamma v\cdot\nabla B), \nabla^{k+1}B\rangle + \langle \nabla^{k+1}(\gamma B\cdot\nabla v), \nabla^{k+1}B\rangle \\
&\quad - \langle \nabla^{k+1}(\nabla\times((\nabla\times B)\times B)), \nabla^{k+1}B\rangle \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10}.
\end{aligned} \tag{3.14}$$

We will estimate each term on the right-hand side. At first, we split J_1 as

$$\begin{aligned}
J_1 &= \frac{\hbar^2}{4\gamma}\left\langle \frac{1}{n+1}\nabla^{k+1}\nabla\Delta n, \nabla^{k+1}v\right\rangle + \frac{\hbar^2}{4\gamma}\sum_{1\le l\le k+1} C_{k+1}^l\left\langle \nabla^l\left(\frac{1}{n+1}\right)\nabla^{k-l+1}\nabla\Delta n, \nabla^{k+1}v\right\rangle \\
&= J_{11} + J_{12}.
\end{aligned} \tag{3.15}$$

By the continuity equation and integration by parts, the first term J_{11} can be rewritten as

$$\begin{aligned} J_{11} &= -\frac{\hbar^2}{4\gamma}\left\langle \nabla\left(\frac{1}{n+1}\right)\nabla^{k+1}\Delta n, \nabla^{k+1}v\right\rangle - \frac{\hbar^2}{4\gamma}\left\langle \frac{1}{n+1}\nabla^{k+1}\Delta n, \nabla^{k+1}\nabla\cdot v\right\rangle \\ &= \frac{\hbar^2}{4\gamma}\left\langle \nabla^2\left(\frac{1}{n+1}\right)\nabla^{k+2}n, \nabla^{k+1}v\right\rangle + \frac{\hbar^2}{4\gamma}\left\langle \nabla\left(\frac{1}{n+1}\right)\nabla^{k+2}n, \nabla^{k+2}v\right\rangle \\ &\quad -\frac{\hbar^2}{4\gamma}\left\langle \frac{1}{n+1}\nabla^{k+1}\Delta n, \nabla^{k+1}\nabla\cdot v\right\rangle, \end{aligned}$$

where the first two terms can be estimated as

$$\begin{aligned} &-\frac{\hbar^2}{4\gamma}\left\langle \nabla\left(\frac{1}{n+1}\right)\nabla^{k+1}\Delta n, \nabla^{k+1}v\right\rangle - \frac{\hbar^2}{4\gamma}\left\langle \frac{1}{n+1}\nabla^{k+1}\Delta n, \nabla^{k+1}\nabla\cdot v\right\rangle \\ &\le C\delta\hbar^2\left(\left\|\nabla^{k+2}n\right\|_{L^2}^2 + \left\|\nabla^{k+2}v\right\|_{L^2}^2\right). \end{aligned}$$

Note that the last term in J_{11} is much more complicated, so we can further decompose it into

$$\begin{aligned} &\frac{\hbar^2}{4\gamma}\left\langle \nabla^{k+2}n\nabla\left(\frac{1}{n+1}\right), \nabla^{k+1}\nabla\cdot v\right\rangle + \frac{\hbar^2}{4\gamma}\left\langle \frac{1}{n+1}\nabla^{k+2}n, \nabla^{k+2}\nabla\cdot v\right\rangle \\ &= \frac{\hbar^2}{4\gamma}\left\langle \nabla^{k+2}n\nabla\left(\frac{1}{n+1}\right), \nabla^{k+1}\nabla\cdot v\right\rangle - \frac{\hbar^2}{4\gamma^2}\left\langle \frac{1}{(n+1)^2}\nabla^{k+2}n, \nabla^{k+2}n_t\right\rangle \\ &\quad -\frac{\hbar^2}{4\gamma^2}\sum_{0\le l\le k+1} C_{k+2}^l\left\langle \frac{1}{n+1}\nabla^l n_t \nabla^{k+2-l}\left(\frac{1}{n+1}\right), \nabla^{k+2}n\right\rangle \\ &\quad -\frac{\hbar^2}{4\gamma}\left\langle \frac{1}{n+1}\nabla^{k+2}n, \nabla^{k+2}\left(\frac{\nabla n\cdot v}{n+1}\right)\right\rangle \\ &:= W_1 + W_2 + W_3 + W_4. \end{aligned}$$

The first two terms W_1 and W_2 can be bounded by

$$W_1 + W_2 \le -\frac{1}{2}\cdot\frac{\hbar^2}{4\gamma^2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{n+1}\nabla^{k+2}n\right\|_{L^2}^2 + C\delta\hbar^2\left(\left\|\nabla^{k+2}n\right\|_{L^2}^2 + \left\|\nabla^{k+2}v\right\|_{L^2}^2\right).$$

For the term W_3, by the continuity equation and the Hölder inequality, we have

$$\begin{aligned} W_3 &\le C\hbar^2\left\|\nabla^{k+2}n\right\|_{L^2}\sum_{0\le l\le k+1}\left\|\nabla^l n_t \nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^2} \\ &\le C\hbar^2\left\|\nabla^{k+2}n\right\|_{L^2}\sum_{0\le l\le k+1}\left(\left\|\nabla^l\nabla\cdot v\nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^2}\right. \\ &\quad \left. + \left\|\nabla^l(\nabla n\cdot v)\nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^2} + \left\|\nabla^l(n\nabla\cdot v)\nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^2}\right). \end{aligned}$$

For the second term of W_3, separating the case of $l = 0,1$ and $k+1$ from the order cases, we bound the summation by

$$
\begin{aligned}
& C\hbar^2 \left\|\nabla^{k+2} n\right\|_{L^2}\left(\left\|\nabla n \cdot v \nabla^{k+2}\left(\frac{1}{n+1}\right)\right\|_{L^2}+\left\|\nabla(\nabla n \cdot v) \nabla^{k+1}\left(\frac{1}{n+1}\right)\right\|_{L^2}\right. \\
& \left.+\left\|\nabla^{k+1}(\nabla n \cdot v) \nabla\left(\frac{1}{n+1}\right)\right\|_{L^2}+\sum_{2 \leq l \leq k}\left\|\nabla^l(\nabla n \cdot v) \nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^2}\right) \\
& \leq C\hbar^2\left\|\nabla^{k+2} n\right\|_{L^2}\left(C\delta\left\|\nabla^{k+2} n\right\|_{L^2}+C\delta\left\|\nabla^{k+2} n\right\|_{L^2}+C\delta\left\|\nabla^{k+1}(\nabla n \cdot v)\right\|_{L^2}\right. \\
& \left.+\sum_{2 \leq l \leq k}\left\|\nabla^l(\nabla n \cdot v)\right\|_{L^2}\left\|\nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^\infty}\right) \\
& \leq C\delta\hbar^2\left\|\nabla^{k+2} n\right\|_{L^2}\left(\left\|\nabla^{k+2} n\right\|_{L^2}+\left\|\nabla^{k+1}(\nabla n \cdot v)\right\|_{L^2}+\sum_{2 \leq l \leq k}\left\|\nabla^l(\nabla n \cdot v)\right\|_{L^2}\right) \\
& \leq C\delta\hbar^2\left(\left\|\nabla^{k+2} v\right\|_{L^2}^2+\left\|\nabla^2 n\right\|_{H^k}^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& \left\|\nabla^{k+1}(\nabla n \cdot v)\right\|_{L^2}=\sum_{0 \leq l \leq k+1}\left\|\nabla^{l+1} n \nabla^{k+1-l} v\right\|_{L^2} \\
& =\left\|\nabla n \nabla^{k+1} v\right\|_{L^2}+\left\|\nabla^2 n \nabla^k v\right\|_{L^2}+\sum_{2 \leq l \leq k+1}\left\|\nabla^{l+1} n \nabla^{k+1-l} v\right\|_{L^2} \\
& \leq C\left(\|\nabla n\|_{L^\infty}\left\|\nabla^{k+1} v\right\|_{L^2}+\left\|\nabla^2 n\right\|_{L^3}\left\|\nabla^k v\right\|_{L^6}+\sum_{2 \leq l \leq k+1}\left\|\nabla^{l+1} n\right\|_{L^2}\left\|\nabla^{k+1-l} v\right\|_{L^\infty}\right) \\
& \leq C\delta\left(\left\|\nabla^{k+2} v\right\|_{L^2}+\left\|\nabla^2 n\right\|_{H^k}\right),
\end{aligned}
$$

and

$$
\sum_{2 \leq l \leq k}\left\|\nabla^l(\nabla n \cdot v)\right\|_{L^2}=\sum_{2 \leq l \leq k} \sum_{0 \leq m \leq l} C_l^m\left\|\nabla^{m+1} n \nabla^{l-m} v\right\|_{L^2} \leq C\delta\left\|\nabla^2 n\right\|_{H^k} .
$$

Similarly, we bound the first and the last term in W_3 by

$$
C\delta\left(\left\|\nabla^{k+2} v\right\|_{L^2}^2+\left\|\nabla^2 n\right\|_{H^k}^2\right).
$$

Collecting these terms, we get

$$
W_3 \leq C\delta\hbar^2\left(\left\|\nabla^{k+2} v\right\|_{L^2}^2+\left\|\nabla^2 n\right\|_{H^k}^2\right).
$$

For the term W_4, we have

$$
\begin{aligned}
W_4= & -\frac{\hbar^2}{4\gamma}\left\langle\nabla^{k+2} n \nabla^{k+3} n, \frac{v}{(n+1)^2}\right\rangle \\
& -\frac{\hbar^2}{4\gamma} \sum_{0 \leq l \leq k+1} C_{k+2}^l\left\langle\nabla^{l+1} n \nabla^{k+2-l}\left(\frac{v}{n+1}\right), \frac{1}{n+1} \nabla^{k+2} n\right\rangle .
\end{aligned}
$$

For the first term of W_4, we have by integration by parts and (3.1) that

$$
\begin{aligned}
& -\frac{1}{2} \cdot \frac{\hbar^2}{4\gamma}\left\langle\nabla\left(\left|\nabla^{k+2} n\right|^2\right), \frac{v}{(n+1)^2}\right\rangle \\
& =\frac{1}{2} \cdot \frac{\hbar^2}{4\gamma}\left\langle\left|\nabla^{k+2} n\right|^2, \nabla \cdot\left(\frac{v}{(n+1)^2}\right)\right\rangle \leq C\delta\hbar^2\left\|\nabla^{k+2} n\right\|_{L^2}^2 .
\end{aligned}
$$

For the second term of W_4, similarly, we separate the case of $l=0,1$ and $k+1$ from the order cases and bound the summation by

$$
\begin{aligned}
&C\hbar^2\left\|\nabla^{k+2}n\right\|_{L^2}\sum_{0\le l\le k+1}\left\|\nabla^{l+1}n\nabla^{k+2-l}\left(\frac{v}{n+1}\right)\right\|_{L^2}\\
&=C\hbar^2\left\|\nabla^{k+2}n\right\|_{L^2}\left(\left\|\nabla n\nabla^{k+2}\left(\frac{v}{n+1}\right)\right\|_{L^2}+\left\|\nabla^{k+1}n\nabla^{2}\left(\frac{v}{n+1}\right)\right\|_{L^2}\right.\\
&\quad\left.+\left\|\nabla^{k+2}n\nabla\left(\frac{v}{n+1}\right)\right\|_{L^2}+\sum_{1\le l\le k-1}\left\|\nabla^{l+1}n\nabla^{k+2-l}\left(\frac{v}{n+1}\right)\right\|_{L^2}\right)\\
&\le C\delta\hbar^2\left\|\nabla^{k+2}n\right\|_{L^2}\left(C\delta\left\|\nabla^{k+2}\left(\frac{v}{n+1}\right)\right\|_{L^2}+C\delta\left\|\nabla^{k+2}n\right\|_{L^2}\right.\\
&\quad\left.+C\delta\sum_{1\le l\le k-1}\left\|\nabla^{k+2-l}\left(\frac{v}{n+1}\right)\right\|_{L^6}\right)\\
&\le C\delta\hbar^2\left(\left\|\nabla^{k+2}v\right\|_{L^2}^2+\left\|\nabla^{2}n\right\|_{H^k}^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\left\|\nabla^{k+2}\left(\frac{v}{n+1}\right)\right\|_{L^2}\le C\sum_{0\le l\le k+2}\left\|\nabla^{l}v\nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^2}\\
&=\left\|v\nabla^{k+2}\left(\frac{1}{n+1}\right)\right\|_{L^2}+\left\|\nabla^{k+2}v\nabla\left(\frac{1}{n+1}\right)\right\|_{L^2}\\
&\quad+\left\|\nabla^{k+1}v\nabla\left(\frac{1}{n+1}\right)\right\|_{L^2}+\sum_{1\le l\le k}\left\|\nabla^{l}v\nabla^{k+2-l}\left(\frac{1}{n+1}\right)\right\|_{L^2}\\
&\le C\delta\left(\left\|\nabla^{k+2}v\right\|_{L^2}+\left\|\nabla^{2}n\right\|_{H^k}\right).
\end{aligned}
$$

Collecting these term, we get

$$
J_{11}\le-\frac{1}{2}\cdot\frac{\hbar^2}{4\gamma^2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{n+1}\nabla^{k+2}n\right\|_{L^2}^2+C\delta\hbar^2\left(\left\|\nabla^{k+2}v\right\|_{L^2}^2+\left\|\nabla^{2}n\right\|_{H^k}^2\right).
$$

For the second term of (3.15), we have by the assumption (3.1), Hölder's inequality, Lemma 5.1, (3.2) and integration by parts that

$$
\begin{aligned}
J_{12}&=-\frac{\hbar^2}{4\gamma}C_{k+1}^{1}\left\langle\nabla^2\left(\frac{1}{n+1}\right)\nabla^{k+2}n,\nabla^{k+1}v\right\rangle\\
&\quad-\frac{\hbar^2}{4\gamma}C_{k+1}^{1}\left\langle\nabla\left(\frac{1}{n+1}\right)\nabla^{k+2}n,\nabla^{k+1}\nabla\cdot v\right\rangle\\
&\quad-\frac{\hbar^2}{4\gamma}C_{k+1}^{2}\left\langle\nabla^2\left(\frac{1}{n+1}\right)\nabla^{k}\Delta n,\nabla^{k+1}\nabla\cdot v\right\rangle\\
&\quad-\frac{\hbar^2}{4\gamma}\sum_{3\le l\le k+1}C_{k+1}^{l}\left\langle\nabla^l\left(\frac{1}{n+1}\right)\nabla^{k-l+1}\nabla\Delta n,\nabla^{k+1}v\right\rangle\\
&\le C\hbar^2\left(\left\|\nabla^2\left(\frac{1}{n+1}\right)\right\|_{L^3}\left\|\nabla^{k+1}v\right\|_{L^6}+\left\|\nabla\left(\frac{1}{n+1}\right)\right\|_{L^\infty}\left\|\nabla^{k+2}v\right\|_{L^2}\right)\left\|\nabla^{k+2}n\right\|_{L^2}\\
&\quad+C\hbar^2\left\|\nabla^2\left(\frac{1}{n+1}\right)\right\|_{L^3}\left\|\nabla^{k+2}n\right\|_{L^2}\left\|\nabla^{k+1}v\right\|_{L^6}\\
&\quad+C\hbar^2\sum_{3\le l\le k+1}\left\|\nabla^l\left(\frac{1}{n+1}\right)\right\|_{L^2}\left\|\nabla^{k-l+2}\Delta n\right\|_{L^3}\left\|\nabla^{k+1}v\right\|_{L^6}\\
&\le C\delta\hbar^2\left(\left\|\nabla^{2}n\right\|_{H^k}^2+\left\|\nabla^{k+2}v\right\|_{L^2}^2\right).
\end{aligned}
$$

Summing up J_{11} and J_{12}, we have

$$J_1 \le -\frac{1}{2}\cdot\frac{\hbar^2}{4\gamma^2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{n+1}\nabla^{k+2}n\right\|_{L^2}^2 + C\delta\hbar^2\left(\left\|\nabla^{k+2}v\right\|_{L^2}^2 + \left\|\nabla^2 n\right\|_{H^k}^2\right).$$

For the term J_2, we can rewrite it as

$$J_2 = -\gamma\left\langle\nabla^{k+1}\left(\nabla n\cdot v\right),\nabla^{k+1}n\right\rangle - \gamma\left\langle\nabla^{k+1}\left(n\cdot\nabla v\right),\nabla^{k+1}n\right\rangle = J_{21}+J_{22}.$$

The first term J_{21} can be bounded by

$$\begin{aligned} J_{21} &= -\gamma\left\langle\nabla^{k+2}n\cdot v,\nabla^{k+1}n\right\rangle - \gamma\sum_{0\le l\le k}\left\langle C_{k+1}^{l}\nabla^{l+1}n\nabla^{k+1-l}v,\nabla^{k+1}n\right\rangle \\ &\le \frac{1}{2}\gamma\left\langle\left|\nabla^{k+1}n\right|^2,\nabla\cdot v\right\rangle + C\left\|\nabla^{k+1}n\right\|_{L^2}\left(\left\|\nabla n\nabla^{k+1}v\right\|_{L^2}\right. \\ &\quad \left. + \left\|\nabla^2 n\nabla^k v\right\|_{L^2} + \sum_{2\le l\le k}\left\|\nabla^{l+1}n\nabla^{k+1-l}v\right\|_{L^2}\right) \\ &\le C\delta\left(\left\|\nabla^{k+1}v\right\|_{L^2}^2 + \left\|\nabla n\right\|_{H^k}^2\right). \end{aligned} \tag{3.16}$$

For the second term J_{22}, similarly, separating the case of $l=0,1$ from the order cases, we bound the summation by

$$J_{22} \le C\delta\left(\left\|\nabla^{k+1}n\right\|_{L^2}^2 + \left\|\nabla v\right\|_{H^{k+1}}^2\right). \tag{3.17}$$

In light of (3.16) and (3.17), we obtain

$$J_2 \le C\delta\left(\left\|\nabla n\right\|_{H^k}^2 + \left\|\nabla v\right\|_{H^{k+1}}^2\right).$$

Recalling from the estimates of J_2, we have

$$J_3 \le C\delta\left\|\nabla v\right\|_{H^{k+1}}^2,$$

$$J_4 \le C\delta\left(\left\|\nabla n\right\|_{H^k}^2 + \left\|\nabla^{k+1}v\right\|_{L^2}^2\right).$$

Let $J_5 = J_{51}+J_{52}+J_{53}$. For the first term J_{51}, we have by (3.1), Lemma 5.1, Hölder's inequality and integration by parts that

$$\begin{aligned} J_{51} &= \frac{\hbar^2}{4\gamma}\left\langle\nabla^k\left(\frac{\nabla n\cdot\nabla^2 n}{(n+1)^2}\right),\nabla^{k+2}v\right\rangle \\ &\le C\hbar^2\sum_{0\le l\le k}\left\|\nabla^{l+2}n\nabla^{k-l}\left(\frac{\nabla n}{(n+1)^2}\right)\right\|_{L^2}\left\|\nabla^{k+2}v\right\|_{L^2} \\ &\le C\hbar^2\left(\left\|\nabla^2 n\right\|_{L^3}\left\|\nabla^k\left(\frac{\nabla n}{(n+1)^2}\right)\right\|_{L^6} + \left\|\nabla^3 n\right\|_{L^2}\left\|\nabla^{k-1}\left(\frac{\nabla n}{(n+1)^2}\right)\right\|_{L^\infty}\right. \\ &\quad \left. + \sum_{2\le l\le k}\left\|\nabla^{l+2}n\right\|_{L^2}\left\|\nabla^{k-l}\left(\frac{\nabla n}{(n+1)^2}\right)\right\|_{L^\infty}\right)\left\|\nabla^{k+2}v\right\|_{L^2} \\ &\le C\delta\hbar^2\left(\left\|\nabla^2 n\right\|_{H^k}^2 + \left\|\nabla^{k+2}v\right\|_{L^2}^2\right). \end{aligned}$$

The same estimates hold for J_{52} and J_{53}. Combining all the estimates for J_5, we get

$$J_5 \le C\delta\hbar^2\left(\left\|\nabla^2 n\right\|_{H^k}^2 + \left\|\nabla^{k+2} v\right\|_{L^2}^2\right).$$

Let $J_6 = J_{61} + J_{62}$. We have by integration by parts and Hölder's inequality that

$$\begin{aligned}
J_{61} &= \mu\left\langle \nabla^k\left(g(n)\Delta v\right), \nabla^{k+2} v\right\rangle \\
&\le C\sum_{0\le l\le 2}\left\|\nabla^l g(n)\nabla^{k-l}\Delta v\right\|_{L^2}\left\|\nabla^{k+2} v\right\|_{L^2} + C\sum_{3\le l\le k+1}\left\|\nabla^l g(n)\nabla^{k-l+2} v\right\|_{L^2}\left\|\nabla^{k+2} v\right\|_{L^2} \\
&\le C\Bigg(\left\|g(n)\right\|_{L^\infty}\left\|\nabla^{k+2} v\right\|_{L^2} + \left\|g'(n)\right\|_{L^3}\left\|\nabla^{k+1} v\right\|_{L^6} + \left\|\nabla^2 g(n)\right\|_{L^\infty}\left\|\nabla^k v\right\|_{L^2} \\
&\quad + \sum_{3\le l\le k+1}\left\|\nabla^l g(n)\right\|_{L^2}\left\|\nabla^{k-l}\Delta v\right\|_{L^\infty}\Bigg)\left\|\nabla^{k+2} v\right\|_{L^2} \\
&\le C\delta\left(\left\|\nabla n\right\|_{H^k}^2 + \left\|\nabla^{k+2} v\right\|_{L^2}^2\right).
\end{aligned}$$

The same estimate holds for J_{62}. Combining all the estimates for J_6, we obtain

$$J_6 \le C\delta\left(\left\|\nabla n\right\|_{H^k}^2 + \left\|\nabla^{k+2} v\right\|_{L^2}^2\right).$$

For the term J_7, we have

$$J_7 \le C\delta\left(\left\|\nabla^{k+1} n\right\|_{L^2}^2 + \left\|\nabla B\right\|_{H^{k+1}}^2 + \left\|\nabla^{k+1} v\right\|_{L^2}^2\right).$$

Similarly, for the terms J_8 and J_9, recalling from the estimate of J_2, we have

$$J_8 \le C\delta\left(\left\|\nabla B\right\|_{H^{k+1}}^2 + \left\|\nabla^{k+1} v\right\|_{L^2}^2\right),$$

$$J_9 \le C\delta\left(\left\|\nabla v\right\|_{H^{k+1}}^2 + \left\|\nabla^{k+1} B\right\|_{L^2}^2\right).$$

Indeed, computing directly, it is easy to deduce

$$(\nabla\times B)\times B = (B\cdot\nabla)B - \frac{1}{2}\nabla\left(|B|^2\right), \tag{3.18}$$

then for the term J_{10}, we have by integration by parts and (3.18) that

$$\begin{aligned}
J_{10} &= -\left\langle \nabla^{k+1} B, \nabla^{k+1}\left[\nabla\times(\nabla\times B)\times B\right]\right\rangle \\
&= \left\langle \nabla^{k+1}(\nabla\times B), \nabla^{k+1}\left[(B\cdot\nabla)B - \frac{1}{2}\nabla\left(|B|^2\right)\right]\right\rangle \\
&\le \left\langle \nabla^{k+1}(\nabla\times B), \nabla^{k+1}\left((B\cdot\nabla)B\right)\right\rangle + \left\langle \nabla^{k+1}(\nabla\times B), \nabla^{k+2}\left(|B|^2\right)\right\rangle.
\end{aligned} \tag{3.19}$$

To estimate the first factor on the right-hand side of (3.19), using Lemma 5.1, 5.2 and Hölder's inequality, we obtain

$$\begin{aligned}
&\left\langle \nabla^{k+1}(\nabla\times B), \nabla^{k+1}\left((B\cdot\nabla)B\right)\right\rangle \\
&= \sum_{0\le l\le k+1} C_{k+1}^l\left\langle \nabla^{k+2} B, \nabla^l B\cdot\nabla^{k+2-l} B\right\rangle \\
&= \left\langle \nabla^{k+2} B, B\cdot\nabla^{k+2-l} B\right\rangle + \left\langle \nabla^{k+2} B, \nabla B\cdot\nabla^{k+2-l} B\right\rangle \\
&\quad + \left\langle \nabla^{k+2} B, \nabla^{k+1} B\cdot\nabla B\right\rangle + \sum_{2\le l\le k}\left\langle \nabla^{k+2} B, \nabla^l B\cdot\nabla^{k+2-l} B\right\rangle
\end{aligned}$$

$$\le C\left\|\nabla^{k+2}B\right\|_{L^2}\left(\left\|B\cdot\nabla^{k+2-l}B\right\|_{L^2}+\left\|\nabla B\cdot\nabla^{k+2-l}B\right\|_{L^2}\right.$$
$$\left.+\left\|\nabla^{k+1}B\cdot\nabla B\right\|_{L^2}+\sum_{2\le l\le k}\left\|\nabla^{l}B\cdot\nabla^{k+2-l}B\right\|_{L^2}\right)$$
$$\le C\delta\left(\left\|\nabla^{k+2}B\right\|_{L^2}^2+\left\|\nabla B\right\|_{H^{k+1}}^2\right).$$

The similar estimate holds for the second factor on the right-hand side of (3.19). Thus, for the term J_{10}, we have

$$J_{10}\le C\delta\left(\left\|\nabla v\right\|_{H^{k+1}}^2+\left\|\nabla B\right\|_{H^{k+1}}^2\right).$$

Consequently, summing up $J_1\sim J_{10}$, by the smallness of δ, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^{k+1}(n,v,B)\right\|_{L^2}^2+\frac{\hbar^2}{4\gamma^2}\left\|\frac{1}{n+1}\nabla^{k+2}n\right\|_{L^2}^2\right)+C\left(\left\|\nabla^{k+2}v\right\|_{L^2}^2+\left\|\nabla^{k+2}B\right\|_{L^2}^2\right)\\&\le C\delta\left(\left\|\nabla n\right\|_{H^k}^2+\hbar^2\left\|\nabla^2 n\right\|_{H^k}^2+\left\|\nabla v\right\|_{H^{k+1}}^2+\left\|\nabla B\right\|_{H^{k+1}}^2\right).\end{aligned}\tag{3.20}$$

Summing up above estimates for from $k=0$ to $k=3$, by the smallness of δ, we get (3.13). □

Next, we derive the dissipation estimate for n.

Lemma 3.3 *Let* (n,v,B) *be a smooth solution to* (2.1)*, then we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{3}\left\langle\nabla^k v,\nabla^{k+l}n\right\rangle+\left\|\nabla n\right\|_{H^3}^2\right)+C\left(\left\|\nabla n\right\|_{H^3}^2+\hbar^2\left\|\nabla^2 n\right\|_{H^3}^2\right)\\&\le C\left\|\nabla v\right\|_{H^3}^2+C\delta\left(\left\|\nabla v\right\|_{H^4}^2+\left\|\nabla B\right\|_{H^4}^2\right).\end{aligned}\tag{3.21}$$

Proof. For $0\le k\le 3$, applying ∇^k to $(2.1)_2$, multiplying them by $\nabla^{k+1}n$ and then integrating them over $\mathbb{R}^3$, we have

$$\begin{aligned}&\gamma\left\|\nabla^{k+1}n\right\|_{L^2}^2+\frac{\hbar^2}{4\gamma}\left\|\nabla^{k+2}n\right\|_{L^2}^2\\&=-\left\langle\nabla^k v_t,\nabla^{k+1}n\right\rangle+\mu\left\langle\nabla^k\Delta v,\nabla^{k+1}n\right\rangle-(\mu+\lambda)\left\langle\nabla^{k+1}\nabla\cdot v,\nabla^{k+1}n\right\rangle\\&\quad-\left\langle\gamma\nabla^k(v\cdot\nabla v),\nabla^{k+1}n\right\rangle-\left\langle\nabla^k\left(h(n)\nabla n\right),\nabla^{k+1}n\right\rangle\\&\quad+\left\langle\frac{\hbar^2}{4\gamma}\nabla^k\left(-g(n)\nabla\Delta n-\frac{\nabla n\cdot\nabla^2 n}{(n+1)^2}-\frac{\nabla n\Delta n}{(n+1)^2}+\frac{|\nabla n|^2\nabla n}{(n+1)^3}\right),\nabla^{k+1}n\right\rangle\\&\quad-\left\langle\nabla^k\left(g(n)\left(\mu\Delta v+(\lambda+\mu)\nabla\nabla\cdot v\right)\right),\nabla^{k+1}n\right\rangle+\left\langle\nabla^k\left(\frac{B\cdot\nabla B}{\gamma(n+1)}\right),\nabla^{k+1}n\right\rangle\\&=L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8.\end{aligned}\tag{3.22}$$

Next, we will estimate each term on the right-hand side. First, for the term L_1, by integration by parts twice, (3.1) and the continuity equation, we have

$$\begin{aligned}L_1&=-\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\nabla^k v,\nabla^{k+1}n\right\rangle+\gamma\left\|\nabla^k\nabla\cdot v\right\|_{L^2}^2+\gamma\left\langle\nabla^k\nabla\cdot v,\nabla^k(\nabla n\cdot v)\right\rangle\\&\quad+\gamma\left\langle\nabla^k\nabla\cdot v,\nabla^k(n\nabla\cdot v)\right\rangle\\&\le-\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\nabla^k v,\nabla^{k+1}n\right\rangle+\gamma\left\|\nabla^k\nabla\cdot v\right\|_{L^2}^2\\&\quad+C\left(\left\|\nabla^k(\nabla n\cdot v)\right\|_{L^2}+\left\|\nabla^k(n\nabla\cdot v)\right\|_{L^2}\right)\left\|\nabla^{k+1}v\right\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\le -\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla^k v, \nabla^{k+1} n \right\rangle + \gamma \left\| \nabla^k \nabla \cdot v \right\|_{L^2}^2 \\
&\quad + C\Bigg(\left\| \nabla n \right\|_{L^3} \left\| \nabla^k v \right\|_{L^6} + \sum_{1\le l\le k} \left\| \nabla^{l+1} n \right\|_{L^2} \left\| \nabla^{k-l} v \right\|_{L^\infty} \\
&\quad + \left\| \nabla \cdot v \right\|_{L^3} \left\| \nabla^k n \right\|_{L^6} + \sum_{1\le l\le k} \left\| \nabla^l \nabla \cdot v \right\|_{L^2} \left\| \nabla^{k-l} n \right\|_{L^\infty} \Bigg) \left\| \nabla^{k+1} v \right\|_{L^2} \\
&\le -\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla^k v, \nabla^{k+1} n \right\rangle + \gamma \left\| \nabla^k \nabla \cdot v \right\|_{L^2}^2 + C\delta\left(\left\| \nabla v \right\|_{H^k}^2 + \left\| \nabla n \right\|_{H^k}^2 \right).
\end{aligned}
$$

For the terms L_2 and L_3, similarly as the estimate of J_{21}, we obtain

$$
L_2 \le -\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \nabla^{k+1} n \right\|_{L^2}^2 + C\delta\left(\left\| \nabla n \right\|_{H^k}^2 + \left\| \nabla^{k+1} v \right\|_{L^2}^2 \right),
$$

$$
L_3 \le -\frac{\mu+\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \nabla^{k+1} n \right\|_{L^2}^2 + C\delta\left(\left\| \nabla n \right\|_{H^k}^2 + \left\| \nabla^{k+1} v \right\|_{L^2}^2 \right).
$$

Similarly for the terms L_4 and L_5, we recall from the estimate of J_2 to have

$$
L_4 \le C\delta\left(\left\| \nabla v \right\|_{H^k}^2 + \left\| \nabla^{k+1} n \right\|_{L^2}^2 \right),
$$

$$
L_5 \le C\delta \left\| \nabla n \right\|_{H^k}^2 .
$$

Let $L_6 = L_{61} + L_{62} + L_{63} + L_{64}$. For the terms L_{61}, we have by integration by parts and Hölder's inequality that

$$
\begin{aligned}
L_{61} &= \frac{\hbar^2}{4\gamma}\left\langle g'(n)\nabla^{k+2} n, \nabla^{k+1} n \right\rangle + \frac{\hbar^2}{4\gamma}\left\langle g(n)\nabla^{k+2} n, \nabla^{k+2} n \right\rangle \\
&\quad -\frac{\hbar^2}{4\gamma}\sum_{1\le l\le k} C_k^l \left\langle \nabla^l g(n) \nabla^{k-l+1}\Delta n, \nabla^{k+1} n \right\rangle \\
&\le C\hbar^2 \Bigg(\left\| g'(n) \right\|_{L^\infty} \left\| \nabla^{k+2} n \right\|_{L^2} \left\| \nabla^{k+1} n \right\|_{L^2} + \left\| g(n) \right\|_{L^\infty} \left\| \nabla^{k+2} n \right\|_{L^2}^2 \\
&\quad + \left(\left\| g'(n) \right\|_{L^\infty} \left\| \nabla^k \Delta n \right\|_{L^2} + \sum_{2\le l\le k} \left\| \nabla^l g(n) \nabla^{k-l+1}\Delta n \right\|_{L^2} \right) \left\| \nabla^{k+1} n \right\|_{L^2} \Bigg) \\
&\le C\delta\hbar^2 \left\| \nabla^2 n \right\|_{H^k}^2 .
\end{aligned}
$$

The same estimates hold for the other three terms of L_6. Combing all the estimates for L_6, we have

$$
L_6 \le C\delta\hbar^2 \left\| \nabla^2 n \right\|_{H^k}^2 .
$$

Finally, Combing with J_6 and J_7, we get

$$
L_7 \le C\delta\left(\left\| \nabla n \right\|_{H^k}^2 + \left\| \nabla^{k+2} v \right\|_{L^2}^2 \right),
$$

$$
L_8 \le C\delta\left(\left\| \nabla B \right\|_{H^k}^2 + \left\| \nabla^{k+1} n \right\|_{L^2}^2 \right).
$$

In light of $L_1 \sim L_8$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\langle \nabla^k v, \nabla^{k+1} n \right\rangle + (2\mu+\lambda)\left\| \nabla^{k+1} n \right\|_{L^2}^2 \right) + C\left(\left\| \nabla^{k+1} n \right\|_{L^2}^2 + \hbar^2 \left\| \nabla^{k+2} n \right\|_{L^2}^2 \right) \\
&\le \gamma \left\| \nabla^{k+1} v \right\|_{L^2}^2 + C\delta\left(\left\| \nabla n \right\|_{H^k}^2 + \hbar^2 \left\| \nabla^2 n \right\|_{H^k}^2 + \left\| \nabla v \right\|_{H^{k+1}}^2 + \left\| \nabla B \right\|_{H^{k+1}}^2 \right).
\end{aligned}
\tag{3.23}
$$

Summing up above estimates for from $k=0$ to $k=3$, by the smallness of δ, we conclude Lemma 3.3. □

4. Convergence Rates

In this section, we will combine all the energy estimates that we have derived in the previous section to prove Proposition 2.2.

The linearized equations corresponding to $(2.2)_1$-$(2.2)_3$ read

$$\begin{cases} n_t + \gamma\nabla\cdot v = 0, \\ v_t + \gamma\nabla n - \frac{\hbar^2}{4\gamma}\nabla\Delta n - \mu\Delta v - (\lambda+\mu)\nabla\nabla\cdot v = 0, \\ B_t - \Delta B = 0. \end{cases} \tag{4.1}$$

Thus, at the level of the linearization, B is decoupled with (n,v). If we set

$$U(t) = (n(t), v(t)),$$

then the solution to $(4.1)_1$-$(4.1)_2$ can be written as

$$U(t) = E(t)U(0) = \mathrm{e}^{-t\mathbb{A}}U(0),$$

where $\mathbb{A}$ is a matrix-valued differential operator given by

$$\mathbb{A} = \begin{pmatrix} 0 & \gamma\nabla \\ \gamma\nabla - \frac{\hbar^2}{4}\gamma\nabla\Delta & -\mu\Delta - (\mu+\lambda)\nabla\nabla\cdot \end{pmatrix}.$$

The solution semigroup $E(t)$ has the following property on the decay in time, cf. [36].

Lemma 4.1 *Let* $s \ge 0$ *be an integer. Assume that* (n,v) *is the solution of the linearized system for the first two equations in* (2.1) *with the initial data* $n_0 \in H^{s+1}\cap L^1$, $v_0 \in H^s \cap L^1$, *then*

$$\|n(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}}\left(\|(n_0,v_0)\|_{L^1} + \|(n_0,v_0)\|_{L^2}\right),$$

$$\|\nabla^{k+1}n(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k+1}{2}}\left(\|(n_0,v_0)\|_{L^1} + \|(\nabla^{k+1}n_0,\nabla^k v_0)\|_{L^2}\right),$$

$$\|\nabla^{k+1}v(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\left(\|(n_0,v_0)\|_{L^1} + \|(\nabla^{k+1}n_0,\nabla^k v_0)\|_{L^2}\right), \tag{4.2}$$

for $0 \le k \le s$.

We need the following elementary inequality [36]:

Lemma 4.2 *Let* $r_1, r_2 > 0$, *then it holds that*

$$\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_2} \le C(r_1,r_2)(1+t)^{-\min\{r_1,r_2,r_1+r_2-1-\varepsilon\}}, \tag{4.3}$$

for an arbitrarily small $\varepsilon > 0$.

If we denote the nonlinear terms for the first two equations in (2.1) as $M = (F_1, F_2)$, then (2.1) becomes

$$\begin{aligned} U(t) &= E(t)U_0 + \int_0^t E(t-\tau)M(U(\tau),B(\tau))\mathrm{d}\tau, \\ B(t) &= S(t)B_0 + \int_0^t S(t-\tau)F_3(U(\tau),B(\tau))\mathrm{d}\tau, \end{aligned} \tag{4.4}$$

where $S(t)=\mathrm{e}^{-t\Delta}$. Note that for $S(t)$, we have

$$\|S(t)B_0\|_{L^p}\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{k}{2}}\|B_0\|_{L^q},$$

and then there exists a constant C such that

$$\left\|\nabla^k B(t)\right\|_{L^p}\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{k}{2}}\|B_0\|_{L^q}+C\int_0^t(1+t-\tau)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{k}{2}}\|F_3(\tau)\|_{L^q}\,\mathrm{d}\tau,\quad (4.5)$$

for any $t\ge 0$ and $1\le p,q\le\infty$.

Lemma 4.3 *Let* (U,B) *be a smooth solution to* (2.1)*, then*

$$\|\nabla(U,B)(t)\|_{L^2}\le CE_0(1+t)^{-\frac{5}{4}}+C\delta\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|\nabla(U,B)(\tau)\|_{H^2}\,\mathrm{d}\tau,\qquad (4.6)$$

where $E_0=\|n_0\|_{H^4\cap L^1}+\|(v_0,B_0)\|_{H^3\cap L^1}$.

Proof. From Duhamel's principle, it holds that

$$(n,v)(t)=\mathrm{e}^{-t\mathbb{A}}(n_0,v_0)+\int_0^t\mathrm{e}^{-(t-\tau)\mathbb{A}}(F_1,F_2)(\tau)\,\mathrm{d}\tau.$$

Thus from Lemma 3.1 and (4.4), we have

$$\begin{aligned}\|\nabla n(t)\|_{L^2}\le{}& C(1+t)^{-\frac{5}{4}}\left(\|(n_0,v_0)\|_{L^1}+\|\nabla(n_0,v_0)\|_{L^2}\right)\\&+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left(\|(F_1,F_2)(\tau)\|_{L^1}+\|\nabla(F_1,F_2)(\tau)\|_{L^2}\right)\mathrm{d}\tau,\end{aligned}\qquad (4.7)$$

$$\begin{aligned}\|\nabla v(t)\|_{L^2}\le{}& C(1+t)^{-\frac{5}{4}}\left(\|(n_0,v_0)\|_{L^1}+\|\nabla^2 n_0,\nabla v_0\|_{L^2}\right)\\&+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left(\|(F_1,F_2)(\tau)\|_{L^1}+\|\nabla^2F_1,\nabla F_2(\tau)\|_{L^2}\right)\mathrm{d}\tau.\end{aligned}\qquad (4.8)$$

By (3.1), Hölder's inequality and Lemma 5.1, the nonlinear source terms can be estimated as follows:

$$\|(F_1,F_2)(\tau)\|_{L^1}\le C\delta\left(\|\nabla n\|_{H^1}+\|\nabla v\|_{H^1}+\|\nabla B\|_{L^2}\right),\qquad (4.9)$$

$$\|\nabla F_1\|_{H^1}\le C\delta\left(\|\nabla n\|_{H^2}+\|\nabla v\|_{H^2}\right),\qquad (4.10)$$

$$\|(F_1,F_2)(t)\|_{H^1}\le C\delta\|\nabla(n,v,B)\|_{H^2}.\qquad (4.11)$$

Put these estimates into (4.7) and (4.8), we have

$$\|\nabla U(t)\|_{L^2}\le CK_0(1+t)^{-\frac{5}{4}}+C\delta\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|\nabla(U,B)(\tau)\|_{H^2}\,\mathrm{d}\tau,\qquad (4.12)$$

where $K_0=\|n_0\|_{H^4\cap L^1}+\|v_0\|_{H^3\cap L^1}$.

Let $p=2$, $q=1$ and $k=1$ in (4.5), we obtain

$$\begin{aligned}\|\nabla B(t)\|_{L^2}&\le C(1+t)^{-\frac{5}{4}}\|B_0\|_{L^1}+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|F_3(\tau)\|_{L^1}\,\mathrm{d}\tau\\&\le C(1+t)^{-\frac{5}{4}}\|B_0\|_{L^1}+C\delta\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|\nabla(U,B)(\tau)\|_{H^2}\,\mathrm{d}\tau.\end{aligned}\qquad (4.13)$$

Putting (4.12) and (4.13) together, then we complete the proof of Lemma 4.3. □

Now we are in a position to prove Proposition 2.2.

Proof.

Since $\delta>0$ is sufficiently small, from Lemma 3.1 and 3.2, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(n,v,B)\|_{H^4}^2+\hbar^2\|\nabla n\|_{H^4}^2\right)+C_1\|\nabla v\|_{H^4}^2+C_1\|\nabla B\|_{H^4}^2\\&\le C_2\delta\left(\|\nabla n\|_{H^3}^2+\hbar^2\|\nabla^2 n\|_{H^4}^2+\|\nabla v\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right).\end{aligned}\tag{4.14}$$

In view of Lemma 3.3, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{3}\left\langle\nabla^k v,\nabla^{k+l}n\right\rangle+\|\nabla n\|_{H^3}^2\right)+C_3\left(\|\nabla n\|_{H^3}^2+\hbar^2\|\nabla^2 n\|_{H^3}^2\right)\\&\le C_4\|\nabla v\|_{H^3}^2+C_4\delta\left(\|\nabla v\|_{H^4}^2+\|\nabla B\|_{H^4}^2\right).\end{aligned}\tag{4.15}$$

Multiplying (4.14) by $\frac{C_1\delta}{C_4}$, adding it with (4.13) since $\delta>0$ is small, then we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(n,v,B)\|_{H^4}^2+\hbar^2\|\nabla n\|_{H^4}^2+\sum_{k=0}^{3}\left\langle\nabla^k v,\nabla^{k+l}n\right\rangle\right)+C_5\|\nabla(v,B,\hbar_n)\|_{H^4}^2\le 0.$$

We have by Gronwall's inequality that

$$\begin{aligned}&\|(n,v,B)\|_{H^4}^2+\|\hbar\nabla n\|_{H^4}^2+\int_0^t\|\nabla(v,B,\hbar n)(\tau)\|_{H^4}^2\\&\le C\left(\|n_0\|_{H^5}^2+\|v_0\|_{H^4}^2+\|B_0\|_{H^4}^2\right),\end{aligned}\tag{4.16}$$

then (4.16) gives (2.3).

We define the temporal energy functional

$$H(t)=\|\nabla(n,v,B)\|_{H^3}^2+\hbar^2\|\nabla^2 n\|_{H^3}^2+\sum_{k=1}^{3}\left\langle\nabla^k v,\nabla^{k+l}n\right\rangle,$$

where it is noticed that

$$H(t)\sim\|\nabla(n,v,B)\|_{H^3}^2+\hbar^2\|\nabla^2 n\|_{H^3}^2,$$

that is, there exists a constant $C_6>0$ such that

$$\frac{1}{C_6}\left(\|\nabla(n,v,B)\|_{H^3}^2+\hbar^2\|\nabla^2 n\|_{H^3}^2\right)\le H(t)\le C_6\left(\|\nabla(n,v,B)\|_{H^3}^2+\hbar^2\|\nabla^2 n\|_{H^3}^2\right).$$

From Lemma 3.2 and 3.3, we have

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t}+C\|\nabla^2(n,v,B)\|_{H^3}^2\le C\delta\|\nabla(n,v,B)\|_{L^2}^2.$$

Adding $\|\nabla(n,v,B)\|_{L^2}^2+\|\hbar\nabla n\|_{L^2}^2$ to both sides of the inequality above gives

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t}+D_1H(t)\le C\|\nabla(U,B)(t)\|_{L^2}^2,\tag{4.17}$$

where D_1 is a positive constant independent of δ. We define

$$M(t):=\sup_{0\le\tau\le t}(1+\tau)^{\frac{5}{2}}H(t)\tag{4.18}$$

then $M(t)$ satisfies

$$\|\nabla(n,v,B)\|_{H^3}+\|\hbar\nabla^2 n\|_{H^3}\le C\sqrt{H(\tau)}\le C(1+\tau)^{-\frac{5}{4}}\sqrt{M(\tau)},\quad 0\le\tau\le t.$$

From Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}\|\nabla(U,B)(t)\|_{L^2} &\le CE_0(1+t)^{-\frac{5}{4}} + C\delta\int_0^t(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{4}}\,d\tau\sqrt{M(t)} \\ &\le C(1+t)^{-\frac{5}{2}}\left(E_0+\delta\sqrt{M(t)}\right).\end{aligned} \tag{4.19}$$

By Gronwall's inequality, we have from (4.16) that

$$\begin{aligned}H(t) &\le H(0)e^{-D_1 t} + C\int_0^t e^{-D_1(t-\tau)}\|\nabla(U,B)(\tau)\|_{L^2}^2\,d\tau \\ &\le H(0)e^{-D_1 t} + C\int_0^t(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{2}}\,d\tau\left(K_0+\delta\sqrt{M(t)}\right)^2 \\ &\le C(1+t)^{-\frac{5}{2}}\left(H(0)+K_0^2+\delta^2 M(t)\right).\end{aligned} \tag{4.20}$$

Since $M(t)$ is non-decreasing, we have from (4.20) that

$$M(t) \le C\left(H(0)+K_0^2+\delta^2 M(t)\right),$$

which implies that if $\delta > 0$ is small enough, then

$$M(t) \le C\left(H(0)+K_0^2\right) \le CK_0^2.$$

This in turn gives

$$\|\nabla(n,v,B)\|_{H^3} + \|\hbar\nabla^2 n\|_{H^3} \le C(1+t)^{-\frac{5}{4}}. \tag{4.21}$$

From (4.21), we have

$$\|\nabla(n,v,B)\|_{L^2} \le C(1+t)^{-\frac{5}{4}},$$

which also implies from Lemma 5.1 that

$$\|(n,v,B)\|_{L^\infty} \le C\|\nabla(n,v,B)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}.$$

Hence (2.5) and (2.6) are proved. By Sobolev's inequality, we have

$$\|(n,v,B)\|_{L^6} \le C\|\nabla(n,v,B)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}.$$

Next, by (4.2) and (4.5), it follows from the Duhamel's principle that

$$\begin{aligned}&\|(n,v,B)(t)\|_{L^2} \\ &\le C(1+t)^{-\frac{3}{4}}\left(\|(n_0,v_0)\|_{L^1} + \|n_0\|_{H^1} + \|v_0\|_{L^2} + \|B_0\|_{L^1}\right) \\ &\quad + C\int_0^t(1+t)^{-\frac{3}{4}}\left(\|(F_1,F_2)(\tau)\|_{L^1} + \|F_1(\tau)\|_{H^1} + \|F_2(\tau)\|_{L^2} + \|F_3(\tau)\|_{L^1}\right)d\tau \\ &\le CK_0(1+t)^{-\frac{3}{4}} + C\delta\int_0^t(1+t)^{-\frac{3}{4}}\|(\nabla U,\nabla B)(\tau)\|_{H^2}\,d\tau \\ &\le CK_0(1+t)^{-\frac{3}{4}} + C\delta\int_0^t(1+t)^{-\frac{3}{4}}(1+\tau)^{-\frac{5}{4}}\,d\tau \le C(1+t)^{-\frac{3}{4}}.\end{aligned}$$

Hence, for any $2 \le q \le 6$, we have by the interpolation that

$$\|(n,v,B)(t)\|_{L^q} \le \|(n,v,B)(t)\|_{L^2}^{\theta}\|(n,v,B)(t)\|_{L^6}^{1-\theta} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)},$$

where $\theta = \dfrac{6-p}{2p}$, this proves (2.4). On the other hand, using the estimates above (2.1), we have

$$
\begin{aligned}
&\left\|\partial_t (n,v,B)(t)\right\|_{L^2} \\
&\le C\Big\{\left\|\nabla\cdot v\right\|_{L^2} + \left\|F_1\right\|_{L^2} + \left\|\hbar^2\nabla\Delta n\right\|_{L^2} + \left\|\Delta v\right\|_{L^2} \\
&\quad + \left\|\nabla\nabla\cdot v\right\|_{L^2} + \left\|F_2\right\|_{L^2} + \left\|\Delta B\right\|_{L^2} + \left\|F_3\right\|_{L^2}\Big\} \\
&\le C(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Then, for any $0 \le t \le T$ we get (2.7). Therefore, the proof of Proposition 2.2 is complete. □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

In this appendix, we state some useful inequalities in the Sobolev space.

Lemma 5.1 *Let* $f \in H^2(\mathbb{R}^3)$. *Then*

$$\|f\|_{L^\infty} \le C\|\nabla f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{H^1}^{\frac{1}{2}} \le C\|\nabla f\|_{H^1}^{\frac{1}{2}},$$

$$\|f\|_{L^6} \le C\|\nabla f\|_{L^2},$$

$$\|f\|_{L^q} \le C\|f\|_{H^1}, 2 \le q \le 6.$$

Lemma 5.2 *Let* $m \ge 1$ *be an integer, then we have*

$$\left\|\nabla^m(fg)\right\|_{L^p} \le C\|f\|_{L^{p_1}}\left\|\nabla^m g\right\|_{L^{p_2}} + C\left\|\nabla^m f\right\|_{L^{p_3}}\|g\|_{L^{p_4}}, \tag{A.1}$$

and

$$\left\|\nabla^m(fg) - f\nabla^m g\right\|_{L^p} \le C\|\nabla f\|_{L^{p_1}}\left\|\nabla^{m-1} g\right\|_{L^{p_2}} + C\left\|\nabla^m f\right\|_{L^{p_3}}\|g\|_{L^{p_4}}, \tag{A.2}$$

where $p, p_1, p_2, p_3, p_4 \in [1,\infty)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \tag{A.3}$$

Proof. Please refer for instance to [37]. □