

Rigorous Proof of Goldbach's Conjecture

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Abstract

In this article, we use set, function, sieve and number theory to study the prime and composite numbers, prove that the lower limit formula of the number of prime numbers derived from the Euler's function, and find $d(n)$ to count the lower limit formula of the number of prime integer-pairs. We proved that Goldbach's conjecture is correct by mathematical induction. Finally, we proved proof reliance by mathematical analysis and computer data.

Keywords

Set, Function, Sieve, Integer-Pair, Induction, Mathematical Analysis

1. Introduction

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory. It states: every even integer greater than 2 can be expressed as the sum of two primes [1] [3].

In the letter to Euler by Goldbach in 1742, Goldbach put forward the following conjecture: any even number greater than 2 can be written as the sum of two prime numbers. The common conjecture today is the version of Euler. The proposition that any sufficiently large even number can be expressed as a sum of the number of prime factors less than a sum of two numbers that prime factors not exceeding a and b separately. It is denoted as " $a + b$ ". In 1966, Chen Jingrun proved that " $1 + 2$ " was established. But the Goldbach's conjecture has not yet been fully resolved [2].

We are going to take the big even number x on the number line, and then going to go back from x , and to look for the even numbers that are not true for the Goldbach's conjecture, which is the exception even numbers.

The number of exception even numbers before x is written $E(x)$. We hope that no matter how big x is, there's only one exception even numbers. Before x which is 2, namely is just 2 that makes the guess wrong. So the Goldbach's con-

jecture is equivalent to $E(x) = 1$ [3]. We will use the method of exception set to solve the Goldbach's conjecture.

Remarks on Notation

Definition: If A is a set, $\text{card}(A)$ is the number of elements of set A . In addition:

- A_p^n is the set of integers with multiples of prime p . As $A_3^{10} = \{3, 6, 9\}$.
- $A_{2,3,\dots,p_m}^n$ is the set with multiples of primes $2, 3, \dots, p_m$ in $[1, n]$, as $A_{3,5}^{10} = \{3, 5, 6, 9, 10\}$.
- $B_{2,3,\dots,p_m}^n$ is the set of integers without multiples of primes $2, 3, \dots, p_m$ in $[1, n]$, as $B_{2,3}^{10} = \{1, 5, 7\}$.
- $\pi(N)$ is the number of prime numbers in $[1, 2n]$.
- $D(N)$ is the number of prime integer-pairs that sum of two integers is equal to $2n$.
- D^n is a set with elements are prime integer-pairs.

2. Integers

2.1. Prime and Composite Numbers

In number theory, we know the number of primes can calculate by sieve of Eratosthenes and inclusion exclusion formula [3] [5] [6].

$$\begin{aligned} \pi(N) = m - 1 + N - \sum_{1 \leq i_1} \left[\frac{N}{p_{i_1}} \right] + \sum_{1 \leq i_1 \leq i_2 \leq m} \left[\frac{N}{p_{i_1} p_{i_2}} \right] - \dots \\ + \sum_{1 \leq i_1 \leq \dots \leq i_k \leq m} \left[\frac{N}{p_{i_1} \dots p_{i_k}} \right] + \dots + (-1)^m \left[\frac{N}{p_1 \dots p_m} \right] \end{aligned} \tag{1}$$

It can be received exact value by the calculation of the tolerance formula, but it can only be used in small range of positive integers, because the operation is very complicated. For a larger range of data operations, there is inefficiency.

2.2. The Lower Limit of the Number of Prime Numbers

2.2.1. Function $\phi(n)$

Theorem. (Euler function) [3] [5] [6] If n contains any prime number p_1, p_2, \dots, p_k factor, let $\phi(n)$ be the number of positive integers not greater than and prime to n , then

$$\phi(n) = n \left(1 - \frac{1}{p_1} \right) \left(1 - \frac{1}{p_2} \right) \dots \left(1 - \frac{1}{p_k} \right) \tag{2}$$

2.2.2. Function $\phi'(n)$

Lemma 1. [4] [5] [6] (Division with remainder) If a and b are two integers, then integers q and r exist as

$$a = bq + r, 0 < r < b.$$

Theorem 2. If p is any prime number, let A_p^n be a set of all multiples of

prime p in $[1, n]$

$$\text{card}(A_p^n) \leq \frac{n}{p}$$

Proof. If $n = kp + r, 0 \leq r < p$, then

$$\text{card}(A_p^n) \leq \left\lfloor \frac{n}{p} \right\rfloor = k \leq \frac{n}{p} = k + \frac{r}{p}.$$

□

Theorem 3. Let B_p^n be a set of integers without multiples of prime p , then

$$\text{card}(B_p^n) \geq n \left(1 - \frac{1}{p}\right).$$

Proof. By Theorem 2, have

$$\text{card}(A_p^n) \leq \frac{n}{p},$$

then

$$\text{card}(B_p^n) = n - \text{card}(A_p^n) \geq n - \frac{n}{p} = n \left(1 - \frac{1}{p}\right).$$

This $(1 - 1/p)$ is called the eliminating factor of the composite number. □

Theorem 4. If $p_m^2 \leq n \leq p_{m+1}^2, B_{2,3,\dots,p_m}^n$ is the collection of n continuous natural number that except multiples of primes $2, 3, \dots, p_m$, then the function $\phi'(n)$ is

$$\text{card}(B_{2,3,\dots,p_m}^n) \geq \phi'(n) = \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right).$$

Proof. Case 1: if n is multiple of primes $2, 3$, which as n equals 18,

$$\text{card}(B_{2,3}^{18}) = \phi'(18) = \phi(18) = 6.$$

This is say, the function $\phi(n)$ is the accurate value calculated when the n is multiple of $2, 3, \dots, p_m$.

Case 2: if n may be not multiples of primes $2, 3, \dots, p_m$, which as n equals 20,

$$\text{card}(B_{2,3}^{20}) = 7, \phi'(20) \approx 6.3333.$$

When n is not multiple of $2, 3, \dots, p_m$ by Theorem 2, it can be known that $1/p_i$ is the component density of multiples of the prime p_i in $[1, n]$. Then we take $\phi'(n)$ from $\phi(n)$ as below

$$\phi'(n) = n \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{p_m}\right)$$

By Theorem 60 in “An Introduction to the Theory of numbers” [5], we known $\phi'(n)$ is multiplicative. According to Theorem 3, it can be known that $(1 - 1/p)$ is the eliminating factor of the composite number. By Theorem 3, the every eliminating factor $(1 - 1/p)$ less than or equal to the actual value, then its means that, $\phi'(n)$ is lower bound of the number of primes in $[1, n]$. For examples: if $n = 60$,

$\text{card}(B_{2,3,5,7}^{60}) = 14$, $\pi(60) = 17$, $\phi'(60) = 13.71$; if $n = 68$, $\text{card}(B_{2,3,5,7}^{68}) = 16$, $\pi(68) = 19$, $\phi'(68) = 15.54$. Namely

$$\text{card}(B_{2,3,\dots,p_m}^n) \geq \phi'(n).$$

□

Theorem 5. If $p_m^2 \leq n < p_{m+1}^2$,

$$\text{card}(B_{2,3,\dots,p_m}^n) \geq \phi'(n) - m = n \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{p_m}\right) - m \tag{3}$$

Proof. By Theorem 4, we know, if n is a multiple of primes $2, 3, \dots, p_m$,

$$\text{card}(B_{2,3,\dots,p_m}^n) \geq \phi'(n).$$

But when n is not a multiple of primes p_p , there maybe has a error r

$$\left(r = \frac{n}{p_k} - \left\lfloor \frac{n}{p_k} \right\rfloor\right), \text{ and}$$

$$r \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{p_m}\right) < 1.$$

In order to obtain a definite lower limit of the number of primes and simple operation, we find a function that may be associated with the error as an error compensation. That is to subtract 1 is the error compensation for every operation of multiples of p_r ($i = 1, 2, \dots, m$) So, we obtain following formula

$$\text{card}(B_{2,3,\dots,p_m}^n) \geq \phi'(n) - m.$$

□

3. Integer Pairs

$2n$ positive integers be arranged to w rows as **Figure 1** follows.

We obtain n pairs of two integers that sum are just equal to $2n$, it is called “integer-pair”. It is called “prime integer-pair” if two integers are primes.

Obviously, as long as we can prove that there has one or more than one prime integer-pair, the Goldbach’s conjecture problem can be solved [7].

Theorem 6. Let D^n be a set of prime integer-pairs in **Figure 1**. Then

$$\text{card}(D^n) \geq d(n) - m = \left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{p_m}\right) - m. \tag{4}$$

Proof. First, to delete all even integer-pairs in **Figure 1**, we obtain $n/2$ (or $(n + 1/2)$) odd integer-pairs, as **Figure 2** (assume $n - 1$ is odd) noticed that the spacing of each odd prime number is the same as that of the natural number, as shown in **Figure 3**. So the function formula of $\phi'(n)$ can also be applied (except without the 2 factor).

Our approach is:

1) First, those integer-pairs on the top row that contain multiple of $2, 3, \dots, p_m$ are to marked “/”, (suppose $n - 3$ are composite numbers) (see **Figure 4**).



Figure 1. n pairs from $2n$ integers.

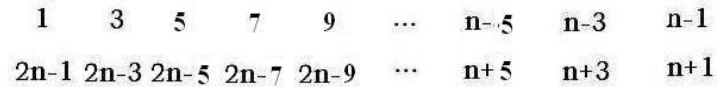


Figure 2. Odd integer-pairs.

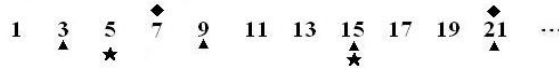


Figure 3. Odd integers.

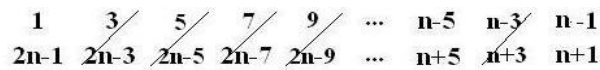


Figure 4. Marking of “/”.

The calculation for the number of integer-pairs marked “/” is by $\phi(n)$ as below

$$\text{card}(B_{2,3,\dots,p_m}^n) \geq \phi'(n) - m = n \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{p_m}\right) - m.$$

by Theorem 5.

2) Next, those integer-pairs on the bottom row that contain multiple of 2, 3, ..., p_m are also to marked “\”, see **Figure 5** (suppose $n - 3$, $n + 3$, $2n - 5$ are composite numbers);

Then, we are deleting all integer-pairs that are marked “/” or “\”. At same times, we are using function $d(n)$ for calculation the number of all deleted integer-pairs as below (by change every numerator 1 to 2 from function $\phi(n)$)

$$\text{card}(D^n) \geq d(n) - m = \frac{n}{2} \left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{5}\right) \dots \left(1 - \frac{2}{p_m}\right) - m.$$

In this way, we counted only one time for every integer-pair that containing one multiple of 2, 3, ..., p_m ; but for other integer-pairs that contained two multiples of 2, 3, ..., p_m (as integer-pairs noted “x”), we repeat counted it by two times, this is called excessive duplicate sieving [8]. Note: this repetitive deletion will make $d(n) - m$ less than the actual number of the remainder with 1 and prime numbers (see **Figure 5**). This does not affect the validity of the proof. Instead, the way to do this is to get a more rigorous the lower limit of $\text{card}(D^n)$ (i.e. $D(N)$), which is exactly what we need. So, the(4) is true. \square

Theorem 7. Let D^n be a set of prime integer-pairs in **Figure 1**. If

$$d(n) - m = \frac{n}{2} \prod_{i=2}^m \left(1 - \frac{2}{p_i}\right) - m \geq 2,$$

is true, then $\text{card}(D^n) \geq 1$.

$$\begin{array}{cccccccc}
 d(n) : & 1 & \cancel{3} & \cancel{5} & \cancel{7} & \dots & \cancel{n-5} & \cancel{n-3} & n-1 \\
 & 2n-1 & 2n-3 & 2n-5 & 2n-7 & \dots & n+5 & n+3 & n+1 \\
 D^n : & & 3 & & 7 & \dots & n-5 & & n-1 \\
 & & 2n-3 & & 2n-7 & \dots & n+5 & & n+1
 \end{array}$$

Figure 5. Marking of “\”.

Proof. Because all integer-pairs that containing of the composite numbers have been deleted, the remnants are just 1 and primes. Because the D^n is a set of prime integer-pairs in n integer-pairs, and

$$\text{card}(D^n) \geq d(n) - m = \frac{n}{2} \prod_{i=2}^m \left(1 - \frac{2}{p_i} \right) - m.$$

In order to get the safe and reliable data, our rule to believe that if $d(n)$ is greater than or equal to 2, the card (D^n) is greater than or equal to 1 (because 1 and three primes can form two integer-pair). So **Theorem 7** is true. \square

4. Proof of Proposition

4.1. Proofs of Some about Theorems

Theorem 8. When $m \geq 60$, $P_m/4 \geq m + 2$.

Proof. Because from m to $m + 1$, $p_{m+1} \geq p_m + 2$. That is to say, the growth rate of P_m greater than that of m .

Use induction on m :

1) When $m = 60$, $P_{60} = 281$, $P_{60}/4 = 70.25$. $70.25 > 60 + 2$ (to win more 8.25).

2) Suppose when $m = k$, $\frac{P_k}{4} \geq k + 8$.

Then, when $m = k + 1$, there have two cases:

a) Most prime numbers are not the twin primes, then

$$\frac{p_{k+1}}{4} \geq \frac{p_k + 4}{4} \geq k + 1 + 8.$$

b) When p_{k+1} and p_k are a pair of twin primes,

$$\frac{p_{k+1}}{4} \geq \frac{p_k + 4}{4} \geq k + 0.5 + 8.$$

The prime numbers behind are not the twin prime number. If the distance is greater than or equal to 4, then the loss of 0.5 in the twin prime numbers will be made up in the increment of successive prime numbers. So as $2n$ goes up, the surplus goes up. This is the argument with the growth rate of P_m is greater than that of m .

This completes the proof of Theorem 8. \square

To understand Theorem 8, see the data obtained by computer programming following **Figure 6**.

Theorem 9. If $p_m^2 + 1 \leq 2n \leq p_{m+1}^2 - 1$, then

$$d(n) - m = \frac{n}{2} \prod_{i=2}^m \left(1 - \frac{2}{p_i} \right) - m \geq 2.$$

m	pm	d	pm/4	m+2	bm
50	229	4	57.25	52	.102037553177845
51	233	6	58.25	53	.101599623765065
52	239	2	59.75	54	.101174520736759
53	241	10	60.25	55	.100754709447395
54	251	6	62.75	56	.100353296262345
55	257	6	64.25	57	9.99628165103519E-02
56	263	6	65.75	58	9.95827297555596E-02
57	269	2	67.25	59	9.92125337341635E-02
58	271	6	67.75	60	9.88464358237053E-02
59	277	4	69.25	61	9.84895894849916E-02
60	281	2	70.25	62	9.81390927252585E-02
61	283	10	70.75	63	9.77923114788795E-02
62	293	14	73.25	64	9.74585493236615E-02
63	307	4	76.75	65	.097141094765604

Figure 6. Data of m and p_m .

Proof. In fact, when $p_m^2 + 1 \leq 2n \leq p_{m+1}^2 - 1$, the $d(n)$ is

$$d(n) = \frac{p_m^2 + 1}{4} \prod_{i=2}^m \left(1 - \frac{2}{p_i}\right) \frac{p_m^2}{4} \times \frac{1}{3} \times \frac{5}{7} \times \frac{9}{11} \times \dots \times \frac{p_m - 2}{p_m}$$

$$> \frac{p_m^2}{4} \times \frac{1}{3} \times \frac{5}{7} \times \frac{9}{11} \times \dots \times \frac{p_m - 4}{p_m - 2} \times \frac{p_m - 2}{p_m} = \frac{p_m^2}{4} \times \frac{1}{p_m}$$

Namely $d(n) \geq \frac{p_m}{4}$.

By Theorem 8, when $m \geq 60$, $P_m/4 \geq m + 2$. So

$$d(n) - m = \frac{n}{2} \prod_{i=2}^m \left(1 - \frac{2}{p_i}\right) - m \geq 2.$$

□

4.2. Proof of the Proposition

Proposition 4.1. Every even number that not less than 4 is the sum of two prime numbers.

Proof. Let $2n$ be any even number larger than 4. That $2n$ integers are arranged form two rows in Figure 7 as below.

Using the method of Section 3, we deleting even integer-pairs and remnants are those odd integer-pairs.

Next to delete those odd integer-pairs of multiple of 2, 3, ..., p_m by the duplicate sieving, as Figure 8.

And by Theorem 6, we obtain function $d(n)$ as

$$d(n) - m = \frac{n}{2} \prod_{i=2}^m \left(1 - \frac{2}{p_i}\right) - m.$$

By Theorem 8 and Theorem 9 we can prove that when $m \geq 60$,

$$d(n) - m = \frac{n}{2} \prod_{i=2}^m \left(1 - \frac{2}{p_i}\right) - m \geq 2.$$

By Theorem 7, it is following

$$\text{card}(D^n) \geq 1.$$

1	2	3	...	n-3	n-2	n-1	n
2n-1	2n-2	2n-3	...	n+3	n+2	n+1	n

Figure 7. n integer-pairs from 2n integers.

1	3	5	7	...	n-5	n-3	n-1
2n-1	2n-3	2n-5	2n-7	...	n+5	n+3	n+1

Figure 8. Deleting of odd integer-pairs.

Namely there is having one or more one of prime integer-pairs exist in n integer-pairs that sum of two primes equals 2n, with when $m \geq 60$,

$$2n \geq p_{60}^2 + 1 = 78962, \text{ (by } p_{60} = 281 \text{)}.$$

For example, $D(78,962) = 542, 78,962 = 43 + 78,919 = 61 + 78,901$.

In addition, for $4 \leq 2n \leq 78,960$, we can prove by a table (to omit). For example:

$$D(78,960) = 1630, 78,960 = 31 + 78,929 = 41 + 78,919;$$

$$D(78,958) = 572, 78,958 = 29 + 78,929 = 41 + 78,917.$$

Hence the Proposition 4.1 is true. □

5. Safety Analysis

Some people may ask: is the method of repeated screening reliable? Is it beyond the range of positive integers?

Now let's do a mathematical analysis of this problem [9] [10] [11].

$$\lim_{n \rightarrow \infty} \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right) = \frac{1}{p_m} = 0,$$

$$\lim_{n \rightarrow \infty} \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right) = \frac{1}{p_m} = 0,$$

$$\prod_{i=2}^m \left(1 - \frac{1}{p_i}\right) = 2 \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right),$$

$$\lim_{n \rightarrow \infty} \prod_{i=2}^m \left(1 - \frac{2}{p_i}\right) = \lim_{n \rightarrow \infty} \frac{2}{p_m} = 0.$$

Because the formulas few a factor of $(1 - 1/2)$, so though every numerator is 2, it will not more than the actual value.

In fact, when $m \geq 60$, $d(n) \geq \frac{p_m}{4} \geq m + 2$ by Theorem 8.9. So the limit of

$\prod_{i=2}^m \left(1 - \frac{2}{p_i}\right)$ is in security range. Obviously, it is not less than 0, and card (D^n)

greater than 1 if p_m (namely $2n$) is enough big. There is also has a space of prime integer-pairs. In other words, the data is safe and reliable. Such as, the safety feasibility of repeated screening method is proved by the mathematical analysis [8] [9]. In addition, Here we see that the function $d(n)$ increases as p_m increases, and $d(n)$ is a monotonic increasing function.

There is a large surplus for the proof of Goldbach’s conjecture, as shown in:

1) By computer calculation, the actual number of prime pairs is less than $d(n) - 1$. That is to say, the error compensation is not need to minus m. So, when $m \geq 5$, $p_m \geq 11$, $p_m \geq 2.75 > 2$, the Proposition 4.1 can be proved.

2) Starting from harsh conditions, the error compensation is required to be $2m$. when $m \geq 1098$, $p_m \geq 8819$, ($2n \geq 77774762$). For example: $D(77774762) = 6808$, $77774762 = 77774743 + 19 = 77774731 + 31$.

So the goldbach’s conjecture is correct absolutely.

6. Computer Test Data

Through computer programming, we get **Table 1** as follows.

1) **Table 1** shows that $\phi(n)$ is less than $\pi(N)$.

This shown the $\phi(n)$ is the lower limit of the number of prime numbers.

2) **Table 1** shows that $d(n)$ is less than $D(N)$ and correct. The general trend of $d(n)$ is upward ; with the increasing of $2n$, the number of prime integer-pairs is increasing. In other words, even number $N (=2n)$ goes to infinity, the Goldbach’s conjecture is still correct.

By computer programming, we obtain **Figure 9**. Those digits represents the number of prime integer-pairs in [150, 300], (the detection interval is 2); 2) the dotted line above represents the function $d(n)$, and below is $d(n) - m$. The solid line shows the $p_m/4$.

Table 1. About the numbers of prime and the numbers of prime pairs.

m	p_m	$2n$	$\phi(n)$	$\pi(N)$	$d(n)$	$D(N)$
2	3	10	1.65	3	0.84	2
3	5	26	3.46	6	1.3	3
4	7	50	5.71	9	1.84	4
5	11	122	12.68	18	1.78	3
6	13	170	16.30	23	4.16	9
7	17	290	26.18	34	6.28	10
8	19	362	30.96	42	7.06	8
9	23	530	43.35	56	9.40	14
10	29	842	66.50	82	13.9	18

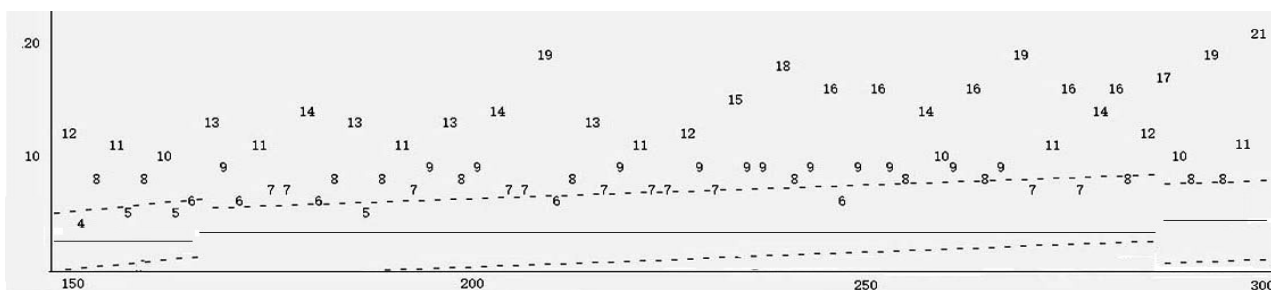


Figure 9. Diagram of the number of prime integer-pairs.

Figure 9 is showing: 1) All digital with the number of prime integer-pairs above the line of $d(n) - m$, which more and more. 2) This is to explain that the function $d(n)$ is the lower bound of the number of prime integer-pairs, $p_m/4$ and $d(n) - m$ are its the reliable lower limit. That means that, when $x \geq 100$, the number of prime integer-pairs must be greater than 1. In other words, no matter how big x is, $E(x)$ is always equal to 1.

7. Conclusions

Through the above proof, we proved that even x greater than 2 can find prime pairs that sum of two primes equal to even x , which proves that the exception set $E(x)$ is always equal to 1. In this way, we proved the Goldbach's conjecture by the method of exception set [3].

In addition, it is worth mentioning that this proving method is equally applicable to solving the same types of problems, such as Angle valley conjecture and a proof of the infinite of twin prime numbers (let every integer-pair be that the difference of two integers is 2). A world mathematical problem unsolved for more than 270 year was finally solved successfully. I have to say that this is a gratifying success case of mathematical theory and computer technology.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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