Journal of Applied Mathematics and Physics, 2018, 6, 1748-1756
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Evolutions of the Ruled Surfaces via the Evolution of Their Directrix Using Quasi Frame along a Space Curve

M. A. Soliman, Nassar H. Abdel-All, R. A. Hussien, Taha Youssef

Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt
Email: nhabdeal2002@yahoo.com, rawya-hussein@yahoo.com, tahaussef@yahoo.com

How to cite this paper: Soliman, M.A., Abdel-All, N.H., Hussien, R.A. and Youssef, T. (2018) Evolutions of the Ruled Surfaces via Their Evolution of the Directrix Using Quasi Frame along a Space Curve. *Journal of Applied Mathematics and Physics*, **6**, 1748-1756.
https://doi.org/10.4236/jamp.2018.68149

Received: July 25, 2018
Accepted: August 27, 2018
Published: August 30, 2018

Abstract

In this paper, evolutions of ruled surfaces generated by the quasi normal and quasi binormal vector fields of space curve are presented. These evolutions of the ruled surfaces depend on the evolutions of their directrix using quasi frame along a space curve.

Keywords

Ruled Surfaces, Quasi Frame, Frenet Frame, Evolution of Space Curves

1. Introduction

Recently, the study of the motion of inelastic plane curves has arisen in a number of diverse engineering applications. Chirikjian and Burdick [1] describe the motion of a planar hyper redundant (or snake-like) robot as the flow of a plane curve, while Brockett [2] explicitly proposes the idea of an inelastic string machine as a robotic device. Jagadeesan Jayender and Kirby G. Vosburgh [3] showed that the colonoscope can be modeled as a set of infinitesimal rigid links along a backbone curve defined in terms of the Frenet-Serret frame. They used evolution of space curve theory in their model. Kalantar, *et al.* [4] they considered a collection of robots which can act as markers on an imaginary curve moving according to the local curvature and the external environmental force projected unto the normal to the curve.

The geometric link between integrable equation and the motion of curves may be said to have its origin in an analysis by Da Rios [5] in 1906. He obtained partial differential equation that governing the moving curves. Takao [6], showed that the Da Rios equations may be mapped to produce the celebrated nonlinear Schrodinger equation. Lamb [7], later in 1977, linked the motion of curves with modified Korteweg-de Vries, the sine-Gordon and nonlinear Schrödinger equations. Lakshmanan *et al.* [8], derived the Heisenberg spin chain equation via the spatial motion of a space curve.

In recent times, Santini and Doliwa [9] linked the motion of inextensible curves with solitonic systems. [10] studied evolution of the translation surfaces and their generating curves in E^3 and obtained the evolution equations of the fundamental quantities and the Christoffel symbols for the translation surfaces. [11] studied generated surfaces via inextensible flows of curves in R^3. They constructed and plotted the surfaces generated from the motion inextensible curves in R^3. D. Y. Kwona and F. C. Park [12] [13] studied evolution of inelastic plane curves and inextensible flows of curves and developable ruled surfaces. They get partial differential equation that governing the flow of curves and the flow of ruled surfaces. Dariush Latifi and Asadollah Razavi [14] obtained necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equation involving the curvature and torsion. T. Körpinar and E. Turhan [15] investigated inextensible flows of tangent developable surfaces in Euclidean 3-space E^3 and obtained results for minimal tangent developable surfaces in Euclidean 3-space. R. Mukherjee and R. Balakrishnan [16] linked moving curves with sine-Gordon equation and displayed the evolving curve via the numerical integration of the Serret-Frenet equations.

In this paper, we introduce a different approach to this problem. The evolution of curves is represented by two sets of quasi Serret-Frenet equations for tangent, quasi normal and quasi binormal vectors to the curve. By applying compatibility condition on these vectors, three partial differential equations for the curvatures $\kappa_1, \kappa_2, \kappa_3$ are derived. We derive system of partial differential equations governing the time evolution of the curvatures of the evolving curve. Ruled surface is constructed on the evolving curve where the generator is quasi normal and quasi binormal vectors to the curve. The coefficients of the first, second fundamental forms, Gaussian curvatures, mean curvatures are obtained.

The article is organized as follows. In Section 2, we introduce differential geometry of curves focusing on Serret-Frenet frame and quasi frame along a space curve. In Section 3, the evolution of curves is represented by two sets of quasi Serret-Frenet equations for tangent, quasi normal and quasi binormal vectors to the curve. By applying compatibility condition on these vectors, three partial differential equations for the curvatures $\kappa_1, \kappa_2, \kappa_3$ are derived. In Section 4, Ruled surface is constructed on the evolving curve where the generator is quasi normal and quasi binormal vectors to the curve. The coefficients of the first, second fundamental forms, Gaussian curvatures, mean curvatures are obtained.

2. Frenet Frame and Quasi Frame along a Space Curve

There is a more moving frame that can be associated to a space curve in space such as Frenet frame [17], Bishop frame [18] [19] [20], Kepler frame [21]. In this section we define a new frame along a space curve as an alternant to Frenet frame which called quasi Frame [22]. Also geometric proprieties for the Frenet frame and quasi frame along a space curve are presented.

Let $\boldsymbol{r}=\boldsymbol{r}(s)$ be a vector valued function of a regular space curve represented with its arc-length s, the vectors associated to the curve are

$$\boldsymbol{T}=\frac{\boldsymbol{r}'(s)}{\|\boldsymbol{r}'(s)\|},\quad \boldsymbol{N}=\frac{\boldsymbol{T}'(s)}{\|\boldsymbol{T}'(s)\|},\quad \boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}, \tag{1}$$

where $\boldsymbol{T}$ is the tangent vector, $\boldsymbol{N}$ is the normal vector, $\boldsymbol{B}$ is the binomial vector.

The curvature κ_1 and the torsion κ_2 are given by

$$\kappa_1=\frac{\|\boldsymbol{r}'(s)\wedge\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(s)\|^3},\quad \kappa_2=\frac{det(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')}{\|\boldsymbol{r}(s)\times\boldsymbol{r}''(s)\|^2}. \tag{2}$$

The Frenet frame $(\boldsymbol{T},\boldsymbol{N},\boldsymbol{B})$ vary along $\boldsymbol{r}$ according to the well-known Serret-Frenet relations [17]

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\boldsymbol{T}\\ \boldsymbol{N}\\ \boldsymbol{B}\end{pmatrix}=\begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{T}\\ \boldsymbol{N}\\ \boldsymbol{B}\end{pmatrix}. \tag{3}$$

The quasi frame of a regular space curve $\boldsymbol{r}=\boldsymbol{r}(s)$ is given by

$$\boldsymbol{T}_q=\boldsymbol{T},\quad \boldsymbol{N}_q=\frac{\boldsymbol{T}\times\boldsymbol{k}}{\|\boldsymbol{T}\times\boldsymbol{k}\|},\quad \boldsymbol{B}_q=\boldsymbol{T}\times\boldsymbol{N}_q, \tag{4}$$

where $\boldsymbol{k}$ is the projection vector can be chosen as $\boldsymbol{k}=(1,0,0)$ or $\boldsymbol{k}=(0,1,0)$ or $\boldsymbol{k}=(0,0,1)$.

Let θ is the angel between the normal $\boldsymbol{N}$ and quasi normal $\boldsymbol{N}_q$. Then, the relation between two frames is given by

$$\begin{pmatrix}\boldsymbol{T}_q\\ \boldsymbol{N}_q\\ \boldsymbol{B}_q\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta(s) & \sin\theta(s)\\ 0 & -\sin\theta(s) & \cos\theta(s)\end{pmatrix}\begin{pmatrix}\boldsymbol{T}\\ \boldsymbol{N}\\ \boldsymbol{B}\end{pmatrix}. \tag{5}$$

Thus,

$$\begin{pmatrix}\boldsymbol{T}\\ \boldsymbol{N}\\ \boldsymbol{B}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta(s) & -\sin\theta(s)\\ 0 & \sin\theta(s) & \cos\theta(s)\end{pmatrix}\begin{pmatrix}\boldsymbol{T}_q\\ \boldsymbol{N}_q\\ \boldsymbol{B}_q\end{pmatrix}. \tag{6}$$

A short calculation using Equations (3), (5) and (6) shows that the variation of quasi frame is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \boldsymbol{T}_q \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T}_q \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{pmatrix}, \tag{7}$$

where the quasi curvatures are

$$\begin{aligned} \kappa_1 &= \kappa \cos\theta, \\ \kappa_2 &= -\kappa \sin\theta, \\ \kappa_3 &= \theta' + \tau. \end{aligned} \tag{8}$$

its well known that if we have the curvature and the torsion of a space curve as a functions of arc-length parameter, then by integrating Serret-Frenet we can reconstruct the curve up to its position in the space and this is an immediate consequence of the of the fundamental existence theorem for space curves [23]. Similarly, if we have the quasi curvatures κ_1, κ_2 and κ_3, then we can reconstruct the curve in the space via the integration of quasi Serret-Frenet Equation (9).

3. Evolution of a Space Curve with Time by Quasi Frame

In this section we study the evolution of a regular space curve using quasi frame. We derive time evolution equation for quasi frame and quasi curvatures. The variations of quasi Serret-Frenet with respect to s and t are similar to [24] [25]

$$\frac{\partial}{\partial s}\begin{pmatrix} \boldsymbol{T}_q \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T}_q \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{pmatrix}, \tag{9}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{T}_q \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{pmatrix} = \begin{pmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & \nu \\ -\mu & -\nu & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T}_q \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{pmatrix}. \tag{10}$$

We can write the Serret-Frenet equations and the equation of the evolution in the matrix form as follows. Defining

$$q = \begin{pmatrix} \boldsymbol{T}_q \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & \nu \\ -\mu & -\nu & 0 \end{pmatrix}. \tag{11}$$

The Serret-Frenet equations and the equation of the evolution can be written concisely as

$$\begin{aligned} \frac{\partial q}{\partial s} &= Aq, \\ \frac{\partial q}{\partial t} &= Bq. \end{aligned} \tag{12}$$

Applying the compatibility condition

$$\frac{\partial}{\partial t}\frac{\partial q}{\partial s}=\frac{\partial}{\partial s}\frac{\partial q}{\partial t}, \tag{13}$$

a short calculation using Equations (9), (10) and (11) leads to

$$\frac{\partial A}{\partial t}-\frac{\partial B}{\partial s}+\left[A,B\right]=0_{3\times 3} \tag{14}$$

where $\left[A,B\right]=AB-BA$ is called Lie bracket of *A* and *B*, using Equation (11) leads to

$$\begin{pmatrix} 0 & \left(\frac{\partial \kappa_1}{\partial t}-\nu\kappa_2+\mu\kappa_3-\frac{\partial \lambda}{\partial s}\right) & \left(\frac{\partial \kappa_2}{\partial t}+\nu\kappa_1-\lambda\kappa_3+\frac{\partial \mu}{\partial s}\right) \\ -\left(\frac{\partial \kappa_1}{\partial t}-\nu\kappa_2+\mu\kappa_3-\frac{\partial \lambda}{\partial s}\right) & 0 & \left(\frac{\partial \kappa_3}{\partial t}-\mu\kappa_1+\lambda\kappa_2-\frac{\partial \nu}{\partial s}\right) \\ -\left(\frac{\partial \kappa_2}{\partial t}+\nu\kappa_1-\lambda\kappa_3+\frac{\partial \mu}{\partial s}\right) & -\left(\frac{\partial \kappa_3}{\partial t}-\mu\kappa_1+\lambda\kappa_2-\frac{\partial \nu}{\partial s}\right) & 0 \end{pmatrix}=0_{3\times 3}. \tag{15}$$

Thus the compatibility conditions becomes

$$\begin{aligned} \frac{\partial \kappa_1}{\partial t}&=\nu\kappa_2-\mu\kappa_3+\frac{\partial \lambda}{\partial s}, \\ \frac{\partial \kappa_2}{\partial t}&=\lambda\kappa_3-\nu\kappa_1-\frac{\partial \mu}{\partial s}, \\ \frac{\partial \kappa_3}{\partial t}&=\mu\kappa_1-\lambda\kappa_2+\frac{\partial \nu}{\partial s}. \end{aligned} \tag{16}$$

The set of Equation (16) is the main result of this paper they give a complete description of the motion of curves via quasi frame. These equations represent evolution equations for quasi curvatures of the evolving curve. For a given (λ,μ,ν) we can integrant 16 to get the $(\kappa_1,\kappa_2,\kappa_3)$ and by integrating 9 we can get the evolving curve in space.

4. Evolution of Ruled Surfaces Depends on Their Directrix by Quasi Frame of a Space Curve

We provide a general scheme for studying evolution of ruled surfaces using an approach different from the one proposed by [13] [15]. We apply our method by using quasi frame along a space curve. Evolutions of ruled surfaces generated by the quasi normal and quasi binormal vector fields of space curve are presented. These evolutions of the ruled surfaces depend on the evolutions of their directrix using quasi frame along a space curve.

4.1. Evolution of Quasi Normal Ruled Surface

The equation of surfaces generated by quasi normal is [26]

$$\boldsymbol{\psi}(u,v,t)=\boldsymbol{r}(u,t)+v\boldsymbol{N}_q(u,t). \tag{17}$$

The tangent space to the surface $\boldsymbol{\psi}$ is,

$$\begin{aligned} \boldsymbol{\psi}_u&=(1-v\kappa_1)\boldsymbol{T}+v\kappa_3\boldsymbol{B}_q, \\ \boldsymbol{\psi}_v&=\boldsymbol{N}_q, \end{aligned} \tag{18}$$

the subscripts s and u stand for partial derivatives.

The normal to ψ is,

$$N_{\psi}=\frac{\psi_s\wedge\psi_t}{|\psi_t\wedge\psi_t|}=\frac{-v\kappa_3 T+(1-v\kappa_1)B_q}{\sqrt{1-2v\kappa_1+v^2\kappa_1^2+v^2\kappa_3^2}}. \tag{19}$$

The second derivative is calculated and given by

$$\begin{aligned}\psi_{uu}&=-(v\kappa_2\kappa_3+v\kappa_{1u})T+(\kappa_1-v\kappa_1^2-v\kappa_3^2)+N_q+(\kappa_2-v\kappa_1\kappa_2+v\kappa_{3u})B_q,\\ \psi_{uv}&=-\kappa_1 T+\kappa_3 B_q,\\ \psi_{vv}&=0.\end{aligned} \tag{20}$$

where $\kappa_{iu}=\frac{\partial\kappa i}{\partial u}$ and $\kappa_{it}=\frac{\partial\kappa i}{\partial t}, i=1,2,3.$

If we compute components of the first fundamental form, we have

$$\begin{aligned}g_{11}&=1-2v\kappa_1+v^2\kappa_1^2+v^2\kappa_3^2,\\ g_{12}&=0,\\ g_{22}&=1.\end{aligned} \tag{21}$$

The fundamental metric is

$$g=1-2v\kappa_1+v^2\kappa_1^2+v^2\kappa_3^2. \tag{22}$$

If we compute components of the second fundamental form, we have

$$\begin{aligned}l_{11}&=\frac{\kappa_2-2v\kappa_1\kappa_2+v^2\kappa_1^2\kappa_2+v^2\kappa_2\kappa_3^2+v^2\kappa_3\kappa_{1u}+v\kappa_{3u}-v^2\kappa_1\kappa_{3t}}{\sqrt{1-2v\kappa_1+v^2\kappa_1^2++v^2\kappa_3^2}},\\ l_{12}&=\frac{\kappa_3}{\sqrt{1-2v\kappa_1+v^2\kappa_1^2++v^2\kappa_3^2}},\\ l_{22}&=0.\end{aligned} \tag{23}$$

The Gaussian curvature K and the mean curvature H are calculated and given by,

$$\begin{aligned}K&=\frac{l_{11}l_{22}-l_{12}^2}{g_{11}g_{22}-g_{12}^2}=-\frac{\kappa_3^2}{(-1+v\kappa_1^2+v^2\kappa_3^2)^2},\\ H&=\frac{l_{11}g_{22}-2l_{12}g_{12}+l_{22}g_{11}}{2(g_{11}g_{22}-g_{12}^2)}=\frac{\kappa_2(-1+v\kappa_1^2+v^2\kappa_3^2)^2+v(v\kappa_3\kappa_{1u}+\kappa_{3t}(1-v\kappa_1))}{2(-1+v\kappa_1^2+v^2\kappa_3^2)^{3/2}}.\end{aligned} \tag{24}$$

4.2. Evolution of Quasi Binormal Ruled Surface

The equation of surfaces generated by quasi normal is

$$\psi(u,v,t)=r(u,t)+vB_q(u,t). \tag{25}$$

The tangent space to the surface ψ is,

$$\begin{aligned}\psi_u&=(1-v\kappa_2)T-v\kappa_3 N_q,\\ \psi_v&=B_q.\end{aligned} \tag{26}$$

the subscripts s and u stand for partial derivatives.

The normal to ψ is,

$$\boldsymbol{N}_{\psi}=\frac{\boldsymbol{\psi}_{s}\wedge\boldsymbol{\psi}_{t}}{\left|\boldsymbol{\psi}_{t}\wedge\boldsymbol{\psi}_{t}\right|}=\frac{-v\kappa_{3}\boldsymbol{T}+\left(-1+v\kappa_{2}\right)\boldsymbol{N}_{q}}{\sqrt{1-2v\kappa_{2}+v^{2}\kappa_{2}^{2}++v^{2}\kappa_{3}^{2}}}. \tag{27}$$

The second derivative is calculated and given by

$$\begin{aligned}
\psi_{uu}&=-\left(v\kappa_{1}\kappa_{3}-v\kappa_{2u}\right)\boldsymbol{T}+\left(\kappa_{1}-v\kappa_{1}\kappa_{2}-v\kappa_{3u}\right)\boldsymbol{N}_{q}+\left(\kappa_{2}-v\kappa_{2}^{2}-v\kappa_{3}^{2}\right)\boldsymbol{B}_{q}\\
\psi_{uv}&=-\kappa_{2}\boldsymbol{T}-\kappa_{3}\boldsymbol{N}_{q}\\
\psi_{vv}&=0
\end{aligned} \tag{28}$$

If we compute components of the first fundamental form, we have

$$\begin{aligned}
g_{11}&=1-2v\kappa_{2}+v^{2}\kappa_{2}^{2}+v^{2}\kappa_{3}^{2}\\
g_{12}&=0\\
g_{22}&=1
\end{aligned} \tag{29}$$

If we compute components of the second fundamental form, we have

$$\begin{aligned}
l_{11}&=\frac{-\kappa_{1}+2v\kappa_{1}\kappa_{2}-v^{2}\kappa_{1}\kappa_{2}^{2}-v^{2}\kappa_{1}\kappa_{3}^{2}+v^{2}\kappa_{3}\kappa_{2u}+v\kappa_{3u}-v^{2}\kappa_{2}\kappa_{3u}}{\sqrt{1-2v\kappa_{2}+v^{2}\kappa_{2}^{2}+v^{2}\kappa_{3}^{2}}1-2v\kappa_{2}+v^{2}\kappa_{2}^{2}++v^{2}\kappa_{3}^{2}}\\
l_{12}&=\frac{\kappa_{3}}{\sqrt{1-2v\kappa_{1}+v^{2}\kappa_{1}^{2}++v^{2}\kappa_{3}^{2}}1-2v\kappa_{2}+v^{2}\kappa_{2}^{2}++v^{2}\kappa_{3}^{2}}\\
l_{22}&=0
\end{aligned} \tag{30}$$

The Gaussian curvature K and the mean curvature H are calculated and given by,

$$\begin{aligned}
K&=\frac{l_{11}l_{22}-l_{12}^{2}}{g_{11}g_{22}-g_{12}^{2}}=-\frac{\kappa_{3}^{2}}{\left(-1+v\kappa_{2}^{2}+v^{2}\kappa_{3}^{2}\right)^{2}},\\
H&=\frac{l_{11}g_{22}-2l_{12}g_{12}+l_{22}g_{11}}{2\left(g_{11}g_{22}-g_{12}^{2}\right)}\\
&=\frac{-\kappa_{1}+2v\kappa_{1}\kappa_{2}-v^{2}\kappa_{1}\kappa_{2}^{2}-v^{2}\kappa_{1}\kappa_{3}^{2}+v^{2}\kappa_{3}\kappa_{2u}+v\kappa_{3u}-v^{2}\kappa_{2}\kappa_{3u}}{2\left(1-2v\kappa_{2}+v^{2}\kappa_{2}^{2}+v^{2}\kappa_{3}^{2}\right)^{3/2}}.
\end{aligned} \tag{31}$$

5. Conclusion

In this paper, evolutions of ruled surfaces generated by the quasi normal and quasi binormal vector fields of space curve are presented. These evolutions of the ruled surfaces depend on the evolutions of their directrix using quasi frame along a space curve.

Acknowledgements

The authors would like to thank the referees for their helpful comments and suggestions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Chirikjian, G. and Burdick, J. (1994) A Modal Approach to Hyper-Redundant Manipulator Kinematics. *IEEE Transactions on Robotics and Automation*, **10**, 343-354. https://doi.org/10.1109/70.294209

[2] Brockett, R. (1991) Notes on Motion Control. Harvard University ES258 Class Notes. Harvard University, Cambridge.

[3] Jayender, J. and Vosburgh, K.G. (2010) New Kinematic Metric for Quantifying Surgical Skill for Flexible Instrument Manipulation. *Information Processing in Computer-Assisted Interventions*, Springer, Berlin.

[4] Shahab, K. and Uwe, Z. (2005) Control of Contour Formations of Autonomous Vehicles by General Curve Evolution Theory. *Proceedings of the* 2005 *Australasian Conference on Robotics and Automation*, Sydney, 5-7 December 2005, 1-8. http://hdl.handle.net/1885/76694

[5] Da Rios, L.S. (1906) Sul moto d'un liquido indefinito con un filetto vorticoso. *Gordon and Breach*, **22**, 117-135.

[6] Galdi, G.P. (1971) Motion of Distorted Vortex Rings. *Journal of the Physical Society of Japan*, **31**, 591-599. https://doi.org/10.1143/JPSJ.31.591

[7] Lamb Jr., G.L. (1977) Solitons on Moving Space Curves. *Journal of Mathematical Physics*, **18**, 1654-1661. https://doi.org/10.1063/1.523453

[8] Lakshmanan, M., Ruijgrok, Th.W. and Thompson, C.J. (1976) On the Dynamics of a Continuum Spin System. *Physica A*, **577**, 577-590. https://doi.org/10.1016/0378-4371(76)90106-0

[9] Doliwa, A. and Santini, P. (1994) An Elementary Geometric Characterisation of the Integrable Motions of a Curve. *Physics Letters A*, **185**, 373-384. https://doi.org/10.1016/0375-9601(94)90170-8

[10] Abd-Ellah, H.N. (2015) Evolution of Translation Surfaces in Euclidean 3-Space. *Applied Mathematics & Information Sciences*, **9**, 661-668.

[11] Hussien, R.A. and Mohamed, S.G. (2016) Generated Surfaces via Inextensible Flows of Curves in R^3. *Journal of Applied Mathematics*, **2016**, Article ID: 6178961. https://doi.org/10.1155/2016/6178961

[12] Kwon, D.Y. and Park, F.C. (1999) Evolution of Inelastic Plane Curves. *Applied Mathematics Letters*, **12**, 115-119. https://doi.org/10.1016/S0893-9659(99)00088-9

[13] Kwon, D.Y. and Park, F.C. (2005) Inextensible Flows of Curves and Developable Surfaces. *Applied Mathematics Letters*, **18**, 1156-1162. https://doi.org/10.1016/j.aml.2005.02.004

[14] Latifi, D. and Razavi, A. (2008) Inextensible Flows of Curves in Minkowskian Space. *Advanced Studies in Theoretical Physics*, **2**, 761-768.

[15] Körpinar, T., Altay, G. and Turhan, E. (2011) New Inextensible Flows of Tangent Developable Surfaces in Euclidian 3-Space E^3. *Revista Notas de Matemática*, **7**, 172-176.

[16] Mukherjee, R. and Balakrishnan, R. (2008) Moving Curves of the Sine-Gordon Equation: New Links. *Physics Letters A*, **372**, 6347-6362. https://doi.org/10.1016/j.physleta.2008.08.070

[17] Eisenhart, L.P. (1909) A Treatise on the Differential Geometry of Curves and Surfaces. Dover Pub. Inc., New York.

[18] Bishop, R.L. (1975) There Is More than One Way to Frame a Curve. *The American Mathematical Monthly*, **82**, 246-251.

https://doi.org/10.1080/00029890.1975.11993807

[19] Ylmaz, S. and Turgut, M. (2010) A New Version of Bishop Frame and an Application to Spherical Images. *Journal of Mathematical Analysis and Applications*, **371**, 764-776.

[20] Bloomenthal, J. (1990) Calculation of Reference Frames along a Space Curve. In: Glassner, A., *Graphics Gems*, Academic Press, Cambridge, 567-571. https://doi.org/10.1016/B978-0-08-050753-8.50124-8

[21] Baur, O. and Grafarend, E.W. (2005) Orbital Rotations of a Satellite. Case Study: GOCE. *Artificial Satellites*, **40**, 87-107.

[22] Dede, M., Ekici, C. and Tozak, H. (2015) Directional Tubular Surfaces. *International Journal of Algebra*, **9**, 527-535. https://doi.org/10.12988/ija.2015.51274

[23] Lipschutz, M.M. (1969) Theory and Problems of Differential Geometry. Schaum's Outline Series.

[24] VassiliouIan, P.J. and Lisle, G. (2000) Geometric Approaches to Differential Equations. Cambridge University Press, Cambridge.

[25] Rogers, C. and Schief, W.K. (2002) Backlund and Darboux Transformations Geometry and Modern Application in Soliton Theory. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511606359

[26] Hussien, R.A. and Youssef, T. (2016) Evolution of Special Ruled Surfaces via the Evolution of Their Directrices in Euclidean 3-Space E^3. *Applied Mathematics & Information Sciences*, **10**, 1949-1956. https://doi.org/10.18576/amis/100536