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Response Rate of a Static Charge outside a High-Dimensional Schwarzschild Black Hole

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Abstract

In present paper, by using the quantization of an electromagnetic field in the background of static spherically symmetric d-dimensional spacetime in the Boulware vacuum, we calculated the response rate of a static charge outside d-dimensional Schwarzschild black hole in the low-frequency regime, which can be expressed as the summation of hypergeometric functions.

Keywords

Response Rate, Schwarzschild Black Hole, High-Dimensional Spacetime

1. Introduction

At present it is believed that the study of quantum field theory in curved spacetime can provide some insights into quantum gravity effects, while the full theory is not available. An important prediction in this field is the thermal evaporation of black hole [1]. This nontrivial effect was soon realized to be closely associated with the existence of an event horizon in Schwarzschild spacetime. One of the difficulties in studying fields in Schwarzschild [2] and other black hole spacetime, even when the fields are non-interacting, stems from the fact that the solutions to the field equations are functions whose properties are not well known. In the low-frequency regime, however, the situation is much simpler and the mode functions of the massless scalar field are well known [3]. Recently, Crispino *et al.* [4] suggested a scheme to quantize the free quantum electrodynamics in static spherically symmetric d-dimensional spacetime and gave out the response rate of a static charge outside the four-dimensional Schwarzschild black hole. The response rate is a quantum concept with no natural analog in classical physics though it is possible to define a corresponding classical quantity mathematically and represents the number of times the source responds to the field per unit time.

In this paper we reviewed the free quantum electrodynamics in static spherically symmetric spacetime of arbitrary dimensions in a modified Feynman gauge [4]. Using the physical modes functions, we calculate the response rate of a static charge outside the d-dimensional Schwarzschild black hole in the Unruh vacuum [5]. Limited to four-dimensional Schwarzschild black hole, the response rate is consistent with the result in Ref. [4].

The paper is organized as follows. In Section 2 we review basic concepts of the electromagnetic field in the arbitrary dimensional spacetime of a spherically symmetric black hole in a modified Feynann gauge. Section 3 devotes to calculate the response rate of a static charge outside a Schwarzschild black hole of arbitrary-dimensions respectively. In Section 4, we summarize the main results.

2. Gupta-Bleuler Quantization in a Modified Feynman Gauge

In this section, we follow the notation of Ref. [4] to study the solutions of field equations for electromagnetic field in an asymptotic flat and static spherically symmetric $(p+2)$-dimensional spacetime. The quantization of electromagnetic field will be carried out in the frame of Gupta-Bleuler formalism in a modified Feynmann gauge.

The line element under considered takes the form

$$\mathrm{d}\tau^2 = f(r)\mathrm{d}t^2 - h(r)\mathrm{d}r^2 - r^2\mathrm{d}s_p^2 \tag{1}$$

with the line element of a unit p-sphere $\mathrm{d}s_p^2$. We assume that both $f(r)$ and $h(r)^{-1}$ have a zero at $r = r_h$ and positive for $r > r_h$.

The Lagrangian density for electromagnetic field in a modified Feynman gauge is

$$\mathcal{L}_F = \sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}G^2\right], \tag{2}$$

and G stands for the modified Feynman gauge

$$G = \nabla^\mu A_\mu + K^\mu A_\mu. \tag{3}$$

here the vector K^μ is independent on electromagnetic field A_μ, and takes the form

$$K^\mu = \left(0, f'/(fh), 0, 0\right). \tag{4}$$

Under this choice, the gauge condition changes into

$$G = \frac{1}{f}\partial_t A_t - \sqrt{\frac{f}{h}}\frac{1}{r^p}\partial_r\left[\frac{r^p}{\sqrt{fh}}A_r\right] - \frac{1}{r^2}\nabla^i A_i \tag{5}$$

From the Lagrangian density for electromagnetic field, the equations of motion are

$$\begin{cases} -\frac{1}{f}\partial_t^2 A_t + \sqrt{\frac{f}{h}}\frac{1}{r^p}\partial_r\left[\frac{r_p}{\sqrt{fh}}\partial_r A_r\right] + \frac{1}{r^2}\nabla^2 A_t = 0 & \text{(6a)} \\ -\frac{1}{f}\partial_t^2 A_r + \frac{1}{f}\partial_r\left[\sqrt{\frac{f}{h}}\frac{f}{r^p}\partial_r\left(\frac{r_p}{\sqrt{fh}}A_r\right)\right] + \frac{1}{r^2}\nabla^2 A_r + \frac{1}{f}\partial_r\left(\frac{f}{r^2}\right)\nabla^i A_i = 0 & \text{(6b)} \\ -\frac{1}{f}\partial_t^2 A_i + \frac{r^{2-p}}{\sqrt{fh}}\partial_r\left(\sqrt{\frac{f}{h}}r^{p-2}\partial_r A_i\right) - \frac{r^2}{fh}\partial_r\left(\frac{f}{r^2}\right)\partial_i A_r & \\ \quad + \frac{1}{r^2}\left[\nabla^j\left(\nabla_j A_i - \nabla_i A_j\right) + \partial_i\left(\nabla^j A_j\right)\right] = 0, \quad (j = 1, \cdots, p) & \text{(6c)} \end{cases}$$

here ∇_i is the covariant derivative on S^p.

We denote the complete set of solutions of Equation (6) by $A_\mu^{(\lambda n;\omega lm)}$, and call a non-physical modes for $\lambda = 0$, physical modes for $\lambda = 1,2,3,\cdots,p$ and a pure-gauge mode for $\lambda = p+1$. The label n represents modes incoming from the past null infinity $(n = \leftarrow)$ and coming out from the past horizon $(n = \rightarrow)$.

Physical Modes 1

For other independent solutions $\lambda = 1,2,\cdots,p$, which represent physical degrees of freedom, the time-component can be taken as zero. They are the linear independent solution of Equation (6) with a gauge condition $G = 0$, and are classified into two types.

The "physical modes 1" solution can be written as [4]

$$\begin{cases} A_t^{(1n;\omega lm)} = 0 & \text{(7a)} \\ A_r^{(1n;\omega lm)} = R_{\omega l}^{(1n)}(r) Y_{lm} \mathrm{e}^{-i\omega t}, (l \geq 1) & \text{(7b)} \\ A_i^{(1n;\omega lm)} = \dfrac{r^{2-p}}{l(l+p-1)} \sqrt{\dfrac{f}{h}} \dfrac{\mathrm{d}}{\mathrm{d}r} \left[\dfrac{r^p}{\sqrt{fh}} R_{\omega l}^{(1n)}(r) \right] \partial_i Y_{lm} \mathrm{e}^{-i\omega t} & \text{(7c)} \end{cases}$$

where $i = 1,2,3,\cdots,p$ and:

$$\left[\frac{\omega^2}{f} - \frac{l(l+p-1)}{r^2} \right] R_{\omega l}^{(1n)}(r) + \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left[\sqrt{\frac{f}{h}} r^{2-p} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{r^p}{\sqrt{fh}} R_{\omega l}^{(1n)}(r) \right) \right] = 0 \tag{8}$$

3. Response Rate of a Static Charge outside a D-Dimensional Schwarzschild Black Hole

In this section, we will calculate the response rate of a static charge outside a d-dimensional $(d = p+2)$ Schwarzschild black hole by following the procedure of Ref. [4]. In this case, the black hole is characterized by $f(r) = h(r)^{-1} = 1 - (r_h/r)^{(p-1)}$.

In order to avoid the indefinite results [6] [7], we use the formula suggested by Crispino *et al.* [8] and assume the static charge located at (r_0, θ_0) with a current density j^μ

$$\begin{cases} j^\mu = (j^t, j^r, 0, \cdots, 0) & \text{(9a)} \\ j^t = \dfrac{\sqrt{2} q \cos Et}{\sqrt{-g}} \delta(r - r_0) \delta(\theta_1 - \theta_{10}) \cdots\cdots \delta(\theta_p - \theta_{p0}) & \text{(9b)} \\ j^r = \dfrac{\sqrt{2} qE \sin Et}{\sqrt{-g}} \Theta(r - r_0) \delta(\theta_1 - \theta_{10}) \cdots\cdots \delta(\theta_p - \theta_{p0}) & \text{(9c)} \end{cases}$$

The step function $\Theta(x)$ is defined by $\Theta(x) = 1, (x > 0)$ and vanishing for $x \leq 0$.

Such current interacts with vector potential A_μ through the Lagrangian $\sqrt{-g} j^\mu A_\mu$. Since $a_{\omega lm}^{((p+1)n)\dagger} |\mathrm{phys}\rangle$ is non-physical states, which excludes the interaction with the pure-gauge particles created by $a_{\omega lm}^{((p+1)n)\dagger}$. We will neglect it. However, the current does interact with the states created by $a_{\omega lm}^{(0n)\dagger}$ but the contribution to physical probabilities maybe taken as zero once the non-physical modes are appropriately chosen. Ref [4] gave the exact form of physical modes 2, we can see that $A_t = A_r = 0$. Therefore, there is no interacting term between the current and physical modes 2, we only need to consider the physical modes 1. To make the counting process more concise, we will limit in the spherical coulomb gauge. The mode function can be written as [4]

$$A_t^{1'n;\omega lm} = \frac{i\omega r^{2-p}}{l(l+p-1)}\sqrt{\frac{f}{h}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^p}{\sqrt{fh}}R_{\omega l}^{1n}\right)Y_{lm}\mathrm{e}^{-i\omega t} \tag{10}$$

$$A_r^{1'n;\omega lm} = \frac{\omega^2 r^2}{l(l+p-1)}\frac{1}{f}R_{\omega l}^{1n}Y_{lm}\mathrm{e}^{-i\omega t}. \tag{11}$$

The Bose-Einstein distribution of the thermal photons coming out of the horizon diverges at zero energy. Then we take the limit $E \to 0$, the proper response rate of the charge can be written [4]

$$\frac{R_{0lm}}{\sqrt{f(r_0)}} = 4\pi \lim_{E\to 0}\frac{\left|\mathcal{T}_{Elm}^{\to}\right|^2}{\sqrt{f(r_0)}\beta E} \tag{12}$$

where $\mathcal{T}_{Elm}^{\to}$ is the index $n = \to$ in transition amplitude $\mathcal{T}_{\omega lm}^{n}$ which has the form

$$\mathcal{T}_{\omega lm}^{n} \equiv \frac{1}{2\pi\delta(\omega - E)}\int \mathrm{d}^{p+2}x\sqrt{-g}\,j^{\mu}\left\langle 1n;\omega lm\right|\hat{A}_{\mu}\left|0\right\rangle \tag{13}$$

Now, we return to calculate $\mathcal{T}_{Elm}^{\to}$. The Equation (6) can be written

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[fr^{2-p}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^p R_{\omega l}^{(1\to)}(r)\right)\right] - \frac{l(l+p-1)}{r^2}R_{\omega l}^{(1\to)}(r) + \frac{\omega^2}{f}R_{\omega l}^{(1\to)}(r) = 0. \tag{14}$$

After introducing the Wheeler tortoise coordinate [4] and a function φ

$$R_{\omega l}^{(1\to)}(r) \equiv \frac{\sqrt{l(l+p-1)}}{\omega}r^{-\frac{p}{2}-1}\varphi_{\omega l}^{(1\to)}(r), \tag{15}$$

the Equation (14) changes into

$$\left(\omega^2 + \frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} - V_1(r^*)\right)\varphi_{\omega l}^{(1\to)}(r) = 0 \tag{16}$$

with

$$V_1\left[r^*(r)\right] = f\frac{l(l+p-1)}{r^2} + f^2\frac{p(p-2)}{4r^2} - ff'\frac{(p-2)}{2r} \tag{17}$$

For the small ω and the condition $\left(r - r_h \ll \omega^2 r_h^3, \left|\omega r^*\right| \ll 1\right)$, the wave coming from the past horizon H^- is almost completely reflected back by the potential toward the horizon

$$\varphi_{\omega l}^{(1\to)} \approx -2\omega r^* + \mathrm{const} \tag{18}$$

For a second order nonlinear Equation (16), it is hard to find the analytic expression for the Wheeler tortoise coordinate r^*. Fortunately, what we need is just the behavior of r^* near horizon. Substituting the expression of $f(r)$, the leading term of the Wheeler tortoise coordinate can be written

$$r^* \approx \frac{r_h}{p-1}\ln(z-1),(z\sim 1) \tag{19}$$

by using the transition

$$z = 2\left(\frac{r}{r_h}\right)^{p-1} - 1 \tag{20}$$

Thus, the boundary condition of $R_{\omega l}^{(1\to)}$ reads

$$R_{\omega l}^{(1\to)} \approx -\frac{2\sqrt{l(l+p-1)}}{p-1} 2^{\frac{1}{p-1}} M^{-\frac{p}{2(p-1)}} \ln(z-1),\left(r-r_h \ll \omega^2 r_h^3, \left|\omega r^*\right| \ll 1\right), \tag{21}$$

which is independent on ω. In terms of variable *z*, the Equation (14) can be written as

$$\begin{aligned}&\left(z^2-1\right)\frac{\mathrm{d}^2 R_{\omega l}^{(1\to)}}{\mathrm{d}z^2} + 2\left[\frac{p}{p-1}(z-1)+1\right]\frac{\mathrm{d}R_{\omega l}^{(1\to)}}{\mathrm{d}z}\\ &-\frac{1}{(p-1)^2}\left[l(l+p-1) - \frac{2p(p-2)}{z+1} - p - \omega^2\frac{z+1}{z-1}\left(M(z+1)\right)^{\frac{2}{p-1}}\right]R_{\omega l}^{(1\to)} = 0.\end{aligned} \tag{22}$$

This equation cannot be analytically solved, but the boundary condition implies that the main contribution is from the ω-independent term. Combining the asymptotic behavior $R_{\omega l}^{(1\to)} \to 0$ as $z\to+\infty$, we find the solution of Equation (22) in small ω limit

$$\begin{aligned}R_{\omega l}^{(1\to)} = &\frac{2\sqrt{l(l+p-1)}}{p-1} 2^{\frac{1}{p-1}} M^{-\frac{p}{2(p-1)}}\\ &\times \mathrm{F}\left(\left[\frac{l}{p-1},\frac{l}{p-1}+2\right],\left[\frac{2l}{p-1}+2\right],\frac{2}{z+1}\right)(z+1)^{\frac{-l-p}{p-1}}\end{aligned} \tag{23}$$

where $\mathrm{F}(\alpha,\beta,\gamma,x)$ is the hypergeometric function and the coefficient has been appropriately chosen to agreement with the boundary condition Equation (21).

From gauge invariance we can see that it is convenient to calculate the amplitude Equation (13) in the spherical Coulomb gauge. Then the contribution from the *r*-component will be suppressed in the low energy limit due to extra factors of ω. So we only need to consider the *t*-component in this limit. By using the definition of Equation (12), we can get

$$\frac{R_{0lm}}{\sqrt{f(r_0)}} = \frac{2^{\frac{2-p}{p-1}}(p-1)q^2}{M\pi l(l+p-1)\sqrt{f(r_0)}}\left(\frac{z_0-1}{z_0+1}\right)^2\left[\frac{\mathrm{d}}{\mathrm{d}z_0}\left(\mathrm{F}_l(z_0)(z_0+1)^{\frac{-l}{p-1}}\right)\right]^2 \left|Y_{lm}\right|^2 \tag{24}$$

Substituting the all parameters for *d*-dimensional black hole

$$z_0 = \frac{r_0^{p-1}}{M} - 1 \tag{25}$$

$$f(r_0) = 1 - \frac{2M}{r_0^{p-1}} \tag{26}$$

$$\mathrm{F}_l\left(z_0\right)=\mathrm{F}\left(\left[\frac{l}{p-1},\frac{l}{p-1}+2\right],\left[\frac{2l}{p-1}+2\right],\frac{2}{z_0+1}\right) \tag{27}$$

the total transition probability per proper time of the charge is given by

$$\begin{aligned} R^{\mathrm{tot}} &= \sum_l \sum_{m=-1}^{l} \frac{R_{0lm}}{\sqrt{f\left(r_0\right)}} \\ &= \sum_l \frac{2^{\frac{2-p}{p-1}}\left(p-1\right)q^2}{M\pi l\left(l+p-1\right)\sqrt{f\left(r_0\right)}}\left(\frac{z_0-1}{z_0+1}\right)^2\left[\frac{\mathrm{d}}{\mathrm{d}z_0}\left(\mathrm{F}_l\left(z_0\right)\left(z_0+1\right)^{\frac{-l}{p-1}}\right)\right]^2\frac{G\left(l\right)}{\Omega_p} \end{aligned} \tag{28}$$

in which the following formula has been used [9]

$$\sum_{m=-1}^{l}\left|Y_{lm}\right|^2=\frac{G\left(l\right)}{\Omega_p} \tag{29}$$

where Ω_p is the volume of S^p and $G\left(l\right)$ is the degeneracy of the eigenvalue $-l\left(l+p-2\right)$ of the Laplacian $\tilde{\Delta}$, which is given by

$$G\left(l\right)=\frac{\left(2l+p-1\right)\left(l+p-2\right)!}{l!\left(p-1\right)!}. \tag{30}$$

when the dimension reduced to four ($p=2$), Equation (23) can be written

$$R_{\omega l}^{(1\to)}=\frac{4}{M}\sqrt{l\left(l+1\right)}\mathrm{F}\left(\left[l,l+2\right],\left[2l+2\right],\frac{2}{z+1}\right)\left(z+1\right)^{-l-2}, \tag{31}$$

and Equation (28) will reduce to a Legendre function of the second kind, which recovers the result in [4].

4. Conclusion

In this paper we compute the total response rate of a static charge outside the d-dimensional Schwarzschild black hole in the Unruh vacuum, by using the free electrodynamics in static spherically symmetric spacetime of arbitrary dimensions in a modified Feynman gauge. This outcome may provide us a chance for further investigating quantum field theory in high-dimensional curved spacetime. For instance, some authors [8] [10] [11] have researched whether or not a quantum version of the equivalence principle could be formulated and show some equivalence for low-frequency quantum phenomena in flat and curved spacetime. The same problem could be reconsidered in high-dimensional spacetime and discuss the dimension dependence of the results.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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