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Lyapunov-Type Inequalities for Conformable BVP

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Abstract

In this paper, we present Lyapunov-type inequality for conformable BVP

$$T_\alpha^a y(t) + q(t) y(t) = 0$$

with the conformable fractional derivative of order $1 < \alpha \le 2$ and $2 < \alpha \le 3$ with corresponding boundary conditions. We obtain the Lyapunov-type inequality by a construction Green's function and get its corresponding maximum value. Application to the corresponding eigenvalue problem is also discussed.

Keywords

Lyapunov-Type Inequalities, Conformable Fractional Derivative, Green's Function, Eigenvalue

1. Introduction

Lyapunov-type inequality is an important and useful tool for studying differential equations. The classical Lyapunov-type inequality for differential equations was studied in [1]:

$$\begin{cases} y''(t) = -q(t) y(t), & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.1}$$

if (1.1) has a nontrivial solution, then

$$\int_a^b |q(s)| \mathrm{d}s > \frac{4}{b-a}. \tag{1.2}$$

Furthermore, the constant 4 in (1.2) is sharp.

More authors paid attention to study Lyapunov-type inequality for differential equations and got many results. In recent years, a series of achievements have been made in the Lyapunov-type inequalities of fractional differential equations. We refer to [2]-[12]. In [3], Ferreira studied the following equations:

$$\begin{cases} {}_a^C D^\alpha y(t) + q(t) y(t) = 0, \quad a < t < b, 1 < \alpha \le 2 \\ y(a) = 0 = y(b), \end{cases} \tag{1.3}$$

if (1.3) has a nontrivial solution, then

$$\int_a^b |q(s)| \mathrm{d}s > \frac{\Gamma(\alpha)\alpha^\alpha}{\left[(\alpha-1)(b-a)\right]^{\alpha-1}}.$$

In [7], Abdeljanad and Baleanu obtained a Lyapunov-type inequality for ABR fractional boundary value problem

$$\begin{cases} \left({}_a^{ABR} D^\alpha y\right)(t) + q(t) y(t) = 0, \quad a < t < b, \ 2 < \alpha \le 3, \\ y(a) = y(b) = 0, \end{cases} \tag{1.4}$$

if (1.4) has a nontrivial solution, then

$$\int_a^b T(s) \mathrm{d}s > \frac{4}{b-a},$$

where

$$T(s) = \left[\frac{3-\alpha}{B(\alpha-2)} |q(t)| + \frac{\alpha-2}{B(\alpha-2)} \left({}_a I^{\alpha-2} |q(s)|\right)(t)\right].$$

In [10], Abdeljawad studied a generalized Lyapunov-type inequalities for conformable BVP

$$\begin{cases} T_\alpha^c x(t) + r(t) x(t) = 0, \quad c < t < d, 1 < \alpha \le 2, \\ x(c) = x(d) = 0, \end{cases} \tag{1.5}$$

if (1.5) has a nontrivial solution, then

$$\int_c^d |r(s)| \mathrm{d}s > \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}(d-c)^{\alpha-1}}.$$

Furthermore, Abdeljawad proved a Lyapunov-type inequalitiy for a sequential conformable BVP

$$\begin{cases} T_\alpha^a \cdot T_\alpha^a x(t) + r(t) x(t) = 0, \quad a < t < b, \dfrac{1}{2} < \alpha \le 1, \\ x(c) = x(d) = 0, \end{cases} \tag{1.6}$$

if (1.6) has a nontrivial solution, then

$$\int_c^d |r(t)| \mathrm{d}s > \frac{3\alpha-1}{(d-c)^{2\alpha-1}} \left(\frac{3\alpha-1}{2\alpha-1}\right)^{\frac{2\alpha-1}{\alpha}}.$$

In this paper, we establish a Lyapunov-type inequalities for conformable BVP

$$\begin{cases} T_\alpha^a y(t) + q(t) y(t) = 0, \quad a < t < b, 1 < \alpha \le 2, \\ y(a) = y'(b) = 0 \end{cases} \tag{1.7}$$

and

$$\begin{cases} T_\alpha^a f(t) + p(t) f(t) = 0, & a < t < b, 2 < \alpha \le 3, \\ f(a) = f'(a) = f'(b) = 0, \end{cases} \tag{1.8}$$

where T_α^a is conformable fractional derivative starting at a of order α, and p, q are real-valued continuous. The introduction and background of conformable fractional are given in [2] [10]. Then, we give the definition and lemma about conformable fractional derivative in the following.

Definition 1.1. [4] Let $n < \alpha \le n+1$. Then

$$\left(I_\alpha^c g\right)(t) = \frac{1}{n!}\int_c^t (t-s)^n (s-c)^{\alpha-n-1} g(s)\,\mathrm{d}s$$

is called the left conformable fractional derivative starting at c of order α.

Lemma 1.1. [4] Let $f:[c,\infty) \to R$ be $(n+1)$ times differentiable for $t > c$, $n < \alpha \le n+1$. Then, we have the following result:

$$\left(I_\alpha^c \cdot T_\alpha^c f\right)(t) = f(t) - \sum_{k=0}^n \frac{f^n(c)(t-c)^k}{k!}.$$

2. A Lyapunov-Type Inequality for Conformable Fractional Derivative of $1 < \alpha \le 2$

Theorem 2.1. $y \in C[a,b]$ is a solution of the BVP (1.7) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t,s) q(s) y(s)\,\mathrm{d}s. \tag{2.1}$$

where $G(t,s)$ is the Green's function defined as

$$G(t,s) = \begin{cases} (t-a)(s-a)^{\alpha-2}, & a \le t \le s \le b, \\ (s-a)^{\alpha-1}, & a \le s \le t \le b. \end{cases} \tag{2.2}$$

Proof. Applying the integral I_a^α in the (1.7), we have

$$I_\alpha^a \cdot T_\alpha^a y(t) = -I_\alpha^a\left(q(t) y(t)\right).$$

Then, using definition 1.1 and lemma 1.1, we obtain

$$y(t) = c_0 + c_1(t-a) - \int_a^t (t-s)(s-a)^{\alpha-2} q(s) y(s)\,\mathrm{d}s. \tag{2.3}$$

Since $y(a) = 0$, we get immediately that $c_0 = 0$.

By the boundary condition $y'(b) = 0$, we obtain

$$c_1 = \int_a^b (s-a)^{\alpha-2} q(s) y(s)\,\mathrm{d}s.$$

Hence, equation (2.3) becomes

$$y(t) = (t-a)\int_a^b (s-a)^{\alpha-2} q(s) y(s)\,\mathrm{d}s - \int_a^t (t-s)(s-a)^{\alpha-2} q(s) y(s)\,\mathrm{d}s. \tag{2.4}$$

Then, equation (2.4) can be written in the form of (2.1), where the Green's function is defined in (2.2).

The proof is completed.

Corollary 2.1. The function G defined in Theorem 2.1 satisfied the following property:

$$\max_{s\in[a,b]} G(t,s) = G(t,t) = (t-a)^{\alpha-1}. \tag{2.5}$$

Proof. We define the function

$$g_1(t,s) = (t-a)(s-a)^{\alpha-2}$$

and

$$g_2(t,s) = (s-a)^{\alpha-1}.$$

For $a \le t \le s \le b$, differentiating $g_1(t,s)$ with respect to *s*, we get

$$g_1'(t,s) = (t-a)(\alpha-2)(s-a)^{\alpha-3} < 0. \tag{2.6}$$

While for $a \le s \le t \le b$, differentiating $g_2(t,s)$ with respect to *s*, we get

$$g_2'(t,s) = (\alpha-1)(s-a)^{\alpha-2} > 0. \tag{2.7}$$

Hence, $g_1(t,s)$ is a decreasing function, $g_2(t,s)$ is an increasing function in *s*. Consequently, G (*t*, *s*) gets the maximum at $s = t$, we obtain (2.5).

Corollary 2.2. If (1.7) has a nontrivial continuous solution, then

$$\int_a^b (t-a)^{\alpha-1} |q(s)| \mathrm{d}s \ge 1. \tag{2.8}$$

Proof. Let $y \in C[a,b]$ be a nontrivial solution of the BVP (1.7), where the norm

$$\|y\| = \sup_{t\in[a,b]} \{|y(t)|\}.$$

Form (2.1), we have

$$\begin{aligned} |y(t)| &\le \int_a^b |G(t,s)||q(s)||y(s)| \mathrm{d}s \\ &\le \int_a^b \max_{s\in[a,b]} G(t,s)|q(s)||y(s)| \mathrm{d}s \\ &\le \int_a^b (t-a)^{\alpha-1} |q(s)||y(s)| \mathrm{d}s. \end{aligned} \tag{2.9}$$

Taking the norm leads to

$$\|y\| \le \left(\int_a^b (t-a)^{\alpha-1} |q(s)| \mathrm{d}s\right)\|y\|.$$

Then,

$$\int_a^b (t-a)^{\alpha-1} |q(s)| \mathrm{d}s \ge 1.$$

This completes the proof.

Corollary 2.3. If the BVP (1.7) has a nontrivial continuous solution, then

$$\int_a^b |q(s)| \mathrm{d}s \ge (b-a)^{1-\alpha}. \tag{2.10}$$

Proof. In (2.8), let

$$f(t) = (t-a)^{\alpha-1}, \quad t \in (a,b).$$

Differentiating $f(t)$ on (a,b), we have

$$f'(t) = (\alpha-1)(t-a)^{\alpha-2} > 0,$$

hence, $f(t)$ is a increasing function, we have

$$\max_{t\in[a,b]} f(t) \le f(b) = (b-a)^{\alpha-1}.$$

Then,

$$\int_a^b (b-a)^{\alpha-1} |q(s)| \mathrm{d}s \ge 1.$$

Hence, we get the inequality (2.10). The proof is complete.

Example 2.1. If the BVP

$$\begin{cases} T_\alpha^a y(t) + \lambda y(t) = 0, \quad 0 < t < 1, 1 < \alpha \le 2, \\ y(0) = y'(1) = 0 \end{cases}$$

has a nontrivial solution, then

$$|\lambda| \ge 1. \tag{2.11}$$

Proof. Assume that λ is an eigenvalue of (1.7). By using Corollary 2.3, we have

$$\int_0^1 |\lambda| \mathrm{d}s = |\lambda| \ge 1.$$

Hence, we get the desired result (2.11). The proof is complete.

3. A Lyapunov-Type Inequality for Conformable Fractional Derivative of $2 < \alpha \le 3$

Theorem 3.1. $f \in C[a,b]$ is a solution of the BVP (1.8) if and only if *f* satisfies the integral equation

$$f(t) = \int_a^b H(t,s) p(s) f(s) \mathrm{d}s. \tag{3.12}$$

where $H(t,s)$ is the Green's function defined as

$$H(t,s) = \begin{cases} \dfrac{(t-a)^2 (b-s)(s-a)^{\alpha-3}}{2(b-a)}, & a \le t \le s \le b, \\ \left[\dfrac{(b-s)(t-a)^2}{2(b-a)} - \dfrac{(t-s)^2}{2}\right](s-a)^{\alpha-3}, & a \le s \le t \le b. \end{cases} \tag{3.13}$$

Proof. Applying the integral I_a^α in the (1.8), we have

$$I_\alpha^a \cdot T_\alpha^a f(t) = -I_\alpha^a (p(t) f(t)).$$

Then, using definition 1.1 and lemma 1.1, we obtain

$$f(t) = a_0 + a_1(t-a) + a_2(t-a)^2 - \frac{1}{2}\int_a^t (t-s)^2 (s-a)^{\alpha-3} p(s) f(s) \mathrm{d}s. \tag{3.14}$$

Since $f(a) = f'(a) = 0$, we get immediately that $a_0 = a_1 = 0$.

By the boundary condition $f'(b) = 0$, we obtain

$$a_2 = \frac{1}{2(b-a)} \int_a^b (b-s)(s-a)^{\alpha-3} p(s) f(s) \mathrm{d}s.$$

Hence, equation (3.14) becomes

$$f(t)=\frac{(t-a)^2}{2(b-a)}\int_a^b (b-s)(s-a)^{\alpha-3} p(s) f(s)\mathrm{d}s \\ -\frac{1}{2}\int_a^t (t-s)^2 (s-a)^{\alpha-3} p(s) f(s)\mathrm{d}s. \tag{3.15}$$

Then equation (3.15) can be written in the form of (3.12), where the Green's function is defined in (3.13). The proof is completed.

Corollary 3.1. The function H defined in Theorem 3.1 satisfied the following property:

$$\max_{t\in[a,b]} H(t,s)=H(b,s)=\frac{(b-a)(b-s)(s-a)^{\alpha-3}}{2},$$

$$\max_{s\in[a,b]} H(b,s)\le\frac{(b-a)(s-a)^{\alpha-3}}{2}.$$

Proof. We define the function

$$h_1(t,s)=\frac{(t-a)^2(b-s)(s-a)^{\alpha-3}}{2(b-a)}$$

and

$$h_2(t,s)=\left[\frac{(b-s)(t-a)^2}{2(b-a)}-\frac{(t-s)^2}{2}\right](s-a)^{\alpha-3}.$$

For $a\le t\le s\le b$, differentiating $h_1(t,s)$ with respect to *t*, we get

$$h_1'(t,s)=\frac{(t-a)(b-s)(s-a)^{\alpha-3}}{b-a}\ge 0. \tag{3.16}$$

Hence, $h_1(t,s)$ is an increasing function in *t*.

While for $a\le s\le t\le b$, differentiating $h_2(t,s)$ with respect to *t*, we get

$$h_2'(t,s)=\left[\frac{(b-s)(t-a)}{b-a}-(t-s)\right](s-a)^{\alpha-3}.$$

Let

$$g(t)=\frac{(b-s)(t-a)}{b-a}-(t-s)=\left(\frac{b-s}{b-a}-1\right)t-\left(\frac{b-s}{b-a}\right)a+s,$$

then, we have

$$g'(t)=\frac{b-s}{b-a}-1<0.$$

Hence,

$$g(t)\ge g(b)=0.$$

That we obtain $h_2(t,s)$ is an increasing function in *t*. Consequently, $H(t,s)$ gets the maximum at $t=b$. We have

$$h_1(b,s)=\frac{(b-a)^2(b-s)(s-a)^{\alpha-3}}{2(b-a)}=\frac{(b-a)(b-s)(s-a)^{\alpha-3}}{2}$$

and

$$h_2(b,s)=\left[\frac{(b-s)(b-a)^2}{2(b-a)}-\frac{(b-s)^2}{2}\right](s-a)^{\alpha-3}=\frac{(b-s)(s-a)^{\alpha-2}}{2}.$$

Hence, $h_1>h_2$, we obtain

$$\max_{t\in[a,b]}H(t,s)=H(b,s)=h_1(b,s)=\frac{(b-a)(b-s)(s-a)^{\alpha-3}}{2}.$$

Furthermore, we have

$$H(b,s)=\frac{(b-a)(b-s)(s-a)^{\alpha-3}}{2}\le\frac{(b-a)^2(s-a)^{\alpha-3}}{2}.$$

Hence,

$$\max_{s\in[a,b]}H(b,s)\le\frac{(b-a)(s-a)^{\alpha-3}}{2}.$$

The proof is completed.

Corollary 3.2. If (1.8) has a nontrivial continuous solution, then

$$\int_a^b(s-a)^{\alpha-3}|p(s)|\,\mathrm{d}s\ge\frac{2}{(b-a)^2}. \tag{3.17}$$

Proof. Let $f\in C[a,b]$ be a nontrivial solution of the BVP (1.8), where the norm

$$\|f\|=\sup_{t\in[a,b]}\{|f(t)|\}.$$

Form (3.1), we have

$$\begin{aligned}|f(t)|&\le\int_a^b|H(t,s)||p(s)||f(s)|\,\mathrm{d}s\\&\le\int_a^b\max_{s\in[a,b]}H(b,s)|p(s)||f(s)|\,\mathrm{d}s\\&\le\int_a^b\frac{(b-a)^2(s-a)^{\alpha-3}}{2}|p(s)||f(s)|\,\mathrm{d}s.\end{aligned} \tag{3.18}$$

Taking the norm leads to

$$\|f\|\le\left(\int_a^b\frac{(b-a)^2(s-a)^{\alpha-3}}{2}|p(s)|\,\mathrm{d}s\right)\|f\|.$$

Then,

$$\int_a^b\frac{(b-a)^2(s-a)^{\alpha-3}}{2}|p(s)|\,\mathrm{d}s\ge1.$$

Hence, we get the inequality in (3.17). This completes the proof.

Example 3.1. If the BVP

$$\begin{cases}T_\alpha^a f(t)+\lambda f(t)=0, & 0<t<1,\ 2<\alpha\le3,\\ f(0)=f'(0)=f'(1)=0\end{cases}$$

has a nontrivial continuous solution, then

$$|\lambda| \ge 2(\alpha - 2). \tag{3.19}$$

Proof. Assume that λ is an eigenvalue of (1.8). By using Corollary 3.2, we have

$$\int_0^1 s^{\alpha-3} |\lambda| \mathrm{d}s \ge 2.$$

Then, we obtain

$$|\lambda| \int_0^1 s^{\alpha-3} \mathrm{d}s = |\lambda| \frac{1}{\alpha - 2} \ge 2.$$

We get the desired result (3.19). The proof is complete.

4. Conclusion

On the base of [10], by changing and increasing the edge value conditions, we establish some new Lyapunov-type inequalities for conformable BVP with the conformable derivative of order $1 < \alpha \le 2$ and $2 < \alpha \le 3$. In Section 2 and Section 3, by Green's function and its corresponding maximum value, we obtain new results about Lyapunov-type inequalities for conformable BVP.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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