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A Generalization of the Clark-Ocone Formula

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Abstract

In this paper, we use a white noise approach to Malliavin calculus to prove the generalization of the Clark-Ocone formula

$$F(\omega)=E[F]+\int_0^T E[D_tF \mid \mathcal{F}_t]\lozenge W(t)\,\mathrm{d}t,$$

where $E[F]$ denotes the generalized expectation, $D_tF(\omega)=\frac{\mathrm{d}F}{\mathrm{d}\omega}$ is the (generalized) Malliavin derivative, $\lozenge$ is the Wick product and $W(t)$ is the 1-dimensional Gaussian white noise.

Keywords

White Noise, Malliavin Calculus, Wick Product, Brownian Motion

1. Introduction

In 1975, Hida introduced the theory of white noise with his lecture note on Brownian functionals [1]. After that H. Holden *et al.* [2] emphasized this theory with stochastic partial differential equations (SPDEs) driven by Brownian motion.

In 1984, Ocone proved the Clark-Ocone formula [3], to give an explicit representation to integral in Itô integral representation theorem in the context of analysis on the Wiener space $\Omega=C_0([0,T])$, the space of all real continuous functions on $[0,T]$ starting at 0. He proved that

$$F(\omega)=E[F]+\int_0^T E[D_tF \mid \mathcal{F}_t]\,\mathrm{d}B(t), \tag{1.1}$$

where D_t is the Malliavin derivative and $B(t)$ is the one dimensional Brownian motion on the Winer space. In [4] the authors proved the generalization of Clark-Ocone formula (see, e.g., [5] [6]). This theorem has many interesting application, for example, computing the replicating portfolio of call option in Black & Scholes type market. They proved that

$$F(\omega)=E[F]+\int_0^T E[D_t F \mid \mathcal{F}_t]\Diamond W(t)\mathrm{d}t, \tag{1.2}$$

where $E[F]$ denotes the generalized expectation, $D_t F(\omega)=\frac{\mathrm{d}F}{\mathrm{d}\omega}$ is the (generalized) Malliavin derivative, $\Diamond$ is the Wick product and $W(t)$ is the one dimensional Gaussian white noise. This formula holds for all $F\in\mathcal{G}^*$, where $\mathcal{G}^*$ is a space of stochastic distribution. In particular, if $F\in L^2(\mu)$ then equation (1.2) turns out to be

$$F(\omega)=E[F]+\int_0^T E[D_t F \mid \mathcal{F}_t]\mathrm{d}B(t).$$

The purpose of this papper is to generalize the well known Clark-Ocone formula to generalized functions of white noise, *i.e.*, to the space $\mathcal{G}^{-\beta}$. The generalization has the following form

$$F(\omega)=E[F]+\int_0^T E[D_t F \mid \mathcal{F}_t]\Diamond W(t)\mathrm{d}t,$$

where $E[F]$ denotes the generalized expectation, $D_t F(\omega)=\frac{\mathrm{d}F}{\mathrm{d}\omega}$ is the (generalized) Malliavin derivative, $\Diamond$ is the Wick product, and $W(t)$ is the 1-dimensional Gaussian white noise.

The paper is organized as follows. In Section 2 and 3, we recall necessary definitions and results from white noise and prove a new results that we will need. Finally in Section 4, we generalize the Clark-Ocone formula, *i.e.*, to the space $\mathcal{G}^{-\beta}$.

2. White Noise

In this section we recall necessary definitions and results from white noise. For more information about white noise analysis (see e.g, [7]-[14]).

Given $\Omega=S(\mathbb{R})$ be the space of tempered distribution on the set $\mathbb{R}$ of real number and let μ be the Gaussian white noise probability measure on Ω such that

$$\int_\Omega \mathrm{e}^{i\langle\omega,\phi\rangle}\mathrm{d}\mu(\omega)=\mathrm{e}^{-\frac{1}{2}\|\phi\|^2}. \tag{2.1}$$

where $\langle\omega,\phi\rangle$ denotes the action of $\omega\in S'(\mathbb{R})$ on ϕ. It follows from (2.1) that

$$E[\langle .,\phi\rangle]=0,\quad E[\langle .,\phi\rangle]^2=\|\phi\|^2, \phi\in S(\mathbb{R})$$

where $E=E_\mu$ denotes the expectation with respect to μ. This isometry allows us to define a Brownian motion $B(t)=B(t,\omega)$ as the continuous version of $\tilde{B}=\tilde{B}(t,\omega)=\langle\omega,\chi_{(0,t)}(.)\rangle$ where

$$\chi_{[0,t]}(s)=\begin{cases}1 & \text{if } 1\le s\le t,\\ -1 & \text{if } -t\le s\le 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then, $\langle\omega,\varphi\rangle=\int_\mathbb{R}\varphi(t)\mathrm{d}B(t)$ for all $\varphi\in L^2(\mathbb{R})$. Let $\mathcal{F}_t$ be the σ algebra generated by $\{B(s,.)\}_{0\le s\le t}$. If $f(t_1,t_2,\cdots,t_n)\in \hat{L}^2(\mathbb{R}^n)$, *i.e.*, f_n is symmetric and

$$\|f_n\|_{L^2(\mathbb{R}^n)}=\int_{\mathbb{R}^n} f_n^2(t_1,\cdots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n<\infty,$$

then the iterated Itô integral is given by

$$\int_{\mathbb{R}^n} f_n \mathrm{d}B^{\otimes n} := n!\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t_n}\cdots\left(\int_{-\infty}^{t_2} f(t_1,\cdots,t_n)\mathrm{d}B(t_1)\right)\cdots\right)\mathrm{d}B(t_n). \tag{2.2}$$

In the following we let

$$h_n(x)=(-1)^n \mathrm{e}^{\frac{x^2}{2}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-\frac{x^2}{2}}\right); n=0,1,2,\cdots \tag{2.3}$$

be the Hermite polynomials and let $\{\xi_n\}_{n=1}^{\infty}$ be the basis of $L^2(\mathbb{R})$ consiting

$$\xi_n(x)=\pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{2}}h_{n-1}(\sqrt{2}x),\ n=1,2,\cdots \tag{2.4}$$

The set of multi-indices $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ of nonnegative integers is denoted by $\mathcal{T}=(\mathbb{N}_0^{\mathbb{N}})_{\mathbb{C}}$. Where $\mathbb{N}=\{1,2,\cdots\}$ is the set of all natural number and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. If $z=(z_1,z_2,\cdots)$ is a sequence of number or function, we use the multi-induces notation

$$z^{\alpha}=z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n} \text{ if } \alpha=(\alpha_1,\cdots,\alpha_n)\in\mathcal{T}$$

Theorem 2.1. ([15]) Let $\varphi_1,\varphi_2,\cdots,\varphi_n$ be are an orthonormal function in $L^2(\Omega)$. Then for all multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathcal{T}$, we have

$$\int_{\mathbb{R}^{|\alpha|}}\varphi^{\hat{\otimes}\alpha}\mathrm{d}B^{\otimes|\alpha|}(x)=h_{\alpha_1}(\langle\omega,\varphi_1\rangle)\cdots h_{\alpha_n}(\langle\omega,\varphi_n\rangle).$$

Corollary 2.2.

$$(H_\alpha \lozenge H_\beta)=H_{\alpha+\beta}(\omega);\alpha,\beta\in\mathcal{T}.$$

where $\lozenge$ denote the Wick product, and extend linearly. Then if $f_n\in\hat{L}^2(\mathbb{R}^n), g_n\in\hat{L}^2(\mathbb{R}^m)$, we have

$$\left(\sum_n\int_{\mathbb{R}^n} f_n\mathrm{d}B^{\otimes n}\right)\lozenge\left(\sum_m\int_{\mathbb{R}^m} g_m\mathrm{d}B^{\otimes m}\right)=\sum_{m,n}\int_{\mathbb{R}^{m+n}} f_n\hat{\otimes}g_m\mathrm{d}B^{\otimes(m+n)}$$

Proof.

$$\begin{aligned}&\int_{\mathbb{R}^{|\alpha|}}\xi^{\hat{\otimes}\alpha}\mathrm{d}B^{\otimes|\alpha|}\lozenge\int_{\mathbb{R}^{|\beta|}}\xi^{\hat{\otimes}\beta}\mathrm{d}B^{\otimes|\beta|}\\&=H_\alpha\lozenge H_\beta=H_{\alpha+\beta}=\int_{\mathbb{R}^{|\alpha+\beta|}}\xi^{\hat{\otimes}(\alpha+\beta)}\mathrm{d}B^{\otimes|\alpha+\beta|}\\&=\int_{\mathbb{R}^{|\alpha+\beta|}}\xi^{\hat{\otimes}\alpha}\hat{\otimes}\xi^{\hat{\otimes}\beta}\mathrm{d}B^{\otimes|\alpha+\beta|}.\end{aligned}$$

3. Stochastic Test Function and Stochastic Distribution (Konddratiev Spaces)

1) Stochastic test function spaces

Suppose $k\in\mathbb{N}$, for $0\le\beta<1$, let $(S)_\beta$ consist of those

$$f=\sum_\alpha c_\alpha H_\alpha,$$

such that

$$\|f\|_{k,\beta} = \sum_{\alpha} c_{\alpha}^{2} (\alpha)^{1+\beta} (2\mathbb{N})^{k\alpha}, \forall k \in \mathbb{N},$$

where

$$(2\mathbb{N})^{k\alpha} = \prod_{i=1}^{m} (2i)^{k\alpha_i}, \text{ for } \alpha = (\alpha_1, \cdots, \alpha_m). \tag{3.1}$$

2) Stochastic distribution

For $0 \le \beta < 1$, let $(S)_{\beta}^{*}$ be the space of Kondratiev space of stochastic distribution, consist of all formal expansions

$$F = \sum_{\alpha} b_{\alpha} H_{\alpha},$$

such that

$$\|F\|_{-q,-\beta} = \sum_{\alpha} b_{\alpha}^{2} (\alpha)^{1-\beta} (2\mathbb{N})^{-q\alpha}, \text{ for some } q \in \mathbb{N},$$

where $(2\mathbb{N})^{\alpha}$ is defined in (3.1).

Note that $(S)_{\beta}^{*}$ is the dual of $(S)_{\beta}$ and we can define the action of $F = \sum_{\alpha} b_{\alpha} H_{\alpha} \in (S)_{\beta}^{*}$ on $f = \sum_{\alpha} c_{\alpha} H_{\alpha} \in (S)_{\beta}$ by

$$\langle F, f \rangle = \sum_{\alpha} \alpha! (b_{\alpha}, c_{\alpha}),$$

where (b_{α}, c_{α}) is the usual inner product in $\mathbb{R}$.

Definition 3.1. Let $F \in (S)_{\beta}^{*}$ be the random variable and let $\gamma \in L^2(\mathbb{R})$. Then we say that F has directional derivative in the direction γ if

$$D_{\gamma} F(\omega) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(F(\omega + \epsilon\gamma) - F(\omega) \right) \tag{3.2}$$

if the limit exist in $F \in (S)_{\beta}^{*}$.

Definition 3.2. A function $\Phi : \mathbb{R} \to (S)_{\beta}^{*}$-integrable if

$$\langle \Phi(.), \phi \rangle \in L^1(\mathbb{R}), \text{ for all } \phi \in (S)_{\beta}.$$

Then the $(S)_{\beta}^{*}$-integrable of $\Phi(t)$, denoted by $\int_{\mathbb{R}} \Phi(t) \mathrm{d}t$, is the unique $(S)_{\beta}^{*}$ element such that

$$\left\langle \int_{\mathbb{R}} \Phi(t) \mathrm{d}t, \phi \right\rangle = \int_{\mathbb{R}} \langle \Phi, \phi \rangle (t) \mathrm{d}t, \phi \in (S)_{\beta}.$$

Definition 3.3. Consider $\varphi(t,\omega) : \mathbb{R} \to (S)_{\beta}^{*}$ such that

$$\varphi(t,\omega)\gamma(t) \text{ is } \varphi(t,\omega)\text{-integrable}$$

and

$$D_{\gamma} F(\omega) = \int_{\mathbb{R}} \varphi(t,\omega) \gamma(t) \mathrm{d}t, \text{ for all } \gamma \in L^2(\mathbb{R}),$$

then we say that F is (Hida) Malliavin differentiable and we put

$$D_t F(\omega) := \frac{\mathrm{d}F}{\mathrm{d}\omega}(t,\omega) = \varphi(t,\omega), t \in \mathbb{R}.$$

D_t is called the Hida-Malliavin derivative or stochastic gradient of F at t.

The set of all differentiable is denoted by $\mathbb{D}$.

Definition 3.4. Consider $F(\omega)=\sum_{\alpha} c_\alpha H_\alpha(\omega)\in(S)^*_\beta$. Then we define the stochastic derivative of F at t by

$$D_t F(\omega):=\frac{\mathrm{d}F}{\mathrm{d}\omega}(t,\omega):=\sum_{\alpha} c_\alpha \sum_i \alpha_i H_{\alpha-\epsilon^{(i)}}(\omega)\cdot\xi_i(t)$$

$$=\sum_{\gamma}\left(\sum_i c_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\xi_i(t)\right)H_\gamma(\omega)$$

Lemma 3.5.

1) Let $F\in(S)^*_\beta$. Then $D_t F\in(S)^*_\beta$ for a.a. $t\in\mathbb{R}$.

2) Suppose $F,F_m\in(S)^*_\beta$ for all $m\in\mathbb{N}$ and

$$F_m\to F \text{ in } (S)^*_\beta.$$

Then there exist a subsequence $\{F_{m_k}\}_{k=1}^{\infty}$ such that

$$D_t F_{m_k}\to D_t F \text{ in } (S)^*_\beta, \text{ for } a.a\ t>0$$

Proof. 1) Suppose $F(\omega)=\sum_{\alpha} c_\alpha H_\alpha(\omega)\in(S)^*_\beta$. Then

$$D_t F(\omega)=\sum_{\alpha} c_\alpha \sum_i \alpha_i H_{\alpha-\epsilon^{(i)}}(\omega)\cdot\xi_i(t)$$

$$=\sum_{\gamma}\left(\sum_i c_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\xi_i(t)\right)H_\gamma(\omega)$$

$$=\sum_{\gamma} g_\gamma(t)H_\gamma(\omega).$$

where $g_\gamma(t)=\sum_i c_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\xi_i(t)$.

We want to prove that for some $q\in\mathbb{N}$,

$$\|D_t F\|^2_{-\beta,-q-1}=\sum_m\left(\sum_{|\gamma|=m} g_\gamma^2(\gamma!)^{1-\beta}\right)(2\mathbb{N})^{-\gamma(q+1)}<\infty \text{ for } a.a.t.$$

Note that

$$\int_{\mathbb{R}} g_\gamma^2(t)\mathrm{d}t=\int_{\mathbb{R}}\left(\sum_i c_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\xi_i(t)\right)^2\mathrm{d}t=\sum c^2_{\gamma+\epsilon^{(i)}}(\gamma_i+1)^2.$$

Moreover,

$$(2\mathbb{N})^{-\gamma q}<(2\mathbb{N})^{-\gamma}=\prod_i(2\cdot i)^{-\gamma_i}\le\prod_i \mathrm{e}^{-\gamma_i(\log 2)}=\mathrm{e}^{-|\tilde{\gamma}|}$$

where $\tilde{\gamma}=(\log 2)\gamma_i$ for all $i\in I$. Hence,

$$\int_{\mathbb{R}}\|D_t F\|^2_{-\beta,-q-1}\mathrm{d}t$$

$$=\sum_{\gamma}\left(c^2_{\gamma+\epsilon^{(i)}}(\gamma_i+1)^2(\gamma!)^{1-\beta}\right)(2\mathbb{N})^{-\gamma(q+1)}$$

$$=\sum_{\gamma,i}(\gamma_i+1)(\gamma!)^{-\beta}(2\mathbb{N})^{-\gamma(q+1)}\sum_{\alpha,|\alpha|=|\gamma|+1} c_\alpha^2\alpha!$$

$$<\sum_m\sum_{|\tilde{\gamma}|=m}(m+1)\mathrm{e}^{-m}\sum_{|\alpha|=(\log 2)^{-1}m+1} c_\alpha^2(\alpha!)(2\mathbb{N})^{-\alpha q}.$$

Using the fact that $(m+1)\mathrm{e}^{-m} \le 1$ for all m, we get

$$\int_{\mathbb{R}} \|D_t F\|_{-\beta,-q-1}^2 \, \mathrm{d}t < \sum_m \left(\sum_{|\alpha|=(\log 2)^{-1} m+1} c_\alpha^2 \alpha! \right) (2\mathbb{N})^{-\alpha q} < \|F\|_{-\beta,-q} < \infty. \quad (3.3)$$

Therefore,

$$D_t F \in (S)_{-\beta,-q-1} \text{ for a.a. } t$$

2) To prove this part, it suffices to prove that if $F_m \to 0$ in $(S)_{-\beta,-q}$, then there exist a subsequence $\{F_{m_k}\}_{k=1}^{\infty}$ such that $D_t F_m \to 0$ in $(S)_\beta^*$ as $k \to \infty$, for a.a. t. We have prove that

$$\int_{\mathbb{R}} \|D_t F\|_{-\beta,-q-1}^2 \, \mathrm{d}t \le \|F_n\|_{-\beta,-q}^2 \to 0.$$

Therefore,

$$\|D_t F_n\|_{-\beta,-q-1} \to 0 \text{ in } L^2(\mathbb{R}).$$

So, there exists a subsequence $\{\|D_t F_n\|\}_{k\ge 1}$ such that $\|D_t F_{n_k}\|_{-\beta,-q-1} \to 0$ for a.a. t as $k \to \infty$. This complete the proof.

Suppose $\xi_1, \xi_2, \cdots$ is the Hermite functions, and put

$$X_i = X_i(\omega) = \langle \omega, \xi_i \rangle = \int_{\mathbb{R}} \xi_i(s) \, \mathrm{d}B(s); i = 1, 2, \cdots \quad (3.4)$$

and

$$X_i^{(t)}(\omega) = \int_0^t \xi_i(s) \, \mathrm{d}B(s); i = 1, 2, \cdots \quad (3.5)$$

and

$$X = (X_1, X_2, \cdots), X^{(t)} = \left(X_1^{(t)}, X_2^{(t)}, \cdots\right).$$

With this notation we have, $X^{\Diamond\alpha}(\omega) = \left(X_1^{\Diamond\alpha_1} \Diamond \cdots \Diamond X_m^{\Diamond\alpha_m}\right)(\omega) = H_\alpha(\omega)$ for all multi indices α where $\alpha = (\alpha_1, \cdots, \alpha_m)$.

Definition 3.6. 1) Let $k \in \mathbb{N}, 0 \le \beta < 1$. We say that

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

belong to the space $\mathcal{G}_k^\beta$ if

$$\|F\|_{\mathcal{G}_k^\beta}^2 = \sum_{n=0}^{\infty} \mathrm{e}^{2kn} (n!)^{1+\beta} \|f_n\|_{L^2(\mathbb{R}^n)}^2 < \infty,$$

we define

$$\mathcal{G}^\beta = \bigcap_{k\in\mathbb{N}} \mathcal{G}_k^\beta,$$

and equip $\mathcal{G}^\beta$ with the projective topology.

2) We say that

$$G = \sum_{n=0}^{\infty} I_n(g_n)$$

belong to the space $\mathcal{G}_{-q}^{-\beta}$ if

$$\|G\|^2_{\mathcal{G}^{-\beta}_{-q}} = \sum_{n=0}^{\infty} \mathrm{e}^{-2qn} (n!)^{1-\beta} \|f_n\|^2_{L^2(\mathbb{R}^n)} < \infty,$$

we define

$$\mathcal{G}^{-\beta} = \bigcap_{q\in\mathbb{N}} \mathcal{G}^{-\beta}_{-q}$$

and equip $\mathcal{G}^{-\beta}$ with the inductive topology. Then $\mathcal{G}^{-\beta}$ is the dual of $\mathcal{G}^{\beta}$, with action

$$\langle G, F\rangle = \sum_{n=0}^{\infty} n! \langle g_n, f_n\rangle.$$

4. The Generalized Clark-Ocone Formula

Now we are prepared to present the main result of this paper. It generalizes the well know Clark-Ocone formula to generalized functions, *i.e.*, to the space $\mathcal{G}^{-\beta}$.

Definition 3.1. Suppose $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^{\beta}$. Then the conditional expectation of F with respect to $\mathcal{F}_t$ is given by

$$E[F \mid \mathcal{F}_t] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n \cdot \chi_{[o,t]^n} \mathrm{d}B^{\otimes n} \tag{4.1}$$

Note that this coincides with usual conditional expectation if $F \in L^2(\mu)$, and

$$\|E[F \mid \mathcal{F}_t]\|_{\mathcal{G}^{\beta}_k} \le \|F\|_{\mathcal{G}^{\beta}_k}, \text{for some } k \in \mathbb{N}. \tag{4.2}$$

In particular

$$E[F \mid \mathcal{F}_t] \in \mathcal{G}^{-\beta} \tag{4.3}$$

Lemma 4.2. Suppose $F, G \in \mathcal{G}^{-\beta}$. Then

$$E[F \lozenge G \mid \mathcal{F}_t] = E[F \mid \mathcal{F}_t] \lozenge E[G \mid \mathcal{F}_t]$$

Proof. Assume that, without loss of generality,

$$F = \int_{\mathbb{R}^n} f_n \mathrm{d}B^{\otimes n} = \sum_{|\alpha|=n} c_\alpha \int_{\mathbb{R}^n} \xi^{\hat{\otimes} n} \mathrm{d}B^{\otimes n}$$

and similarly G. By Corollary 2.2 and Definition 4.1, we have

$$\begin{aligned} E[F \lozenge G \mid \mathcal{F}_t] &= E\left[\int_{\mathbb{R}^{m+n}} f_n \hat{\otimes} g_m \mathrm{d}B^{\otimes(m+n)} \mid \mathcal{F}_t\right] \\ &= \int_{\mathbb{R}^{m+n}} f_n \hat{\otimes} g_m \cdot \chi_{[0,t]^{m+n}} \mathrm{d}B^{\otimes(m+n)} \\ &= \int_{\mathbb{R}^{m+n}} f_n \hat{\otimes} \chi_{[0,t]^n} \hat{\otimes} g_m \cdot \chi_{[0,t]^m} \mathrm{d}B^{\otimes(m+n)} \\ &= E[F \mid \mathcal{F}_t] \lozenge E[G \mid \mathcal{F}_t]. \end{aligned}$$

Lemma 4.3.

Let $F \in \mathcal{G}^{-\beta}$. Then $D_t F \in \mathcal{G}^{-\beta}$ for a.a. $t \in \mathbb{R}$.

Consider $F, F_m \in \mathcal{G}^{-\beta}$ for all $m \in \mathbb{N}$ and

$$F_m \to F \text{ in } \mathcal{G}^{-\beta}.$$

Then there exists a subsequence $\{F_{m_k}\}_{k=1}^{\infty}$ such that

$$D_t F_{m_k} \to D_t F \text{ in } \mathcal{G}^{-\beta}, \text{for } a.a\, t > 0$$

Proof. 1) Suppose $F(\omega)=\sum_{\alpha} c_{\alpha} H_{\alpha}(\omega)\in\mathcal{G}^{-\beta}$. Then

$$\begin{aligned}D_t F(\omega)&=\sum_{\alpha} c_{\alpha}\sum_{i}\alpha_i H_{\alpha-\epsilon^{(i)}}(\omega)\cdot\xi_i(t)\\&=\sum_{\gamma}\left(\sum_{i} c_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\xi_i(t)\right)H_{\gamma}(\omega)\\&=\sum_{\gamma} g_{\gamma}(t)H_{\gamma}(\omega).\end{aligned}$$

where

$$g_{\gamma}(t)=\sum_{i} c_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\xi_i(t).$$

Choose $q<\infty$ such that $\|F\|^2_{\mathcal{G}^{-\beta}_{-q}}=\sum_m\sum_{|\alpha|=m} c^2_{\alpha}(\alpha!)^{1-\beta}\mathrm{e}^{-2qm}<\infty$. We will prove that

$$\|D_t F\|^2_{\mathcal{G}^{-\beta}_{-q-1}}=\sum_{n}\left(\sum_{|\gamma|=n} g^2_{\gamma}(\gamma!)^{1-\beta}\right)\mathrm{e}^{-2(q+1)n}<\infty \text{ for } a.a.t$$

Note that

$$\int_{\mathbb{R}} g^2_{\gamma}(t)\mathrm{d}t=\int_{\mathbb{R}}\left(\sum_{i} c_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\xi_i(t)\right)^2\mathrm{d}t=\sum c^2_{\gamma+\epsilon^{(i)}}(\gamma_i+1)^2.$$

So

$$\begin{aligned}&\sum_{|\gamma|=n}\left(\int_{\mathbb{R}} g^2_{\gamma}(t)\mathrm{d}t\right)(\gamma!)^{1+\beta}\\&=\sum c^2_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\left(\left(\gamma+\epsilon^{(i)}\right)!\right)^{1+\beta}\\&\le\sum_{|\gamma|=n}(n+1)\sum_{i} c^2_{\gamma+\epsilon^{(i)}}(\gamma_i+1)\left(\left(\gamma+\epsilon^{(i)}\right)!\right)^{1+\beta}\\&\le(n+1)\sum_{|\alpha|=|\gamma|+1} c^2_{\alpha}(\alpha!)^{1+\beta}.\end{aligned}$$

Hence, using the fact that $(n+1)\mathrm{e}^{-n}\le 1$ for all n, we get

$$\begin{aligned}&\int_{\mathbb{R}}\|D_t F\|^2_{\mathcal{G}^{-\beta}_{-(q+1)}}\mathrm{d}t\\&=\int_{\mathbb{R}}\sum_{n}\left(\sum_{|\gamma|=n} g^2_{\gamma}(\gamma!)^{1-\beta}\right)\mathrm{e}^{-2(q+1)n}\mathrm{d}t\\&\le\sum_{n}(n+1)\left(\sum_{|\alpha|=|\gamma|+1} c^2_{\alpha}(\alpha!)^{1-\beta}\right)\mathrm{e}^{-2(q+1)n}\\&\le\sum_{n}\left(\sum_{|\alpha|=|\gamma|+1} c^2_{\alpha}(\alpha!)^{1-\beta}\right)\mathrm{e}^{-2qn}\le\|F\|^2_{\mathcal{G}^{-\beta}_{-q}}<\infty.\end{aligned}\tag{4.4}$$

Therefore,

$$\|D_t F\|^2_{\mathcal{G}^{-\beta}_{-(q+1)}}<\infty \text{ for } a.a.t$$

and

$$D_t F\in\mathcal{G}^{-\beta}_{-(q+1)}\subset\mathcal{G}^{-\beta} \text{ for } a.a.t$$

2) It suffices to prove that if $G_m\to 0$ in $\mathcal{G}^{-\beta}_{-q}$, then there exists a subsequence $\{G_{m_k}\}_{k=1}^{\infty}$ such that $D_t G_m\to 0$ in $\mathcal{G}^{-\beta}$ as $k\to\infty$, for a.a. t. By

(4.4) we can see that $\|D_t G_m\|_{\mathcal{G}_{-q}^{-\beta}} \to 0$ in $L^2(\mathbb{R})$. So there exists a subsequence

$$\left\{\|D_t G_m\|_{\mathcal{G}_{-q}^{-\beta}}\right\}_{k=0}^{\infty}$$

such that

$$\|D_t G_m\|_{\mathcal{G}_{-q}^{-\beta}} \to 0 \text{ for } a.a.\, t \text{ as } k \to \infty. \tag{4.5}$$

Therefore,

$$D_t G_{m_k} \to 0 \text{ in } \mathcal{G}^{-\beta} \text{ for } a.a.\, t \text{ as } k \to \infty.$$

The last assertion follows from (4.2).

Theorem 4.4. Suppose λ denote Lebesque measure on $\mathbb{R}$. Let $F(\omega) \in L^2(\mu)$ be $\mathcal{F}_t$-measurable. Then

$$(t,\omega) \to E[D_t F \mid \mathcal{F}_t](\omega) \in L^2(\lambda \times \mu)$$

and

$$F(\omega) = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t]\,\mathrm{d}B(t).$$

Proof. Let $F(\omega) = \sum_{\alpha \in \mathcal{T}} c_\alpha H_\alpha(\omega)$ be the chaos expansion of F and put

$$F_n = \sum_{\alpha \in \mathcal{T}} c_\alpha H_\alpha(\omega) = \sum_{\alpha \in \mathcal{T}_n} c_\alpha X^{\Diamond\alpha},$$

where $\mathcal{T}_n = \{\alpha \in \mathcal{T}; |\alpha| \le n \,\&\, \text{length}(\alpha) \le n\}$. Then by Lemma 3.8 (see [4]), we have

$$F_n(\omega) = E[F_n] + \int_0^T E[D_t F_n \mid \mathcal{F}_t]\,\mathrm{d}B(t).$$

By Itô representation theorem there is a unique $u(t,\omega)$ which is $\mathcal{F}_t$ adapted and such that

$$E\left[\int_0^T u^2(t,\omega)\,\mathrm{d}t\right] < \infty$$

and such that

$$F(\omega) = E[F] + \int_0^T u(t,\omega)\,\mathrm{d}B(t),$$

since $F_n \to F$ in $L^2(\mu)$, we conclude that

$$\begin{aligned} &E\left[\int_0^T \left(E[D_t F_n \mid \mathcal{F}_t] - u(t,\omega)\right)^2 \mathrm{d}t\right] \\ &= E\left[F_n - F - E[F_n] + E[F_n]^2\right] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$E[D_t F_n \mid \mathcal{F}_t] \to u(t,\omega) \text{ in } L^2(\lambda \times \mu),$$

on the other hand, by Lemma 4.1, we have

$$E[D_t F_n \mid \mathcal{F}_t] \to E[D_t \mid \mathcal{F}_t] \text{ in } \mathcal{G}^{-\beta} \text{ for a.a } t.$$

By taking another subsequence, we obtain that

$$E[D_t F_n \mid \mathcal{F}_t] \to u(t,\omega) \text{ in } L^2(\mu) \text{ for a.a } t.$$

We conclude that

$$u(t,\omega) = E[D_t F \mid \mathcal{F}_t] \text{ for a.a } t.$$

This completes the proof.

Lemma 4.5. Suppose $F \in \mathcal{G}^{-\beta}$ and $f \in \mathcal{G}^{\beta}$. Then

$$|\langle F, f \rangle| \le \|F\|_{\mathcal{G}_{-\beta,-q}} \cdot \|f\|_{\beta,\hat{q}};$$

where $\hat{q} = \frac{2q}{\ln 2}$.

Proof. Let $F(\omega) = \sum_\alpha a_\alpha H_\alpha(\omega), f(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega)$. Then

$$|\langle F, f \rangle| = \left|\sum_\alpha a_\alpha b_\alpha \alpha!\right| = \left|\sum_m \left(\sum_{|\alpha|=m} a_\alpha b_\alpha \alpha!\right)\right|$$

$$\le \left(\sum_m \left(\sum_{|\alpha|=m} a_\alpha^2 (\alpha_i)^{1-\beta}\right) \mathrm{e}^{-2qm}\right)^{\frac{1}{2}} \left(\sum_m \left(\sum_{|\alpha|=m} b_\alpha^2 (\alpha_i)^{1+\beta}\right) \mathrm{e}^{2qm}\right)^{\frac{1}{2}}$$

$$\le \|F\|_{\mathcal{G}_{-\beta,-q}} \left(\sum_\alpha b_\alpha^2 (\alpha_i)^{1+\beta} (2\mathbb{N})^{\hat{q}\alpha}\right)^{\frac{1}{2}}$$

$$= \|F\|_{\mathcal{G}_{-q}^{-\beta}} \cdot \|f\|_{\hat{q},\beta}.$$

Lemma 4.6. Suppose $F \in \mathcal{G}^{\beta}, f \in (S)_\beta$. Then

$$\int_{\mathbb{R}} \langle E[D_t F \mid \mathcal{F}_t], f \rangle^2 \,\mathrm{d}t < \infty.$$

Proof. By Lemma 4.3 and (4.4), we have

$$\int_{\mathbb{R}} \langle E[D_t F \mid \mathcal{F}_t], f \rangle^2 \,\mathrm{d}t \le \int_{\mathbb{R}} \|E[D_t F \mid \mathcal{F}_t]\|_{\mathcal{G}_{-q}^{-\beta}}^2 \|f\|_{\hat{q},\beta}$$

$$\le \|f\|_{\hat{q},\beta} \int_{\mathbb{R}} \|[D_t F \mid \mathcal{F}_t]\|_{\mathcal{G}_{-q}^{-\beta}}^2 < \infty, \text{ for some } q \in \mathbb{N}.$$

Lemma 4.7. Let $F_n, F \in \mathcal{G}^{-\beta}$ and $F_n \to F$ in $(S)_\beta^*$. Then

$$\int_0^T E[D_t F_n \mid \mathcal{F}_t] \lozenge W(t) \,\mathrm{d}t \to \int_0^T E[D_t F \mid \mathcal{F}_t] \lozenge W(t) \,\mathrm{d}t. \tag{4.6}$$

Proof. In case of $\beta = 0$ a complete proof is given in [4]. The proof for general $0 \le \beta < 1$ is a simple modification. Note that both integral in (4.6) exist by Lemma 4.7. Hence, by Lemma 4.6 and (4.4), we have

$$\left|\left\langle \int_0^T E[D_t F_n \mid \mathcal{F}_t] \lozenge W(t) \,\mathrm{d}t - \int_0^T E[D_t F \mid \mathcal{F}_t] \lozenge W(t) \,\mathrm{d}t, f \right\rangle\right|$$

$$= \int_0^T \left|\langle E[D_t (F_n - F) \mid \mathcal{F}_t], f \rangle\right| \mathrm{d}t$$

$$\le \sqrt{T} \|f\|_{\hat{q},\beta} \left(\int_0^T \|[D_t (F_n - F) \mid \mathcal{F}_t]\|_{\mathcal{G}_{-q}^{-\beta}}^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty.$$

This completes the proof.

Theorem 4.8. Let $F(\omega) \in \mathcal{G}^{-\beta}$ be $\mathcal{F}_t$-measurable. Then $E[D_t F \mid \mathcal{F}_t] \lozenge W(t)$ is integrable in $(S)_\beta^*$ and

$$F(\omega) = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_T] \lozenge W(t) \,\mathrm{d}t.$$

where, $E[F]$ denotes the generalized exsection of F.

Proof. Let $F_n(\omega)=\sum_{\alpha} c_\alpha H_\alpha(\omega)$. Then, by Lemma 3.8 (see [4]), we have

$$F_n(\omega)=E[F_n]+\int_0^T E[D_t F_n \mid \mathcal{F}_T]\lozenge W(t)\,\mathrm{d}t,$$

therefore,

$$F(\omega)=E[F]+\lim_{n\to\infty}\int_0^T E[D_t F_n \mid \mathcal{F}_T]\lozenge W(t)\,\mathrm{d}t,$$

the limit exist in $\mathcal{G}^{-\beta}$ and hence in $(S)^*_\beta$. The result follows from Lemma 4.7.

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