

# Least Squares Hermitian Problem of Matrix Equation $(AXB, CXD) = (E, F)$ Associated with Indeterminate Admittance Matrices

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## Abstract

For  $A \in \mathbb{C}^{m \times n}$ , if the sum of the elements in each row and the sum of the elements in each column are both equal to 0, then  $A$  is called an indeterminate admittance matrix. If  $A$  is an indeterminate admittance matrix and a Hermitian matrix, then  $A$  is called a Hermitian indeterminate admittance matrix. In this paper, we provide two methods to study the least squares Hermitian indeterminate admittance problem of complex matrix equation  $(AXB, CXD) = (E, F)$ , and give the explicit expressions of least squares Hermitian indeterminate admittance solution with the least norm in each method. We mainly adopt the Moore-Penrose generalized inverse and Kronecker product in Method I and a matrix-vector product in Method II, respectively.

## Keywords

Matrix Equation, Least Squares Solution, Least Norm Solution, Hermitian Indeterminate Admittance Matrices

## 1. Introduction

Firstly, we state some symbols that are used in this paper. The set of all real column vectors with  $n$  coordinates by  $\mathbf{R}^n$ , and the set of all  $m \times n$  real matrices by  $\mathbf{R}^{m \times n}$  are denoted. Let  $\mathbf{SR}^{n \times n}$  and  $\mathbf{ASR}^{n \times n}$  stand for the set of all  $n \times n$  real symmetric matrices and the set of all  $n \times n$  real anti-symmetric matrices, respectively. The set of all  $m \times n$  complex matrices is denoted by  $\mathbf{C}^{m \times n}$ , and  $\mathbf{HC}^{n \times n}$  stands for the set of all  $n \times n$  Hermitian matrices. For  $A \in \mathbf{R}^{m \times n}$ , if the sum of the elements in each row and the sum of the elements in each column are both equal to 0, then  $A$  is called an indeterminate admittance matrix.

$\mathbf{IA}^{n \times n}$  is denoted to be the set of all indeterminate admittance matrices. For  $A \in \mathbf{R}^{n \times n}$ , if  $A$  is not only an indeterminate admittance matrix, but also a symmetry matrix, then  $A$  is called a symmetry indeterminate admittance matrix. Similarly, for  $A \in \mathbf{R}^{n \times n}$ ,  $A$  stands for an anti-symmetric indeterminate admittance matrix if  $A$  is an indeterminate admittance matrix and an anti-symmetric matrix.  $\mathbf{SIA}^{n \times n}$  and  $\mathbf{AIA}^{n \times n}$  are denoted to be the set of all symmetry indeterminate admittance matrices and the set of all anti-symmetric indeterminate admittance matrices, respectively. For  $A \in \mathbf{IA}^{n \times n}$ , if  $A$  is also a Hermitian matrix, then  $A$  is called a Hermitian indeterminate admittance matrix.  $\mathbf{HIA}^{n \times n}$  is denoted to be the set of all Hermitian indeterminate admittance matrices. The transpose matrix, the conjugate transpose matrix and the Moore-Penrose generalized inverse of matrix  $A$  are denoted by  $A^T$ ,  $A^H$  and  $A^+$ , respectively. The identity matrix of order  $n$  is denoted by  $I_n$ . The trace of matrix  $A = (a_{ij}) \in \mathbf{C}^{n \times n}$  is

$$\text{tr}(A) = a_{11} + a_{22} + \dots + a_{nn} = e_1^T A e_1 + e_2^T A e_2 + \dots + e_n^T A e_n,$$

where  $e_j$  is the  $j$ th column of the identity matrix  $I_n$ . The 2-norm of the vector  $x$  by  $\|x\|_2$  is denoted. For  $A, B \in \mathbf{C}^{m \times n}$ , we define the inner product:  $\langle A, B \rangle = \text{tr}(A^H B)$ , then  $\mathbf{C}^{m \times n}$  is a Hilbert inner product space and the norm of a matrix generated by this inner product is the matrix Frobenius norm  $\|\cdot\|$ .

**Definition 1 ([1]).** For matrix  $A \in \mathbf{R}^{m \times n}$ , let  $a_j = (a_{1j}, a_{2j}, \dots, a_{mj}) (j = 1, 2, \dots, n)$ , and denote the following vector by  $\text{vec}(A)$ :

$$\text{vec}(A) = (a_1, a_2, \dots, a_n)^T. \tag{1}$$

**Definition 2 ([1]).** For matrix  $A \in \mathbf{SIA}^{n \times n}$ , let  $a_1 = (a_{11}, a_{21}, \dots, a_{n-1,1})$ ,  $a_2 = (a_{22}, a_{32}, \dots, a_{n-1,2})$ ,  $\dots$ ,  $a_{n-2} = (a_{n-2, n-2}, a_{n-1, n-2})$ ,  $a_{n-1} = a_{n-1, n-1}$ , and denote the following vector by  $\text{vec}_S(A)$ :

$$\text{vec}_S(A) = (a_1, a_2, \dots, a_{n-1})^T. \tag{2}$$

**Definition 3 ([1]).** For matrix  $A \in \mathbf{AIA}^{n \times n}$ , let  $a_1 = (a_{21}, a_{31}, \dots, a_{n-1,1})$ ,  $a_2 = (a_{32}, a_{42}, \dots, a_{n-1,2})$ ,  $\dots$ ,  $a_{n-2} = a_{n-1, n-2}$ , and denote the following vector by  $\text{vec}_A(A)$ :

$$\text{vec}_A(A) = (a_1, a_2, \dots, a_{n-2})^T. \tag{3}$$

It is well known that indeterminate admittance matrices play important roles in circuit modeling and lattices network and so on [2] [3]. In this paper, we mainly discuss the least squares problem associated with indeterminate admittance matrices, and derive it as follows.

**Problem I.** Given  $A \in \mathbf{C}^{m \times n}$ ,  $B \in \mathbf{C}^{n \times s}$ ,  $C \in \mathbf{C}^{m \times n}$ ,  $D \in \mathbf{C}^{n \times s}$ ,  $E \in \mathbf{C}^{m \times s}$ , and  $F \in \mathbf{C}^{m \times s}$ , let

$$\begin{aligned} H_L &= \left\{ X \mid X \in \mathbf{HIA}^{n \times n}, \|AXB - E\|^2 + \|CXD - F\|^2 \right. \\ &= \left. \min_{X_0 \in \mathbf{HIA}^{n \times n}} \|AX_0B - E\|^2 + \|CX_0D - F\|^2 \right\}. \end{aligned}$$

Find  $X_H \in H_L$  such that

$$\|X_H\| = \min_{X \in H_L} \|X\|. \quad (4)$$

The solution  $X_H$  is also called the least squares Hermitian indeterminate admittance solution of complex matrix equation

$$(AXB, CXD) = (E, F) \quad (5)$$

with the least norm.

For studying Problem I mentioned above, we first state some Lemmas.

**Lemma 1.** ([4]) *The matrix equation  $Ax = b$ , with  $A \in \mathbf{R}^{m \times n}$  and  $b \in \mathbf{R}^n$ , has a solution  $x \in \mathbf{R}^n$  if and only if*

$$AA^+b = b, \quad (6)$$

*in this case it has the general solution*

$$x = A^+b + (I - A^+A)y, \quad (7)$$

*where  $y \in \mathbf{R}^n$  is an arbitrary vector.*

**Lemma 2.** ([4]) *The least squares solutions of the matrix equation  $Ax = b$ , with  $A \in \mathbf{R}^{m \times n}$  and  $b \in \mathbf{R}^n$ , can be represented as*

$$x = A^+b + (I - A^+A)y, \quad (8)$$

*where  $y \in \mathbf{R}^n$  is an arbitrary vector, and the least squares solution with the least norm is  $x = A^+b$ .*

Direct and iterative methods on solving the matrix equations associated with the constrained matrix (such as Hermitian matrix, anti-Hermitian matrix, bi-symmetric matrix, reflexive matrix) sets have been widely investigated. See [5]-[25] and references cited therein. Yuan, Liao and Lei [1] derived the least squares symmetric solution with the least norm of real matrix equation  $AXB + CYD = E$  by using the vec-operator, Kronecker product and the Moore-Penrose generalized inverse. In order to avoid the difficulties of the coefficient matrices with large size from the Kronecker product, Yuan and Liao [26] recently improved this method, defined a matrix-vector product, and successfully carried out a special vectorization of the matrix equation  $AXB + CXD = E$  to derive the least squares Hermitian solution with the least norm. Based on these methods, we continue to study Problem I in this paper.

We now briefly introduce the contents of our paper. In Section 2, by using the Moore-Penrose generalized inverse and the Kronecker product, we derive the least squares Hermitian indeterminate admittance solution with the least norm for the complex matrix Equation (5). In Section 3, we firstly discuss a class of linear least squares problem in Hilbert inner product  $\mathbf{C}^{m \times n}$ , and analysis a matrix-vector product of  $\mathbf{C}^{m \times n}$ . Then we present the explicit expression of the solution for the complex matrix Equation (5) by using the method.

## 2. Method I for the Solution of Problem I

In this section, we present the expression of the least square Hermitian indeterminate admittance solution of complex matrix Equation (5) with the least norm by using the Moore-Penrose generalized inverse and the Kronecker product of

matrices.

**Definition 4.** For  $A = (a_{ij}) \in \mathbb{C}^{m \times n}$ ,  $B = (b_{ij}) \in \mathbb{C}^{p \times q}$ , the symbol  $A \otimes B = (a_{ij}B) \in \mathbb{C}^{mp \times nq}$  stands for the Kronecker product of  $A$  and  $B$ .

**Theorem 3.** Suppose  $X \in \mathbb{R}^{n \times n}$  and  $\alpha_i = e_i - e_n$  ( $i = 1, 2, \dots, n$ ). Then

$$1) X \in \mathbf{SIA}^{n \times n} \Leftrightarrow \text{vec}(X) = K_S \text{vec}_S(X), \tag{9}$$

where  $\text{vec}_S(X)$  is represented as (2), and the matrix  $K_S \in \mathbb{R}^{n^2 \times \frac{n(n-1)}{2}}$  is of the following form

$$K_S = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_1 & \cdots & 0 & \alpha_2 & \cdots & \alpha_{n-1} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \alpha_{n-2} & \alpha_{n-1} & 0 \\ 0 & 0 & \cdots & \alpha_1 & 0 & \cdots & \alpha_2 & \cdots & 0 & \alpha_{n-2} & \alpha_{n-1} \\ -\alpha_1 & -\alpha_2 - \alpha_1 & \cdots & -\alpha_{n-1} - \alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} - \alpha_2 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} - \alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \tag{10}$$

$$2) X \in \mathbf{AIA}^{n \times n} \Leftrightarrow \text{vec}(X) = K_A \text{vec}_A(X), \tag{11}$$

where  $\text{vec}_A(X)$  is represented as (3), and the matrix  $K_A \in \mathbb{R}^{n^2 \times \frac{(n-2)(n-1)+1}{2}}$  is of the following form

$$K_A = \begin{bmatrix} -\alpha_2 & -\alpha_3 & \cdots & -\alpha_{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \alpha_1 & 0 & \cdots & 0 & -\alpha_3 & \cdots & -\alpha_{n-1} & 0 & \cdots & 0 \\ 0 & \alpha_1 & \cdots & 0 & \alpha_2 & \cdots & 0 & -\alpha_4 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_{n-1} \\ 0 & 0 & \cdots & \alpha_1 & 0 & \cdots & \alpha_2 & 0 & \cdots & \alpha_{n-2} \\ \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \cdots & \alpha_{n-1} - \alpha_1 & \alpha_3 - \alpha_2 & \cdots & \alpha_{n-1} - \alpha_2 & \alpha_4 - \alpha_3 & \cdots & \alpha_{n-1} - \alpha_{n-2} \end{bmatrix} \tag{12}$$

**Proof.** 1) For  $X \in \mathbf{SIA}^{n \times n}$ ,  $X$  can be expressed as

$$X = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{(n-1)1} & -\sum_{i=1}^{n-1} a_{i1} \\ x_{21} & x_{22} & \cdots & x_{(n-1)2} & -(x_{21} + x_{22} + \cdots + x_{(n-1)2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(n-1)1} & x_{(n-1)2} & \cdots & x_{(n-1)(n-1)} & -\sum_{j=1}^{n-1} a_{(n-1)j} \\ -\sum_{i=1}^{n-1} a_{i1} & -(x_{21} + x_{22} + \cdots + x_{(n-1)2}) & \cdots & -\sum_{j=1}^{n-1} a_{(n-1)j} & \sum_{i,j=1}^{n-1} a_{ij} \end{bmatrix}$$

$$= x_{11}(\alpha_1, 0, 0, \dots, 0, 0, -\alpha_1) + x_{21}(\alpha_2, \alpha_1, 0, \dots, 0, 0, -\alpha_2 - \alpha_1) + x_{31}(\alpha_3, 0, \alpha_1, \dots, 0, 0, -\alpha_3 - \alpha_1) + \cdots + x_{(n-1)1}(\alpha_{n-1}, 0, 0, \dots, 0, \alpha_1, -\alpha_{n-1} - \alpha_1) + x_{22}(0, \alpha_2, 0, \dots, 0, 0, -\alpha_2) + x_{32}(0, \alpha_3, \alpha_2, \dots, 0, 0, -\alpha_3 - \alpha_2) + \cdots + x_{(n-1)2}(0, \alpha_{n-1}, 0, \dots, 0, \alpha_2, -\alpha_{n-1} - \alpha_2) + \cdots + x_{(n-2)(n-2)}(0, 0, 0, \dots, \alpha_{n-2}, 0, -\alpha_{n-2}) + x_{(n-1)(n-2)}(0, 0, 0, \dots, \alpha_{n-1}, \alpha_{n-2}, -\alpha_{n-1} - \alpha_{n-2}) + x_{(n-1)(n-1)}(0, 0, 0, \dots, 0, \alpha_{n-1}, -\alpha_{n-1}).$$

It then follows that

$$\begin{aligned} \text{vec}(X) = & x_{11} \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\alpha_1 \end{bmatrix} + x_{21} \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\alpha_2 - \alpha_1 \end{bmatrix} + x_{31} \begin{bmatrix} \alpha_3 \\ 0 \\ \alpha_1 \\ \vdots \\ 0 \\ 0 \\ -\alpha_3 - \alpha_1 \end{bmatrix} + \dots \\ & + x_{(n-1)1} \begin{bmatrix} \alpha_{n-1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_1 \\ -\alpha_{n-1} - \alpha_1 \end{bmatrix} + x_{22} \begin{bmatrix} 0 \\ \alpha_2 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\alpha_2 \end{bmatrix} + x_{32} \begin{bmatrix} 0 \\ \alpha_3 \\ \alpha_2 \\ \vdots \\ 0 \\ 0 \\ -\alpha_3 - \alpha_2 \end{bmatrix} + \dots \\ & + x_{(n-1)2} \begin{bmatrix} 0 \\ \alpha_{n-1} \\ 0 \\ \vdots \\ 0 \\ \alpha_2 \\ -\alpha_{n-1} - \alpha_2 \end{bmatrix} + \dots + x_{(n-2)(n-2)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_{n-2} \\ -\alpha_{n-2} \end{bmatrix} \\ & + x_{(n-1)(n-2)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \alpha_{n-1} \\ \alpha_{n-2} \\ -\alpha_{n-1} - \alpha_{n-2} \end{bmatrix} + x_{(n-1)(n-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_{n-1} \\ -\alpha_{n-1} \end{bmatrix}. \end{aligned}$$

Thus we have

$$\text{vec}(X) = K_S \text{vec}_S(X).$$

Conversely, if the matrix  $X \in \mathbf{R}^{n \times n}$  satisfies  $\text{vec}(X) = K_S \text{vec}_S(X)$ , then it is easy to see that  $X \in \mathbf{SIA}^{n \times n}$ .

2) For  $X \in \mathbf{AIA}^{n \times n}$ ,  $X$  can be expressed as

$$X = \begin{bmatrix} 0 & -x_{21} & \dots & -x_{(n-1)1} & \sum_{i=1}^{n-1} a_{i1} \\ x_{21} & 0 & \dots & -x_{(n-1)2} & -x_{21} + \dots + x_{(n-1)2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(n-1)1} & x_{(n-1)2} & \dots & 0 & -\sum_{j=1}^{n-1} a_{(n-1)j} \\ -\sum_{i=1}^{n-1} a_{i1} & -(x_{21} + x_{22} + \dots + x_{(n-1)2}) & \dots & -\sum_{j=1}^{n-1} a_{(n-1)j} & 0 \end{bmatrix}.$$

It then follows that

$$\begin{aligned} \text{vec}(X) = & x_{21} \begin{bmatrix} \alpha_2 \\ -\alpha_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \alpha_1 - \alpha_2 \end{bmatrix} + x_{31} \begin{bmatrix} \alpha_3 \\ 0 \\ -\alpha_1 \\ \vdots \\ 0 \\ 0 \\ \alpha_1 - \alpha_3 \end{bmatrix} + \dots + x_{(n-1)1} \begin{bmatrix} \alpha_{n-1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\alpha_1 \\ \alpha_1 - \alpha_{n-1} \end{bmatrix} \\ & + x_{32} \begin{bmatrix} 0 \\ \alpha_3 \\ -\alpha_2 \\ \vdots \\ 0 \\ 0 \\ \alpha_2 - \alpha_3 \end{bmatrix} + \dots + x_{(n-1)2} \begin{bmatrix} 0 \\ \alpha_{n-1} \\ 0 \\ \vdots \\ 0 \\ -\alpha_2 \\ \alpha_2 - \alpha_{n-1} \end{bmatrix} + \dots + x_{(n-1)(n-2)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \alpha_{n-1} \\ -\alpha_{n-2} \\ \alpha_{n-2} - \alpha_{n-1} \end{bmatrix}. \end{aligned}$$

Thus we have

$$\text{vec}(X) = K_A \text{vec}_A(X).$$

Conversely, if the matrix  $X \in \mathbf{R}^{n \times n}$  satisfies  $\text{vec}(X) = K_A \text{vec}_A(X)$ , then it is easy to see that  $X \in \mathbf{AIA}^{n \times n}$ . The proof is completed.

**Theorem 4.** Suppose  $X = \text{Re } X + \sqrt{-1} \text{Im } X \in \mathbf{C}^{n \times n}$ , then

$$X \in \mathbf{HIA}^{n \times n} \Leftrightarrow \text{vec}(X) = K_S \text{vec}_S(\text{Re } X) + \sqrt{-1} K_A \text{vec}_A(\text{Im } X) \quad (13)$$

where  $\text{vec}_S(\text{Re } X)$  and  $\text{vec}_A(\text{Im } X)$  are represented as (2) and (3), and the matrix  $K_S, K_A$  are in the forms (10) and (12).

**Proof.** For  $X = \text{Re } X + \sqrt{-1} \text{Im } X \in \mathbf{HIA}^{n \times n}$ , then  $X^H = X$ , we have

$$(\text{Re } X + \sqrt{-1} \text{Im } X)^H = \text{Re } X + \sqrt{-1} \text{Im } X.$$

Thus we can get  $(\text{Re } X)^T = \text{Re } X, (\text{Im } X)^T = -\text{Im } X$ . Then  $\text{Re } X \in \mathbf{SIA}^{n \times n}$  and  $\text{Im } X \in \mathbf{AIA}^{n \times n}$ . By (9) and (11),

$$\begin{aligned} \text{vec}(X) &= \text{vec}(\text{Re } X) + \sqrt{-1} \text{vec}(\text{Im } X) \\ &= K_S \text{vec}_S(\text{Re } X) + \sqrt{-1} K_A \text{vec}_A(\text{Im } X). \end{aligned}$$

Conversely, if the matrix  $X \in \mathbf{C}^{n \times n}$  satisfies  $\text{vec}(X) = K_S \text{vec}_S(\text{Re } X) + \sqrt{-1} K_A \text{vec}_A(\text{Im } X)$ , then it is easy to see that  $X \in \mathbf{HIA}^{n \times n}$ . The proof is completed.

We now consider Problem I by using the Moore-Penrose generalized inverse and Kronecker product of matrices.

**Theorem 5.** Given  $A \in \mathbf{C}^{m \times n}$ ,  $B \in \mathbf{C}^{n \times s}$ ,  $C \in \mathbf{C}^{m \times n}$ ,  $D \in \mathbf{C}^{n \times s}$ ,  $E \in \mathbf{C}^{m \times s}$ , and  $F \in \mathbf{C}^{m \times s}$ ,  $K_S, K_A$  are defined as (10) and (12),  $\text{vec}(X), \text{vec}_S(X), \text{vec}_A(X)$  are defined as (1), (2) and (3). Then the set  $H_L$  of the problem can be expressed as

$$H_L = \left\{ X \mid \text{vec}(X) = \left( K_S, \sqrt{-1} K_A \right) \left[ P_0^+ k_0 + (I - P_0^+ P_0) y \right] \right\}, \quad (14)$$

where

$$P = \begin{bmatrix} (B^T \otimes A)K_S & \sqrt{-1}(B^T \otimes A)K_A \\ (D^T \otimes C)K_S & \sqrt{-1}(D^T \otimes C)K_A \end{bmatrix},$$

$$k = \begin{bmatrix} \text{vec}(E) \\ \text{vec}(F) \end{bmatrix}, \quad P_0 = \begin{bmatrix} \text{Re } P \\ \text{Im } P \end{bmatrix}, \quad k_0 = \begin{bmatrix} \text{Re } k \\ \text{Im } k \end{bmatrix},$$

where  $y$  is an arbitrary vector.

Furthermore, the unique least squares Hermitian indeterminate admittance solution with the least norm  $X_H \in H_L$  can be expressed as

$$\text{vec}(X_H) = \begin{bmatrix} K_S & \sqrt{-1}K_A \end{bmatrix} P_0^+ k_0. \tag{15}$$

**Proof.** By Theorem 4, we can get

$$\begin{aligned} & \|AXB - E\|^2 + \|CXD - F\|^2 \\ &= \|\text{vec}(AXB) - \text{vec}(E)\|_2^2 + \|\text{vec}(CXD) - \text{vec}(F)\|_2^2 \\ &= \|(B^T \otimes A)\text{vec}(X) - \text{vec}(E)\|_2^2 + \|(D^T \otimes C)\text{vec}(X) - \text{vec}(F)\|_2^2 \\ &= \|(B^T \otimes A)\text{vec}(\text{Re } X) + \sqrt{-1}(B^T \otimes A)\text{vec}(\text{Im } X) - \text{vec}(E)\|_2^2 \\ &\quad + \|(D^T \otimes C)\text{vec}(\text{Re } X) + \sqrt{-1}(D^T \otimes C)\text{vec}(\text{Im } X) - \text{vec}(F)\|_2^2 \\ &= \left\| \begin{bmatrix} B^T \otimes A & \sqrt{-1}(B^T \otimes A) \\ D^T \otimes C & \sqrt{-1}(D^T \otimes C) \end{bmatrix} \begin{bmatrix} \text{vec}(\text{Re } X) \\ \text{vec}(\text{Im } X) \end{bmatrix} - \begin{bmatrix} \text{vec}(E) \\ \text{vec}(F) \end{bmatrix} \right\|_2^2 \\ &= \left\| \begin{bmatrix} (B^T \otimes A)K_S & \sqrt{-1}(B^T \otimes A)K_A \\ (D^T \otimes C)K_S & \sqrt{-1}(D^T \otimes C)K_A \end{bmatrix} \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - \begin{bmatrix} \text{vec}(E) \\ \text{vec}(F) \end{bmatrix} \right\|_2^2 \\ &= \left\| P \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - k \right\|_2^2 \\ &= \left\| \begin{bmatrix} \text{Re } P + \sqrt{-1} \text{Im } P \\ \text{Re } P - \sqrt{-1} \text{Im } P \end{bmatrix} \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - \begin{bmatrix} \text{Re } k + \sqrt{-1} \text{Im } k \\ \text{Re } k - \sqrt{-1} \text{Im } k \end{bmatrix} \right\|_2^2 \\ &= \left\| \begin{bmatrix} \text{Re } P \\ \text{Im } P \end{bmatrix} \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - \begin{bmatrix} \text{Re } k \\ \text{Im } k \end{bmatrix} \right\|_2^2 \\ &= \left\| \begin{bmatrix} \text{Re } P \\ \text{Im } P \end{bmatrix} \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - \begin{bmatrix} \text{Re } k \\ \text{Im } k \end{bmatrix} \right\|_2^2 \\ &= \left\| \begin{bmatrix} \text{Re } P \\ \text{Im } P \end{bmatrix} \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - \begin{bmatrix} \text{Re } k \\ \text{Im } k \end{bmatrix} \right\|_2^2 \\ &= \left\| P_0 \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - k_0 \right\|_2^2. \end{aligned}$$

Thus, by Lemma 2,

$$\begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} = P_0^+ k_0 + (I - P_0^+ P_0) y.$$

By Theorem 2, it follows that

$$\text{vec}(X) = (K_S, \sqrt{-1}K_A) [P_0^+ k_0 + (I - P_0^+ P_0) y].$$

Thus we have

$$\text{vec}(X_H) = [K_S, \sqrt{-1}K_A] P_0^+ k_0.$$

The proof is completed.

We now discuss the consistency of the complex matrix Equation (5). By Lemma 1 and Theorem 3, we can get the following conclusions.

**Corollary 6.** *The matrix Equation (5) has a solution  $X \in \mathbf{HIA}^{n \times n}$  if and only if*

$$P_0 P_0^+ k_0 = k_0. \tag{16}$$

In this case, denote by  $H_E$  the solution set of (5). Then

$$H_E = \left\{ X \mid \text{vec}(X) = (K_S, \sqrt{-1}K_A) [P_0^+ k_0 + (I - P_0^+ P_0) y] \right\}. \tag{17}$$

Furthermore, if (16) holds, then the matrix Equation (5) has a unique solution  $X \in H_E$  if and only if

$$\text{rank}(P) = n^2 - 2n + 1. \tag{18}$$

In this case,

$$H_E = \left\{ X \mid \text{vec}(X) = (K_S, \sqrt{-1}K_A) P_0^+ k_0 \right\}. \tag{19}$$

The least norm problem

$$\|X_H\| = \min_{X \in H_E} \|X\|$$

has a unique solution  $X_H \in H_E$  and  $X_H$  can be expressed as (15).

### 3. Method II for the Solution of Problem I

The method for solving Problem I used in this section is from [26]. We concisely recall it as follows.

**Definition 5.** Let  $x = (x_1, x_2, \dots, x_k)^T \in \mathbf{C}^k$ ,  $y = (y_1, y_2, \dots, y_k)^T \in \mathbf{C}^k$  and  $A = (A_1, A_2, \dots, A_k)$ ,  $A_i \in \mathbf{C}^{m \times n}$  ( $i = 1, 2, \dots, k$ ). Define

- 1)  $A \odot x = x_1 A_1 + x_2 A_2 + \dots + x_k A_k \in \mathbf{C}^{m \times n}$ ;
- 2)  $A \odot (x, y) = (A \odot x, A \odot y)$ .

Let  $P = (P_1, P_2, \dots, P_k)$ ,  $P_i \in \mathbf{C}^{m \times n}$  ( $i = 1, 2, \dots, k$ ), and  $a, b \in \mathbf{C}$ . By Definition 5, we have the following facts which are useful in this paper.

- 1)  $x^H \odot y = x^H y = (x, y)$ ;
- 2)  $A \odot x + P \odot x = (A + P) \odot x$ ;
- 3)  $A \odot (ax + by) = a(A \odot x) + b(A \odot y)$ ;



4)  $(aA + bB) \odot x = a(A \odot x) + b(B \odot x);$

5)  $(A, B) \odot \begin{bmatrix} x \\ y \end{bmatrix} = A \odot x + B \odot y;$

6)  $x \odot y$  is no meaning.

Suppose  $B = (B_1, B_2, \dots, B_s) \in \mathbf{C}^{k \times s}$ ,  $B_i \in \mathbf{C}^k$ ,  $(i = 1, 2, \dots, s)$ ,  
 $C = (C_1, C_2, \dots, C_t) \in \mathbf{C}^{k \times t}$ ,  $C_i \in \mathbf{C}^k$ ,  $(i = 1, 2, \dots, t)$ ,  $D \in \mathbf{C}^{l \times m}$ . Then

7)  $A \odot B = (A \odot B_1, A \odot B_2, \dots, A \odot B_s);$

8)  $A \odot (B, C) = (A \odot B, A \odot C);$

9)  $D(A \odot x) = (DA) \odot x;$

10)  $(A \odot x)B = (A_1B, A_2B, \dots, A_kB) \odot x.$

Suppose  $E \in \mathbf{C}^{s \times m}$ ,  $F = (F_1, F_2, \dots, F_k) \in \mathbf{C}^{m \times kn}$ ,  $F_i \in \mathbf{C}^{m \times n}$ ,  $(i = 1, 2, \dots, k)$ ,  
 $G \in \mathbf{C}^{n \times s}$ . Then

11)  $E[(F_1, F_2, \dots, F_k) \odot x]G = (EF_1G, EF_2G, \dots, EF_kG) \odot x.$

Suppose  $M_1 \in \mathbf{C}^{m \times s}$ ,  $M_2 \in \mathbf{C}^{m \times t}$ ,  $M_3 \in \mathbf{C}^{n \times s}$ ,  $M_4 \in \mathbf{C}^{n \times t}$ .  $x_1, x_2 \in \mathbf{C}$ . Then

12)  $\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (M_1, M_2) \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ (M_3, M_4) \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix};$

13)  $\begin{bmatrix} M_1 & 0 \\ 0 & M_4 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M_1x_1 \\ M_4x_2 \end{bmatrix}.$

**Lemma 7.** ([26]) Given matrices  $A_1, A_2, \dots, A_l \in \mathbf{C}^{m \times n}$  and  $B \in \mathbf{C}^{m \times n}$ , let

$$D_i = \begin{bmatrix} \text{Re}(A_i) \\ \text{Im}(A_i) \end{bmatrix}, \quad B_0 = \begin{bmatrix} \text{Re}(B) \\ \text{Im}(B) \end{bmatrix}. \tag{20}$$

Let  $k = (k_1, k_2, \dots, k_l)^T \in \mathbf{R}^n$  that satisfies

$$k_1A_1 + k_2A_2 + \dots + k_lA_l = B. \tag{21}$$

If the matrix Equation (21) is consistent, then the solution set of the matrix Equation (21) is exactly the solution set of the following consistent system

$$\begin{bmatrix} \langle D_1, D_1 \rangle & \langle D_1, D_2 \rangle & \dots & \langle D_1, D_l \rangle \\ \langle D_2, D_1 \rangle & \langle D_2, D_2 \rangle & \dots & \langle D_2, D_l \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle D_l, D_1 \rangle & \langle D_l, D_2 \rangle & \dots & \langle D_l, D_l \rangle \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_l \end{bmatrix} = \begin{bmatrix} \langle D_1, B_0 \rangle \\ \langle D_2, B_0 \rangle \\ \vdots \\ \langle D_l, B_0 \rangle \end{bmatrix}. \tag{22}$$

**Lemma 8.** ([26]) Given  $B \in \mathbf{C}^{m \times n}$  and the matrices  $A_1, A_2, \dots, A_l \in \mathbf{C}^{m \times n}$ , let

$$k = (k_1, k_2, \dots, k_l)^T \in \mathbf{R}^n$$

such that

$$\|k_1A_1 + k_2A_2 + \dots + k_lA_l - B\| = \min \tag{23}$$

Then the solution set of (23) is the solution set of the system (22).

We now analyze the structure of the complex matrix equation  $(AXB, CXD) = (E, F)$  over  $X \in \mathbf{HIA}^{n \times n}$  with the new product that we have presented.

Let

$$E_{ij} = (e_{st}) \in \mathbf{R}^{n \times n}, \tag{24}$$

where

$$e_{st} = \begin{cases} 1 & (s, t) = (i, j), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$L_S = (E_{11} - E_{n1} - E_{1n} + E_{nn}, E_{21} + E_{12} - E_{n1} - E_{2n} - E_{1n} - E_{n2} + E_{nn}, \dots, E_{n-1,1} + E_{1,n-1} - E_{n1} - E_{n-1,n} - E_{n,n-1} - E_{1n} + E_{nn}, E_{22} - E_{2n} - E_{n2} + E_{nn}, \dots, E_{n-1,n-1} - E_{n,n-1} - E_{n-1,n} + E_{nn}). \quad (25)$$

Note that  $L_S \in \mathbf{R}^{n \times \frac{n^2(n-1)}{2}}$ .

Let

$$L_A = (E_{11} + E_{22} + \dots + E_{nn}, E_{21} - E_{12} - E_{n1} - E_{2n} + E_{n2} + E_{1n}, E_{31} - E_{13} - E_{n1} - E_{3n} + E_{1n} + E_{n3}, \dots, E_{n-1,1} - E_{1,n-1} - E_{n1} - E_{n-1,n} + E_{1n} + E_{n,n-1}, E_{32} - E_{23} - E_{3n} - E_{n2} + E_{2n} + E_{n3}, \dots, E_{n-1,n-2} - E_{n-2,n-1} - E_{n-1,n} - E_{n,n-2} + E_{n,n-1} + E_{n-2,n}). \quad (26)$$

Note that  $L_A \in \mathbf{R}^{n \times \frac{n(n-2)(n-1)+n}{2}}$ . We can get the following lemmas.

**Lemma 9.** Suppose  $X \in \mathbf{R}^{n \times n}$ , then

$$X \in \mathbf{SIA}^{n \times n} \Leftrightarrow X = L_S \odot \text{vec}_S(X), \quad (27)$$

where  $\text{vec}_S(X)$  is represented as (2), and the matrix  $L_S$  is in the form (25).

**Lemma 10.** Suppose  $X \in \mathbf{R}^{n \times n}$ , then

$$X \in \mathbf{AIA}^{n \times n} \Leftrightarrow X = L_A \odot \text{vec}_A(X), \quad (28)$$

where  $\text{vec}_A(X)$  is represented as (3), and the matrix  $L_A$  is in the form (26).

**Lemma 11.** Suppose  $X = \text{Re } X + \sqrt{-1} \text{Im } X \in \mathbf{C}^{n \times n}$ , then

$$X \in \mathbf{HIA}^{n \times n} \Leftrightarrow X = L_S \odot \text{vec}_S(\text{Re } X) + \sqrt{-1} L_A \odot \text{vec}_A(\text{Im } X), \quad (29)$$

where  $\text{vec}_S(\text{Re } X)$  and  $\text{vec}_A(\text{Im } X)$  are represented as (2) and (3). The matrix  $L_S, L_A$  are in the form (25) and (26).

**Theorem 12.** Suppose  $A \in \mathbf{C}^{m \times n}$ ,  $B \in \mathbf{C}^{n \times s}$ ,  $C \in \mathbf{C}^{m \times n}$ ,  $D \in \mathbf{C}^{n \times s}$  and  $X = \text{Re } X + \sqrt{-1} \text{Im } X \in \mathbf{HIA}^{n \times n}$ . Let  $A = (A_1, A_2, \dots, A_n)$  and  $C = (C_1, C_2, \dots, C_n)$ , where  $A_i \in \mathbf{C}^m$  is the  $i$ th column vector of matrix  $A$ , and  $C_i \in \mathbf{C}^m$  is the  $i$ th column vector of matrix  $C$ ,

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix}, \quad (30)$$

where  $B_j \in \mathbf{C}^s$  is the  $j$ th row vector of matrix  $B$ , and  $D_j \in \mathbf{C}^s$  is the  $j$ th row vector of matrix  $D$ . Then

$$1) \quad AE_{ij}B = A_iB_j \quad (31)$$

2) Let  $F_{ij}, G_{ij} \in \mathbf{C}^{n \times n}$ ,  $(i, j = 1, 2, \dots, n), i \geq j$ , where

$$F_{ij} = \begin{cases} A_i B_j - A_n B_j - A_i B_n + A_n B_n, & i = j, \\ A_i B_j + A_j B_i - A_i B_n - A_n B_j - A_j B_n - A_n B_i + A_n B_n, & i > j, \end{cases}$$

$$G_{ij} = \begin{cases} 0, & i = j, \\ \sqrt{-1}(A_i B_j - A_j B_i - A_n B_j - A_i B_n + A_j B_n + A_n B_i), & i > j. \end{cases}$$

Thus

$$AXB = (F_{11}, F_{21}, \dots, F_{n-1,1}, F_{22}, F_{32}, \dots, F_{n-1,2}, \dots, F_{n-1,n-1}, G_{11}, G_{21}, G_{21}, G_{31}, \dots, \quad (32)$$

$$G_{n-1,1}, G_{32}, \dots, G_{n-1,2}, G_{43}, \dots, G_{n-1,n-2}) \odot \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix}. \quad (33)$$

3) Let  $M_{ij}, N_{ij} \in \mathbb{C}^{n \times n}, (i, j = 1, 2, \dots, n), i \geq j$ , where

$$M_{ij} = \begin{cases} C_i D_j - C_n D_j - C_i D_n + C_n D_n, & i = j, \\ C_i D_j + C_j D_i - C_i D_n - C_n D_j - C_j D_n - C_n D_i + C_n D_n, & i > j, \end{cases}$$

$$N_{ij} = \begin{cases} 0, & i = j, \\ \sqrt{-1}(C_i D_j - C_j D_i - C_n D_j - C_i D_n + C_j D_n + C_n D_i), & i > j. \end{cases}$$

Thus

$$CXD = (M_{11}, M_{21}, \dots, M_{n-1,1}, M_{22}, M_{32}, \dots, M_{n-1,2}, \dots, M_{n-1,n-1}, N_{11}, N_{21}, N_{21}, N_{31}, \quad (34)$$

$$\dots, N_{n-1,1}, N_{32}, \dots, N_{n-1,2}, N_{43}, \dots, N_{n-1,n-2}) \odot \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix}. \quad (35)$$

**Proof. 1)**

$$AE_{ij}B = (A_1, A_2, \dots, A_n)E_{ij}B = (0, \dots, A_i, \dots, 0)B = A_i B_j;$$

2) By (1), Definition 5 and Lemma 7, we can get

$$\begin{aligned} AXB &= A[L_S \odot \text{vec}_S(\text{Re } X) + \sqrt{-1}L_A \odot \text{vec}_A(\text{Im } X)]B \\ &= A[L_S \odot \text{vec}_S(\text{Re } X)]B + \sqrt{-1}A[L_A \odot \text{vec}_A(\text{Im } X)]B \\ &= [(AL_S) \odot \text{vec}_S(\text{Re } X)]B + \sqrt{-1}[(AL_A) \odot \text{vec}_A(\text{Im } X)]B \\ &= [A(E_{11} - E_{n1} - E_{1n} + E_{nn})B, A(E_{21} + E_{12} - E_{n1} - E_{2n} - E_{1n} - E_{n2} + E_{nn})B, \\ &\quad \dots, A(E_{n-1,1} + E_{1,n-1} - E_{n1} - E_{n-1,n} - E_{n,n-1} - E_{1n} + E_{nn})B, A(E_{22} - E_{2n} - E_{n2} \\ &\quad + E_{nn})B, \dots, A(E_{n-1,n-1} - E_{n,n-1} - E_{n-1,n} + E_{nn})B] \odot \text{vec}_S(\text{Re } X) \\ &\quad + \sqrt{-1}[A(E_{11} + E_{22} + \dots + E_{nn})B, A(E_{21} - E_{12} - E_{n1} - E_{2n} + E_{n2} + E_{1n})B, \\ &\quad A(E_{31} - E_{13} - E_{n1} - E_{3n} + E_{1n} + E_{n3})B, \dots, A(E_{n-1,1} - E_{1,n-1} - E_{n1} - E_{n-1,n} \\ &\quad + E_{1n} + E_{n,n-1})B, A(E_{32} - E_{23} - E_{3n} - E_{n2} + E_{2n} + E_{n3})B, \dots, A(E_{n-1,n-2} \\ &\quad - E_{n-2,n-1} - E_{n-1,n} - E_{n,n-2} + E_{n,n-1} + E_{n-2,n})B] \odot \text{vec}_A(\text{Im } X) \\ &= (F_{11}, F_{21}, \dots, F_{n-1,1}, F_{22}, F_{32}, \dots, F_{n-1,2}, \dots, F_{n-1,n-1}) \odot \text{vec}_S(\text{Re } X) \\ &\quad + (G_{11}, G_{21}, G_{31}, \dots, G_{n-1,1}, G_{32}, \dots, G_{n-1,2}, G_{43}, \dots, G_{n-1,n-2}) \odot \text{vec}_A(\text{Im } X) \\ &= (F_{11}, F_{21}, \dots, F_{n-1,1}, F_{22}, F_{32}, \dots, F_{n-1,2}, \dots, F_{n-1,n-1}, G_{11}, G_{21}, G_{21}, G_{31}, \dots, \\ &\quad G_{n-1,1}, G_{32}, \dots, G_{n-1,2}, G_{43}, \dots, G_{n-1,n-2}) \odot \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix}. \end{aligned}$$

3) The proof is similar to that of (2), so we omit it.

The proof is completed.

We now use Lemmas 7 - 11, and Theorem 12 to consider the least squares Hermitian indeterminate admittance solution for the matrix Equation (5). The following notations and lemmas are necessary for deriving the solutions.

For  $A \in \mathbf{C}^{m \times n}$ ,  $B \in \mathbf{C}^{n \times s}$ ,  $C \in \mathbf{C}^{m \times n}$ ,  $D \in \mathbf{C}^{n \times s}$ ,  $E \in \mathbf{C}^{m \times s}$ , and  $F \in \mathbf{C}^{m \times s}$ , let

$$\begin{aligned}
 S_{ij} &= \begin{bmatrix} F_{ij} \\ M_{ij} \end{bmatrix}, \quad T_{ij} = \begin{bmatrix} G_{ij} \\ N_{ij} \end{bmatrix}, \quad R = \begin{bmatrix} E \\ F \end{bmatrix}. \\
 \hat{S}_{ij} &= \begin{bmatrix} \text{Re}(S_{ij}) \\ \text{Im}(S_{ij}) \end{bmatrix}, \quad \hat{T}_{ij} = \begin{bmatrix} \text{Re}(T_{ij}) \\ \text{Im}(T_{ij}) \end{bmatrix}, \quad R_0 = \begin{bmatrix} \text{Re}(R) \\ \text{Im}(R) \end{bmatrix}, \\
 U &= \begin{bmatrix} S & W \\ W^T & T \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},
 \end{aligned} \tag{36}$$

where  $n \geq i \geq j \geq 1$ ,

$$\begin{aligned}
 S &= \begin{bmatrix} \langle \hat{S}_{11}, \hat{S}_{11} \rangle & \langle \hat{S}_{11}, \hat{S}_{21} \rangle & \cdots & \langle \hat{S}_{11}, \hat{S}_{n-1, n-1} \rangle \\ \langle \hat{S}_{21}, \hat{S}_{11} \rangle & \langle \hat{S}_{21}, \hat{S}_{21} \rangle & \cdots & \langle \hat{S}_{21}, \hat{S}_{n-1, n-1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \hat{S}_{n-1, n-1}, \hat{S}_{11} \rangle & \langle \hat{S}_{n-1, n-1}, \hat{S}_{21} \rangle & \cdots & \langle \hat{S}_{n-1, n-1}, \hat{S}_{n-1, n-1} \rangle \end{bmatrix}, \\
 T &= \begin{bmatrix} \langle \hat{T}_{11}, \hat{T}_{21} \rangle & \langle \hat{T}_{11}, \hat{T}_{31} \rangle & \cdots & \langle \hat{T}_{11}, \hat{T}_{n-1, n-2} \rangle \\ \langle \hat{T}_{21}, \hat{T}_{11} \rangle & \langle \hat{T}_{21}, \hat{T}_{21} \rangle & \cdots & \langle \hat{T}_{21}, \hat{T}_{n-1, n-2} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \hat{T}_{n-1, n-2}, \hat{T}_{11} \rangle & \langle \hat{T}_{n-1, n-2}, \hat{T}_{21} \rangle & \cdots & \langle \hat{T}_{n-1, n-2}, \hat{T}_{n-1, n-2} \rangle \end{bmatrix}, \\
 W &= \begin{bmatrix} \langle \hat{S}_{11}, \hat{T}_{11} \rangle & \langle \hat{S}_{11}, \hat{T}_{21} \rangle & \cdots & \langle \hat{S}_{11}, \hat{T}_{n-1, n-2} \rangle \\ \langle \hat{S}_{21}, \hat{T}_{11} \rangle & \langle \hat{S}_{21}, \hat{T}_{21} \rangle & \cdots & \langle \hat{S}_{21}, \hat{T}_{n-1, n-2} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \hat{S}_{n-1, n-1}, \hat{T}_{11} \rangle & \langle \hat{S}_{n-1, n-1}, \hat{T}_{21} \rangle & \cdots & \langle \hat{S}_{n-1, n-1}, \hat{T}_{n-1, n-2} \rangle \end{bmatrix}, \\
 v_1 &= \begin{bmatrix} \langle \hat{S}_{11}, R_0 \rangle \\ \langle \hat{S}_{21}, R_0 \rangle \\ \vdots \\ \langle \hat{S}_{n-1, n-1}, R_0 \rangle \end{bmatrix}, \quad v_2 = \begin{bmatrix} \langle \hat{T}_{11}, R_0 \rangle \\ \langle \hat{T}_{21}, R_0 \rangle \\ \vdots \\ \langle \hat{T}_{n-1, n-2}, R_0 \rangle \end{bmatrix}.
 \end{aligned}$$

**Theorem 13.** Let  $A \in \mathbf{C}^{m \times n}$ ,  $B \in \mathbf{C}^{n \times s}$ ,  $C \in \mathbf{C}^{m \times n}$ ,  $D \in \mathbf{C}^{n \times s}$ ,  $E \in \mathbf{C}^{m \times s}$ , and  $F \in \mathbf{C}^{m \times s}$ ,  $L_S, L_A$  are defined as (25) and (26),  $\text{vec}_S(X), \text{vec}_A(X)$  are defined as (2) and (3). let  $U, v$  be as in (36). Then  $H_L$  can be expressed as

$$H_L = \left\{ X \mid X = (L_S, \sqrt{-1}L_A) \odot [U^+v + (I - U^+U)y] \right\}, \tag{37}$$

where  $y$  is an arbitrary vector.

Furthermore, the unique least squares Hermitian indeterminate admittance solution with the least norm  $X_H \in H_L$  can be expressed as

$$X_H = (L_S, \sqrt{-1}L_A) \odot (U^+v) \tag{38}$$

**Proof.** By Theorem 4, we can get

$$\begin{aligned} & \|AXB - E\|^2 + \|CXD - F\|^2 \\ &= \left\| \begin{bmatrix} AXB - E \\ CXD - F \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} AXB \\ CXD \end{bmatrix} - \begin{bmatrix} E \\ F \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} (F_{11}, F_{21}, \dots, F_{n-1, n-1}, G_{11}, G_{21}, G_{31}, \dots, G_{n-1, n-2}) \odot \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} \\ (M_{11}, M_{21}, \dots, M_{n-1, n-1}, N_{11}, N_{21}, N_{31}, \dots, N_{n-1, n-2}) \odot \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} \end{bmatrix} - R \right\|^2 \\ &= \left\| (S_{11}, S_{21}, \dots, S_{n-1, n-1}, T_{11}, T_{21}, T_{31}, \dots, T_{n-1, n-2}) \odot \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} - R \right\|^2. \end{aligned}$$

Then by Lemma 11, the least squares problem

$$\|AXB - E\|^2 + \|CXD - F\|^2 = \min$$

with respect to the Hermitian indeterminate admittance matrix  $X$  is equivalent to the following consistent matrix equation

$$U \begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} = v.$$

Thus, by Lemma 2,  $\|AXB - E\|^2 + \|CXD - F\|^2 = \min$  if and only if

$$\begin{bmatrix} \text{vec}_S(\text{Re } X) \\ \text{vec}_A(\text{Im } X) \end{bmatrix} = U^+v + (I - U^+U)y.$$

From Lemma 11, it follows that

$$X = (L_S, \sqrt{-1}L_A) \odot [U^+v + (I - U^+U)y].$$

where  $y$  is an arbitrary vector. it yields that

$$X_H = (L_S, \sqrt{-1}L_A) \odot (U^+v).$$

The proof is completed.

We now discuss the consistency of the complex matrix Equation (5). By Lemma 1 and Theorem 13, we can get the following conclusions.

**Corollary 14.** The matrix Equation (5) has a solution  $X \in \mathbf{HIA}^{n \times n}$  if and only if

$$UU^+v = v. \tag{39}$$

In this case, denote by  $H_E$  the solution set of(5). Then

$$H_E = \left\{ X \mid X = (L_S, \sqrt{-1}L_A) \odot [U^+v + (I - U^+U)y] \right\}, \tag{40}$$

where  $y$  is an arbitrary vector.

Furthermore, if (39) holds, then the matrix Equation (5) has a unique solution  $X \in H_E$  if and only if

$$\text{rank}(U) = \frac{n(n-1)}{2} + \frac{(n-1)(n-2)+1}{2}. \quad (41)$$

In this case,

$$H_E = \left\{ X \mid X = (L_S, \sqrt{-1}L_A) \odot (U^+v) \right\}. \quad (42)$$

The least norm problem

$$\|X_H\| = \min_{X \in H_E} \|X\|$$

has a unique solution  $X_H \in H_E$  and  $X_H$  can be expressed as (38).

## 4. Conclusion

In this paper, we mainly consider the least squares Hermitian indeterminate admittance problem of the complex matrix equation  $(AXB, CXD) = (E, F)$ . We derive the explicit solution of this complex matrix equation over  $X \in \mathbf{HIA}^{n \times n}$ . The paper provide a direct method to solve the least squares admittance problem of complex matrix equation  $(AXB, CXD) = (E, F)$ . More works such as iterative methods, error analysis and numerical stability need to be investigated in future.

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## References

- [1] Yuan, S.-F., Liao, A.-P. and Lei, Y. (2007) Least Squares Symmetric Solution of the Matrix Equation  $AXB + CYD = E$  with the Least Norm. *Mathematica Numerica Sinica*, **29**, 203-216. (In Chinese)
- [2] Elena, A.S., Carlos, C.P. and Teresa, M.M. (2017) Equivalent Circuits for Nonsymmetric Reciprocal Two Ports Based on Eigenstate Formulation. *IEEE Transactions on Microwave Theory and Techniques*, **65**, 4812-4822. <https://doi.org/10.1109/TMTT.2017.2708103>
- [3] Haigh, D.G., Clarke, T.J.W. and Radmore, P.M. (2006) Symbolic Framework for Linear Active Circuits Based on Port Equivalence Using Limit Variables. *IEEE Transactions on Circuits and Systems I*, **53**, 2011-2024. <https://doi.org/10.1109/TCSI.2006.882815>
- [4] Ben-Israel, A. and Greville, T.N.E. (1974) *Generalized Inverses: Theory and Applications*. John Wiley and Sons, New York.
- [5] Chen, J.-L. and Chen, X.-H. (2001) *Special Matrices*. Qinghua University Press, Beijing. (In Chinese)
- [6] Dehghan, M. and Hajarian, M. (2014) Finite Iterative Methods for Solving Systems of Linear Matrix Equations over Reflexive and Anti-Reflexive Matrices. *Bulletin of the Iranian Mathematical Society*, **40**, 295-323.

- [7] Dehghan, M. and Hajarian, M. (2013) Construction of an Efficient Iterative Method for Solving Generalized Coupled Sylvester Matrix Equations. *Transactions of the Institute of Measurement and Control*, **35**, 961-970. <https://doi.org/10.1177/0142331212465105>
- [8] Dehghan, M. and Hajarian, M. (2012) Iterative Algorithms for the Generalized Centro-Symmetric and Central Anti-Symmetric Solutions of General Coupled Matrix Equation. *Engineering Computations*, **29**, 528-560. <https://doi.org/10.1108/02644401211235870>
- [9] Dehghan, M. and Hajarian, M. (2012) Solving Coupled Matrix Equations over Generalized Bisymmetric Matrices. *International Journal of Control, Automation and Systems*, **10**, 9005-9012. <https://doi.org/10.1007/s12555-012-0506-2>
- [10] Dehghan, M. and Hajarian, M. (2012) On the Generalized Reflexive and Anti-Reflexive Solutions to a System of Matrix Equations. *Linear Algebra and Its Applications*, **437**, 2793-2812. <https://doi.org/10.1016/j.laa.2012.07.004>
- [11] Dehghan, M. and Hajarian, M. (2011) The (R, S)-Symmetric and (R, S)-Skew Symmetric Solutions of the Pair of Matrix Equations  $A_1XB_1 = C_1$  and  $A_2XB_2 = C_2$ . *Bulletin of the Iranian Mathematical Society*, **37**, 273-283.
- [12] Hajarian, M. and Dehghan, M. (2011) The Generalized Centro-Symmetric and Least Squares Generalized Centro-Symmetric Solutions of the Matrix Equation  $AYB + CY^TD = E$ . *Mathematical Methods in the Applied Sciences*, **34**, 1562-1579. <https://doi.org/10.1002/mma.1459>
- [13] Hernández, V. and Gassó, M. (1989) Explicit Solution of the Matrix Equation  $AXB - CXD = E$ . *Linear Algebra and Its Applications*, **121**, 333-344. [https://doi.org/10.1016/0024-3795\(89\)90708-8](https://doi.org/10.1016/0024-3795(89)90708-8)
- [14] Liao, A.-P., Bai, Z.-Z. and Lei, Y. (2006) Best Approximate Solution of Matrix Equation  $AXB + CYD = E$ . *SIAM Journal on Matrix Analysis and Applications*, **27**, 675-688. <https://doi.org/10.1137/040615791>
- [15] Liao, A.-P. and Lei, Y. (2005) Least Squares Solution with the Minimum-Norm for the Matrix Equation  $(AXB, GXH) = (C, D)$ . *Computers & Mathematics with Applications*, **50**, 539-549. <https://doi.org/10.1016/j.camwa.2005.02.011>
- [16] Magnus, J.R. (1983) L-Structured Matrices and Linear Matrix Equations. *Linear and Multilinear Algebra*, **14**, 67-88. <https://doi.org/10.1080/03081088308817543>
- [17] Mansour, A. (2010) Solvability of  $AXB - CXD = E$  in the Operators Algebra  $B(H)$ . *Lobachevskii Journal of Mathematics*, **31**, 257-261. <https://doi.org/10.1134/S1995080210030091>
- [18] Hajarian, M. (2014) Matrix form of the CGS Method for Solving General Coupled Matrix Equations. *Applied Mathematics Letters*, **34**, 37-42. <https://doi.org/10.1016/j.aml.2014.03.013>
- [19] Mitra, S.K. (1977) The Matrix Equation  $AXB + CXD = E$ . *SIAM Journal on Applied Mathematics*, **32**, 823-825. <https://doi.org/10.1137/0132070>
- [20] Peng, Z.-Y. and Peng, Y.-X. (2006) An Efficient Iterative Method for Solving the Matrix Equation  $AXB + CXD = E$ . *Numerical Linear Algebra with Applications*, **13**, 473-485. <https://doi.org/10.1002/nla.470>
- [21] Sheng, X.-P. and Chen, G.-L. (2010) An Iterative Method for the Symmetric and Skew Symmetric Solutions of a Linear Matrix Equation  $AXB + CXD = E$ . *Journal of Computational and Applied Mathematics*, **233**, 3030-3040. <https://doi.org/10.1016/j.cam.2009.11.052>
- [22] Shi, S.Y. and Chen, Y. (2003) Least Squares Solution of Matrix Equation  $AXB +$

$CXD = E$ . *SIAM Journal on Matrix Analysis and Applications*, **24**, 802-808.

<https://doi.org/10.1137/S0895479802401059>

- [23] Tian, Y.-G. (2000) The Solvability of Two Linear Matrix Equations. *Linear and Multilinear Algebra*, **48**, 123-147. <https://doi.org/10.1080/03081080008818664>
- [24] Wang, Q.-W. and He, Z.-H. (2013) Solvability Conditions and General Solution for Mixed Sylvester Equations. *Automatica*, **49**, 2713-2719. <https://doi.org/10.1016/j.automatica.2013.06.009>
- [25] Xu, G.-P., Wei, M.-S. and Zheng, D.-S. (1998) On Solutions of Matrix Equation  $AXB + CXD = F$ . *Linear Algebra and Its Applications*, **279**, 93-109. [https://doi.org/10.1016/S0024-3795\(97\)10099-4](https://doi.org/10.1016/S0024-3795(97)10099-4)
- [26] Yuan, S.-F. and Liao, A.-P. (2014) Least Squares Hermitian Solution of the Complex Matrix Equation  $AXB + CXD = E$  with the Least Norm. *Journal of the Franklin Institute*, **351**, 4978-4997. <https://doi.org/10.1016/j.jfranklin.2014.08.003>