

BTZ Quasinormal Frequencies as Poles of Green's Function

Alfredo López-Ortega*, Daniel Mata-Pacheco

Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Mexico City, México

Email: *alopez@ipn.mx

How to cite this paper: López-Ortega, A. and Mata-Pacheco, D. (2018) BTZ Quasinormal Frequencies as Poles of Green's Function. *Journal of Applied Mathematics and Physics*, 6, 1170-1178.

<https://doi.org/10.4236/jamp.2018.66099>

Received: May 7, 2018

Accepted: June 8, 2018

Published: June 11, 2018

Copyright © 2018 by authors and Scientific Research Publishing Inc. This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

Based on the well known fact that the quasinormal frequencies are the poles of the frequency domain Green's function we describe a method that allows us to calculate exactly the quasinormal frequencies of the Klein-Gordon field moving in the three-dimensional rotating Bañados-Teitelboim-Zanelli black hole. These quasinormal frequencies are already published and widely explored in several applications, but we use this example to expound the proposed method of computation. We think that the described procedure can be useful to calculate exactly the quasinormal frequencies of classical fields propagating in other backgrounds. Furthermore, we compare with previous results and discuss some related facts.

Keywords

Quasinormal Modes, Klein-Gordon Field, BTZ Black Hole

1. Introduction

Several dynamical aspects of the black hole perturbations are encoded in their quasinormal modes (QNM), namely in the solutions to their equations of motion that are purely ingoing near the event horizon. Also, in the far region they satisfy a boundary condition determined by the asymptotic structure of the spacetime, for example, for asymptotically flat black holes, it is imposed that the perturbation is purely outgoing in the far region, whereas for asymptotically anti-de Sitter black holes a Dirichlet type boundary condition is commonly imposed in the far region [1], [2]. Owing to these boundary conditions, the QNM possess complex frequencies, the quasinormal frequencies (QNF). As is well known, the real part of the QNF is the oscillation frequency of the perturbation and its imaginary part determines the decay time [1], [2].

Motivated by the AdS-CFT correspondence, the QNF spectrum of asymptotically anti-de Sitter spacetimes has been studied, because they determine the decay times of the dual quantum field theories living at the boundary [3], [4]. As a consequence, the QNF spectrum of asymptotically anti-de Sitter black holes is widely explored, [1], [2], and we find numerical computations in many references, see for example Refs. [3], [5]-[16], whereas in Refs. [4], [17]-[28], we encounter exact calculations of the QNF for several test fields.

Among the asymptotically anti-de Sitter solutions, the three-dimensional Bañados-Teitelboim-Zanelli black hole (BTZ black hole in what follows) [29], [30] plays a special role, since in this background we can obtain several exact results [31]. An example is the exact determination of the QNF spectra for the Klein-Gordon, Dirac, and massive vector fields propagating on it [4], [17], [18]. Here we use the widely known example of exact calculation for the QNF of the Klein-Gordon field moving in the BTZ black hole to describe an alternative method to compute exactly the proper oscillations of classical fields propagating in black holes. The proposed method is based on the work by Leaver of Ref. [32] that defines the QNF as the poles of the frequency domain Green's function. As far as we know this method is not previously exploited to compute exactly the QNF of black holes and in what follows we describe this procedure.

2. QNF of the BTZ Black Hole

As is well known, in the coordinates (t, r, ϕ) the metric of the three-dimensional rotating BTZ black hole is¹ [29], [30]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2, \quad (1)$$

where the function f is equal to

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad (2)$$

the factor l^2 is related to the negative cosmological constant by $\Lambda = -1/l^2$ and we point out that the mass M and the angular momentum J of the black hole are determined by

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+r_-}{l}, \quad (3)$$

with r_{\pm} denoting the outer and inner radii of the black hole [29], [30].

In analogy to Refs. [17], [18], we define the QNM of the BTZ black hole (1) as the modes that satisfy the boundary conditions:

- 1) The field is ingoing near the outer horizon.
- 2) The field goes to zero as $r \rightarrow \infty$.

For the cases in which an exact solution of the radial equation is available, the usual method for computing exactly the QNF is to impose the boundary conditions on the radial function and then we calculate exactly the QNF from the

¹As in Ref. [18], we choose units such that $8G = 1$.

conditions on the radial solutions [17]-[27]. Nevertheless a different method based on the results of Ref. [32] is possible and in what follows we describe this procedure. Since the work by Leaver of Ref. [32], it is known that the QNF correspond to the poles of Green’s function in the frequency domain (see also Refs. [1] [2]). Based on this picture and on the results by Birmingham [18] for the QNM of the Klein-Gordon field propagating in the three-dimensional rotating BTZ black hole [29], [30] we calculate exactly their QNF and compare with the previous results of Ref. [18].² As far as we know this method is not used previously in the exact computation of the QNF for asymptotically anti-de Sitter black holes [1], [2], [17]-[28].

For the Schrödinger type equation [1], [2]

$$\frac{d^2\Phi}{dx^2} + \omega^2\Phi - V\Phi = 0, \tag{4}$$

where Φ is related to the radial function of the field, ω denotes the frequency and V is the effective potential that is characteristic of the field under study, it is well known that its frequency domain Green’s function is equal to [32]

$$\tilde{G}(x_1, x_2, \omega) = \begin{cases} \frac{\Phi_+(\omega, x_1) \Phi_\infty(\omega, x_2)}{W_x(\omega)}, & x_1 < x_2; \\ \frac{\Phi_+(\omega, x_2) \Phi_\infty(\omega, x_1)}{W_x(\omega)}, & x_2 < x_1, \end{cases} \tag{5}$$

where Φ_+ is the solution of the Schrödinger type equation satisfying the boundary condition near the horizon of the black hole. In a similar way Φ_∞ fulfills the boundary condition as $r \rightarrow \infty$, and

$$W_x(\Phi_\infty, \Phi_+) = W_x(\omega) = \Phi_\infty \frac{d\Phi_+}{dx} - \Phi_+ \frac{d\Phi_\infty}{dx}, \tag{6}$$

is the Wronskian of these solutions. In the rest of this section, for the Klein-Gordon field we determine the Wronskian (6) of the solutions satisfying the boundary conditions at the event horizon and at the asymptotic region. Taking as a basis the expression (5) for the frequency domain Green’s function we calculate exactly the corresponding QNF by finding the zeros of its Wronskian, that is, the poles of the frequency domain Green’s function.

The Klein-Gordon equation takes the form³

$$\left(\square^2 - \frac{\mu^2}{l^2} \right) \Psi = \left(\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \right) - \frac{\mu^2}{l^2} \right) \Psi = 0, \tag{7}$$

where \square^2 is the d’Alembertian operator, μ is related to the mass of the field, $g^{\mu\nu}$ denotes the inverse metric, g its determinant and Ψ denotes the Klein-Gordon field. As shown in Ref. [18] if we take a separable solution

²For the exact calculation of the QNF spectrum for the Klein-Gordon field moving in the static BTZ black hole see Ref. [17].

³In the rest of this work we follow closely the notation of Ref. [18], but we make some small changes.

$$\Psi = R(r) e^{-i\omega t} e^{im\phi}, \quad (8)$$

where m is the angular eigenvalue, then the Klein-Gordon equation simplifies to the following differential equation for the radial function R

$$\frac{d}{dr} \left(f \frac{dR}{dr} \right) + \frac{1}{r} f \frac{dR}{dr} + \left(\frac{\omega^2}{f} - \frac{J\omega m}{fr^2} - \frac{(r^2/l^2 - M)m^2}{fr^2} \right) R = 0. \quad (9)$$

We notice that the previous expression can be transformed to a Schrödinger type equation of the form (4) as follows

$$\frac{d^2 \hat{R}}{dx^2} + \omega^2 \hat{R} - \left\{ \frac{J\omega m}{r^2} + \frac{(r^2/l^2 - M)m^2}{r^2} - \frac{f}{g} \frac{d}{dr} \left(f \frac{dg}{dr} \right) - \frac{f}{g} \frac{dg}{dr} \right\} \hat{R} = 0, \quad (10)$$

with

$$f \frac{d}{dr} = \frac{d}{dx}, \quad g = \frac{1}{\sqrt{r}}, \quad R = g\hat{R}. \quad (11)$$

In Equation (10) the expression in curly braces is the effective potential of the Klein-Gordon field in the rotating BTZ black hole.

In the variable z defined by

$$z = \frac{r^2 - r_+^2}{r^2 - r_-^2}, \quad (12)$$

we find that the radial Equation (9) simplifies to [18]

$$z(1-z) \frac{d^2 R}{dz^2} + (1-z) \frac{dR}{dz} + \left(\frac{A}{z} + B + \frac{C}{1-z} \right) R = 0, \quad (13)$$

where

$$A = \frac{l^4}{4(r_+^2 - r_-^2)^2} \left(\omega r_+ - \frac{m}{l} r_- \right)^2, \quad C = -\frac{\mu^2}{4}, \quad (14)$$

$$B = -\frac{l^4}{4(r_+^2 - r_-^2)^2} \left(\omega r_- - \frac{m}{l} r_+ \right)^2.$$

Taking the radial function R in the form

$$R = z^\alpha (1-z)^\beta F, \quad (15)$$

we obtain that the function F must be a solution to the hypergeometric differential equation [33] [34] [35]

$$z(1-z) \frac{d^2 F}{dz^2} + [c - (a+b+1)z] \frac{dF}{dz} - abF = 0, \quad (16)$$

when the quantities α and β are solutions to the algebraic equations

$$\alpha^2 + A = 0, \quad \beta^2 - \beta + C = 0, \quad (17)$$

and the parameters of the hypergeometric equation are equal to

$$a = \alpha + \beta + i\sqrt{-B}, \quad b = \alpha + \beta - i\sqrt{-B}, \quad c = 2\alpha + 1. \quad (18)$$

As in Ref. [18], in what follows we choose $\alpha = -i\sqrt{A}$ and $\beta = (1 - \sqrt{1 + \mu^2})/2$.

Hence, around $z = 0$ the solutions of the radial equation take the form

$$R = z^\alpha (1-z)^\beta \left(C_1 {}_2F_1(a, b; c; z) + C_2 z^{1-c} {}_2F_1(a-c+1, b-c+1; 2-c; z) \right), \quad (19)$$

where C_1, C_2 are constants and ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function [33] [34] [35]. In a straightforward way we can verify that the solution satisfying the boundary condition a) near the outer horizon is [18]

$$R_+ = C_1 z^\alpha (1-z)^\beta {}_2F_1(a, b; c; z). \quad (20)$$

Instead of following the usual way to calculate exactly the QNF [18], here we determine the solutions to the radial Equation (9) as $r \rightarrow \infty$, that is, around $z = 1$. In what follows, it is convenient to utilize the variable u defined by

$$u = 1 - z. \quad (21)$$

Using this variable we find that the function F of the Formula (15) must be solution of a hypergeometric differential Equation (16) with parameters [33]

$$\tilde{a} = a, \quad \tilde{b} = b, \quad \tilde{c} = 1 + a + b - c, \quad (22)$$

and therefore the radial function takes the form

$$R = (1-u)^\alpha u^\beta \left(K_1 {}_2F_1(a, b; a+b-c+1; u) + K_2 u^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; u) \right), \quad (23)$$

where K_1 and K_2 are constants. Analyzing its behavior as $r \rightarrow \infty$ we obtain that the radial function fulfilling the boundary condition b) at the asymptotic region is

$$R_\infty = K_2 (1-u)^\alpha u^\beta u^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; u), \quad (24)$$

since the term proportional to K_1 diverges as $u \rightarrow 0$.

Considering Equation (11) we obtain that for the BTZ black hole $\Phi_+ = \sqrt{r} R_+$ and $\Phi_\infty = \sqrt{r} R_\infty$ and hence we get that their Wronskian is equal to

$$W_x(\Phi_\infty, \Phi_+) = r W_x(R_\infty, R_+). \quad (25)$$

Owing to the hypergeometric differential Equation (16) is of second order, we know that among any three solutions there is a linear relation [33] [34] [35]. An example of this type is the relation among its two solutions around $z = 1$ and one of its solutions around $z = 0$, usually known as Kummer property that takes the form

$$\begin{aligned} & {}_2F_1(a, b; c; z) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z) \\ &+ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-z). \end{aligned} \quad (26)$$

From the Kummer property of the hypergeometric function and considering that the Wronskian of the solutions to the hypergeometric equation is [35]

$$W_z \left({}_2F_1(a, b; a+b-c+1; 1-z), (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-z) \right) \\ = (a+b-c)z^{-c}(1-z)^{c-a-b-1}, \quad (27)$$

we find that the Wronskian of the radial solutions Φ_+ and Φ_∞ is equal to

$$W_x(\Phi_\infty, \Phi_+) = -\frac{2}{l^2} K_2 C_1 (r_+^2 - r_-^2) (a+b-c) \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \quad (28)$$

We notice that $W_x(\Phi_\infty, \Phi_+)$ does not depend on the coordinate x (or r), as we expect [32]. From the last expression we find that the zeros of the Wronskian (that is, the poles of Green's function) are located at [34]

$$c-a = -n, \quad c-b = -n, \quad n = 0, 1, 2, 3, \dots, \quad (29)$$

and therefore the QNF of the Klein-Gordon field in the rotating BTZ black hole are equal to [18]

$$\omega = \frac{m}{l} - \frac{2i}{l^2} (r_+ - r_-) \left(n + \frac{1}{2} + \frac{\sqrt{1+\mu^2}}{2} \right), \\ \omega = -\frac{m}{l} - \frac{2i}{l^2} (r_+ + r_-) \left(n + \frac{1}{2} + \frac{\sqrt{1+\mu^2}}{2} \right), \quad (30)$$

that is, the proposed method produces the same QNF that the procedure used in Ref. [18].

3. Conclusions

The QNF of the rotating BTZ black hole have been useful in several applications [4], [17], [18], [31], and this is an example showing the utility of the exact results in the research line of black hole perturbations and related areas. In this paper our main objective is the description of the method given in the previous section and based on the results of Ref. [32]. We believe that the proposed method can be used to calculate exactly the QNF of classical fields propagating in other black holes and this procedure is an useful addition to the tools for computing exactly the QNF of spacetimes, because we think that the possibility of calculating the physical quantities with different methods contributes to a better understanding of their properties. Furthermore, notice that using the proposed method we can verify the exact results that are obtained with the usual procedure of Refs. [4], [17]-[28].

It is convenient to mention that for the Klein-Gordon field propagating in the rotating BTZ black hole, the proposed procedure yields the same values for the QNF as the usual method of Refs. [17]-[28], but the developed procedure makes explicit the interpretation of the QNF as the poles of the frequency domain Green's function and this fact is not manifest in the procedure of Refs. [17]-[28]. We believe that this characteristic is a relevant addition of the method previously described in Section 2 and can be useful in the AdS-CFT correspondence [3], [4]. Another feature of the proposed method is that for the solution satisfying the boundary condition near the horizon and for the solution fulfilling boundary

condition at the asymptotic region, the procedure makes clear that at the QNF their Wronskian is equal to zero. A fact that is commonly stated [1], [2], but not explicitly verified.

As far as we are aware this method is not previously used in the exact determination of the QNF for black holes, and we believe that this procedure can be a useful tool in the study of spacetime perturbations. Doubtless the applicability of the method that we expound in the present work to the determination of the QNM for other gravitational systems deserves further research.

Acknowledgements

This work was supported by CONACYT México, SNI México, COFAA-IPN, EDI-IPN, and the Research Project IPN SIP-20181408.

References

- [1] Kokkotas, K.D. and Schmidt, B.G. (1999) Quasi-Normal Modes of Stars and Black Holes. *Living Reviews in Relativity*, **2**, 2. <https://doi.org/10.12942/lrr-1999-2>
- [2] Berti, E., Cardoso, V. and Starinets, A.O. (2009) Quasinormal Modes of Black Holes and Black Branes. *Classical and Quantum Gravity*, **26**, Article ID: 163001. <https://doi.org/10.1088/0264-9381/26/16/163001>
- [3] Horowitz, G.T. and Hubeny, V.E. (2000) Quasinormal Modes of AdS Black Holes and the Approach to Thermal Equilibrium. *Physical Review D*, **62**, Article ID: 024027. <https://doi.org/10.1103/PhysRevD.62.024027>
- [4] Birmingham, D., Sachs, I. and Solodukhin, S.N. (2002) Conformal Field Theory Interpretation of Black Hole Quasi-Normal Modes. *Physical Review Letters*, **88**, Article ID: 151301.
- [5] Cardoso, V. and Lemos, J.P.S. (2001) Quasinormal Modes of Schwarzschild Anti-De Sitter Black Holes: Electromagnetic and Gravitational Perturbations. *Physical Review D*, **64**, Article ID: 084017. <https://doi.org/10.1103/PhysRevD.64.084017>
- [6] Cardoso, V. and Lemos, J.P.S. (2001) Quasinormal Modes of Toroidal, Cylindrical and Planar Black Holes in Anti-De Sitter Space-Times. *Classical and Quantum Gravity*, **18**, 5257. <https://doi.org/10.1088/0264-9381/18/23/319>
- [7] Cardoso, V., Konoplya, R. and Lemos, J.P.S. (2003) Quasinormal Frequencies of Schwarzschild Black Holes in Anti-De Sitter Space-Times: A Complete Study on the Asymptotic Behavior. *Physical Review D*, **68**, Article ID: 044024. <https://doi.org/10.1103/PhysRevD.68.044024>
- [8] Berti, E. and Kokkotas, K.D. (2003) Quasinormal Modes of Reissner-Nordstrom-anti-de Sitter Black Holes: Scalar, Electromagnetic and Gravitational Perturbations. *Physical Review D*, **67**, Article ID: 064020. <https://doi.org/10.1103/PhysRevD.67.064020>
- [9] Lopez-Ortega, A. (2006) Electromagnetic Quasinormal Modes of D-Dimensional Black Holes. *General Relativity and Gravitation*, **38**, 1747. <https://doi.org/10.1007/s10714-006-0358-2>
- [10] Giammatteo, M. and Jing, J.L. (2005) Dirac Quasinormal Frequencies in Schwarzschild-AdS Space-Time. *Physical Review D*, **71**, Article ID: 024007. <https://doi.org/10.1103/PhysRevD.71.024007>
- [11] Jing, J.L. and Pan, Q.Y. (2005) Dirac Quasinormal Frequencies of Reissner-Nordstrom Black Hole in Anti-De Sitter Spacetime. *Physical Review D*, **71**, Ar-

title ID: 124011.

- [12] Starinets, A.O. (2002) Quasinormal Modes of near Extremal Black Branes. *Physical Review D*, **66**, Article ID: 124013. <https://doi.org/10.1103/PhysRevD.66.124013>
- [13] Nunez, A. and Starinets, A.O. (2003) Ad/CFT Correspondence, Quasinormal Modes, and Thermal Correlators in $N = 4$ SYM. *Physical Review D*, **67**, Article ID: 124013. <https://doi.org/10.1103/PhysRevD.67.124013>
- [14] Kovtun, P.K. and Starinets, A.O. (2005) Quasinormal Modes and Holography. *Physical Review D*, **72**, Article ID: 086009. <https://doi.org/10.1103/PhysRevD.72.086009>
- [15] Musiri, S., Ness, S. and Siopsis, G. (2006) Perturbative Calculation of Quasi-Normal Modes of AdS Schwarzschild Black Holes. *Physical Review D*, **73**, Article ID: 064001. <https://doi.org/10.1103/PhysRevD.73.064001>
- [16] Miranda, A.S., Morgan, J. and Zanchin, V.T. (2008) Quasinormal Modes of Plane-Symmetric Black Holes According to the AdS/CFT Correspondence. *Journal of High Energy Physics*, **0811**, 030. <https://doi.org/10.1088/1126-6708/2008/11/030>
- [17] Cardoso, V. and Lemos, J.P.S. (2001) Scalar, Electromagnetic and Weyl Perturbations of BTZ Black Holes: Quasinormal Modes. *Physical Review D*, **63**, Article ID: 124015.
- [18] Birmingham, D. (2001) Choptuik Scaling and Quasinormal Modes in the AdS/CFT Correspondence. *Physical Review D*, **64**, Article ID: 064024.
- [19] Cordero, R., Lopez-Ortega, A. and Vega-Acevedo, I. (2012) Quasinormal Frequencies of Asymptotically Anti-De Sitter Black Holes in Two Dimensions. *General Relativity and Gravitation*, **44**, 917-940. <https://doi.org/10.1007/s10714-011-1316-1>
- [20] Gomez-Navarro, D.V. and Lopez-Ortega, A. (2017) Electromagnetic Quasinormal Modes of Five-Dimensional Topological Black Holes. *Revista Mexicana de Física*, **63**, 541-548.
- [21] Aros, R., Martinez, C., Troncoso, R. and Zanelli, J. (2003) Quasinormal Modes for Massless Topological Black Holes. *Physical Review D*, **67**, Article ID: 044014. <https://doi.org/10.1103/PhysRevD.67.044014>
- [22] Birmingham, D. and Mokhtari, S. (2006) Exact Gravitational Quasinormal Frequencies of Topological Black Holes. *Physical Review D*, **74**, Article ID: 084026.
- [23] Gupta, K.S., Juri, T. and Samsarov, A. (2017) Noncommutative Duality and Fermionic Quasinormal Modes of the BTZ Black Hole. *Journal of High Energy Physics*, **17**, 107. [https://doi.org/10.1007/JHEP06\(2017\)107](https://doi.org/10.1007/JHEP06(2017)107)
- [24] Kandemir, B.S. and Ertem, Ü. (2017) Quasinormal Modes of BTZ Black Hole and Hawking-Like Radiation in Graphene. *Annalen der Physik*, **529**, Article ID: 1600330. <https://doi.org/10.1002/andp.201600330>
- [25] Lopez-Ortega, A. (2008) Electromagnetic Quasinormal Modes of D-Dimensional Black Holes II. *General Relativity and Gravitation*, **40**, 1379-1401. <https://doi.org/10.1007/s10714-007-0538-8>
- [26] Lopez-Ortega, A. (2010) Quasinormal Frequencies of the Dirac Field in the Massless Topological Black Hole. *Revista Mexicana de Física*, **56**, 44-53.
- [27] Lopez-Ortega, A. (2014) Electromagnetic Quasinormal Modes of an Asymptotically Lifshitz Black Hole. *General Relativity and Gravitation*, **46**, 1756. <https://doi.org/10.1007/s10714-014-1756-5>
- [28] Oliva, J. and Troncoso, R. (2010) Exact Quasinormal Modes for a Special Class of Black Holes. *Physical Review D*, **82**, Article ID: 027502. <https://doi.org/10.1103/PhysRevD.82.027502>

- [29] Banados, M., Teitelboim, C. and Zanelli, J. (1992) The Black Hole in Three-Dimensional Space-Time. *Physical Review Letters*, **69**, 1849-1851.
<https://doi.org/10.1103/PhysRevLett.69.1849>
- [30] Banados, M., Henneaux, M., Teitelboim, C. and Zanelli, J. (1993) Geometry of the (2+1) Black Hole. *Physical Review D*, **48**, 1506-1525.
<https://doi.org/10.1103/PhysRevD.48.1506>
- [31] Birmingham, D., Sachs, I. and Sen, S. (2001) Exact Results for the BTZ Black Hole. *International Journal of Modern Physics D*, **10**, 833-857.
<https://doi.org/10.1142/S0218271801001207>
- [32] Leaver, E.W. (1986) Spectral Decomposition of the Perturbation Response of the Schwarzschild Geometry. *Physical Review D*, **34**, 384-408.
<https://doi.org/10.1103/PhysRevD.34.384>
- [33] Wang, Z.X. and Guo, D.R. (1989) *Special Functions*. World Scientific Publishing, Singapore.
- [34] Abramowitz, M. and Stegun, I.A. (1965) *Handbook of Mathematical Functions, Graphs, and Mathematical Table*. Dover Publications, New York.
- [35] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. (2010) *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York.