The Inertial Manifolds for a Class of Higher-Order Coupled Kirchhoff-Type Equations

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Abstract

In this paper, we mainly deal with a class of higher-order coupled Kirchhoff-type equations. At first, we take advantage of Hadamard’s graph to get the equivalent form of the original equations. Then, the inertial manifolds are proved by using spectral gap condition. The main result we gained is that the inertial manifolds are established under the proper assumptions of $M(s)$ and $g_i(u,v), i=1,2$.

Keywords

Higher-Order Coupled Kirchhoff-Type Equations, Inertial Manifold, Hadamard’s Graph, Spectral Gap Condition

1. Introduction

This paper mainly deals with existence of inertial manifolds for a class of higher-order coupled Kirchhoff-type equations:

\begin{align}
    u_t + M(\|\nabla u\|^2 + \|\nabla v\|^2)(-\Delta) u + \beta (-\Delta) v + g_1(u,v) &= f_1(x), \\
    v_t + M(\|\nabla u\|^2 + \|\nabla v\|^2)(-\Delta) v + \beta (-\Delta) v + g_2(u,v) &= f_2(x), \\
    u(x,0) &= u_0(x), u_t(x,0) = u_i(x), x \in \Omega, \\
    v(x,0) &= v_0(x), v_t(x,0) = v_i(x), x \in \Omega, \\
    u\big|_{\partial \Omega} &= 0, \frac{\partial u}{\partial \nu}\big|_{\partial \Omega} = 0, i=1,2,3,\cdots, m-1, \\
    v\big|_{\partial \Omega} &= 0, \frac{\partial v}{\partial \nu}\big|_{\partial \Omega} = 0, i=1,2,3,\cdots, m-1,
\end{align}


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where $m > 1$ is an integer constant, $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth Dirichlet boundary $\partial \Omega$, $u_0(x), u_1(x), v_0(x), v_1(x)$ are the initial value, $\mu_i$ and $v_i$ are the unit outward normal on $\partial \Omega$, $M(s)$ is a nonnegative $C^1$ function, $(-\Delta)^m u_i$ and $(-\Delta)^m v_i$ are strongly damping, $g_1(u, v)$ and $g_2(u, v)$ are nonlinear source terms, $f_1(x)$ and $f_2(x)$ are given forcing functions.

It is significant to establish inertial manifolds for the study of the long-time behavior of infinite dimensional dynamical systems, because it is an important bridge between infinite-dimensional dynamic system and finite-dimensional dynamical system.

To better carry out our work, let’s recall some results regarding wave equations.

Jingzhu Wu and Guoguang Lin [1] studied the following two-dimensional strong damping Boussinesq equation while $\alpha > 2$:  
\begin{align*}
  u_{tt} - \alpha \Delta u_t - \Delta u + u^{2+\varepsilon} &= f(x, y), (x, y) \in \Omega, \\
  u(x, y, 0) &= u_0(x, y), (x, y) \in \Omega, \\
  u(x, y, t) &= u(x + \pi, y, t) = u(x, y + \pi, t) = 0, (x, y) \in \Omega,
\end{align*}
where $\Omega = (0, \pi) \times (0, \pi) \subset \mathbb{R} \times \mathbb{R}, t > 0$. They obtained result that is existence of inertial manifolds.

Guigui Xu, Libo Wang and Guoguang Lin [2] investigated the strongly damped wave equation:  
\begin{align*}
  u_{tt} - \alpha \Delta u_t + \beta \Delta^2 u - \gamma \Delta u_t + g(u) &= f(x, t), (x, t) \in \Omega \times \mathbb{R}^+, \\
  u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \\
  u \bigr|_{\partial \Omega} &= 0, \Delta u \bigr|_{\partial \Omega} = 0, (x, t) \in \partial \Omega \times \mathbb{R}^+.
\end{align*}
They gave some assumptions for the nonlinearity term $g(u)$ to satisfy the following inequalities:

(A1) $\liminf_{s \to \infty} \frac{G(s)}{s^2} \geq 0, s \in \mathbb{R}, G(s) = \int_0^s g(r) dr$.

(A2) There is positive constant $C_1$ such that  
$\liminf_{s \to \infty} \frac{s g(s) - C_1 G(s)}{s^2} \geq 0, s \in \mathbb{R}$.

According to the above assumptions, they proved the inertial manifolds by using the Hadamard’s graph transformation method.

Ruijin Lou, Penhui Lv, Guoguang Lin [3] considered a class of generalized nonlinear Kirchhoff-Sine-Gordon equation:  
\begin{align*}
  u_{tt} - \beta \Delta u_t + \alpha u_t - \phi \left( \| \nabla u \| \right)^2 \Delta u + g(\sin u) &= f(x), \\
  u(x, 0) &= 0, x \in \Omega, t \geq 0, \\
  u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), x \in \Omega.
\end{align*}
Under some reasonable assumptions, they obtained some results that are
squeezing property of the nonlinear semigroup associated with this equation and the existence of exponential attractors and inertial manifolds.

Lin Chen, Wei Wang and Guoguang Lin [4] studied higher-order Kirchhoff-type equation with nonlinear strong dissipation in $n$ dimensional space:

$$u_m + (-\Delta)^m u_i + \phi(\|\nabla u\|) (-\Delta)^m u + g(u) = f(x), x \in \Omega, t > 0, m > 1,$$

$$u(x,t) = 0, \frac{\partial u}{\partial v} = 0, i = 1, 2, \cdots, m - 1, x \in \partial \Omega, t > 0,$$

$$u(x,0) = u_0(x), u_i(x,0) = u_i(x).$$

For the above equation, they made some suitable assumptions about $\phi(s)$ and $g(u)$ to get existence of exponential attractors and inertial manifolds.

In this article, we first take advantage of Hadamard’s graph to transform equations (1.1)-(1.2) into an equivalent one-order system of form. Then, we proved the inertial manifolds by using spectral gap condition.

2. Preliminaries

We denote the some simple symbols, $\| \|$ and $(,)$ stand for norm and inner product of $H$ and $H^n = H^n(\Omega)$, $H^n_0(\Omega) = H^n(\Omega) \cap H^n_0(\Omega)$, $H^n_0(\Omega) = H^n(\Omega) \cap H^n_0(\Omega)$, $H^n_0(\Omega) = H^n(\Omega), f_i = f_i(x), (i = 1, 2),

$$v = \|\nabla u\|^2 + \|\nabla v\|^2, u_k = \|\nabla u_k\|^2 + \|\nabla v_k\|^2, (k = 1, 2, \cdots), \|\| = \|\|_{L^1(\Omega)},$$

$$X = H^n_0(\Omega) \times H^n_0(\Omega) \times L^2(\Omega) \times L^2(\Omega), \|\|_c = \|\|_{L^c(\Omega)}.$$

are various positive constants.

Next, we give some assumptions needed for problem (1.1)-(1.6).

$$(H1) \quad g_i(u,v), (i = 1, 2), M(s) \in C^1(R).$$

$$(H2) \quad \varepsilon \leq m_0 \leq M(s) \leq m_1 = \frac{B^2 \mu_1^2 - 1}{4}.$$

Then, we give the basic concepts below.

**Definition 2.1.** [6] An inertial manifold $\mu$ is a finite-dimensional manifold enjoying the following three properties:

1) $\mu$ is Lipschitz,
2) $\mu$ is positively invariant for the semi-group $\{S(t)\}_{t \geq 0}$, i.e., $S(t) \mu \subset \mu, \forall t \geq 0$,
3) $\mu$ attracts exponentially all the orbits of the solution.

**Definition 2.2.** [6] Let $A : X \to X$ be an operator and assume that $F \in C^1(X, X)$ satisfies the Lipschitz condition:

$$\|F(U) - F(V)\|_1 \leq L_1 \|U - V\|_1, U, V \in X,$$

where $X = H^n_0(\Omega) \times H^n_0(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. The operator $A$ is said to satisfy the spectral gap condition relative to $F$, if the point spectrum of the operator $A$ can be divided into two parts $\sigma_1$ and $\sigma_2$, of which $\sigma_1$ is finite, and such that, if

$$\Lambda_1 = \sup\{\text{Re} \lambda \lambda \in \sigma_1\}, \Lambda_2 = \sup\{\text{Re} \lambda \lambda \in \sigma_2\},$$

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and
\[ X_i = \text{span}\{\omega_j \mid j \in \sigma_i\}, i = 1, 2. \] (2.5)

Then
\[ \Lambda_2 - \Lambda_1 > 4\lambda_k. \] (2.6)

And the orthogonal decomposition
\[ X = X_1 \oplus X_2, \] (2.7)
holds with continuous orthogonal projections \( P_1 : X \to X_1 \) and \( P_2 : X \to X_2 \).

**Lemma 2.1.** [7] Let the eigenvalues \( \lambda_j^+, j \geq 1 \) be arranged in nondecreasing order, for all \( m \in \mathbb{N} \), there is \( N \geq m \) such that \( \lambda_{N}^+ \) and \( \lambda_{N+m}^+ \) are consecutive.

### 3. Inertial Manifold

In this section, we use the Hadamard’s graph transformation method to prove the existence of inertial manifolds of problem (1.1)-(1.6) when \( N \) is sufficiently large.

Equations (1.1)-(1.2) are equivalent to the following one order evolution equation:
\[ U_t + HU = F(U), \] (3.1)
where \( U = (u, v, p, q), p = u_t, q = v_t, \)
\[ H = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \\ M(v)(-\Delta)^{\mu} & 0 & \beta(-\Delta)^{\mu} & 0 \\ 0 & M(v)(-\Delta)^{\mu} & 0 & \beta(-\Delta)^{\mu} \end{pmatrix}, \] (3.2)
\[ F(U) = \begin{pmatrix} f_1(x) - g_1(u, v) \\ f_2(x) - g_2(u, v) \end{pmatrix}, \] (3.3)
\[ D(H) = \{ (u, v, p, q) \in H^{2m} \times H^{2m} \times H^{\mu} \times H^{\mu} \}. \] (3.4)

In \( X \), we denote the usual graph norm, which is introduced by the scalar product as the following form
\[ (U, V)_X = (M(s) \cdot \nabla^\mu u, \nabla^\mu \overline{u}) + (M(s) \cdot \nabla^\mu v, \nabla^\mu \overline{v}) + (\overline{z}_1, p) + (\overline{z}_2, q) \] (3.5)
where \( U = (u, v, p, q), V = (y_1, y_2, z_1, z_2) \). \( \overline{y}_1, \overline{y}_2, \overline{z}_1, \overline{z}_2 \) denote the conjugation of \( y_1, y_2, z_1, z_2 \) respectively, \( u, v, y_1, y_2 \in H^{2m}_0(\Omega), p, q, z_1, z_2 \in H^{\mu}_0(\Omega) \).

For \( U \in D(H) \), we have
\[ (HU, U)_X = (-M(v)(-\Delta)^{\mu} u + \beta(-\Delta)^{\mu} p) + (\overline{p}, M(v)(-\Delta)^{\mu} u + \beta(-\Delta)^{\mu} p) \]
\[ + (\overline{q}, M(v)(-\Delta)^{\mu} v + \beta(-\Delta)^{\mu} q) \]
\[ = \beta \|\nabla^\mu p\|^2 + \beta \|\nabla^\mu q\|^2 \geq 0. \] (3.6)
Therefore, the operator $H$ in (3.2) is monotone, and $(HU, U)_X$ is a nonnegative and real number.

To obtain the eigenvalues of $H$, we consider the following eigenvalue equation:

$$HU = \lambda U, U = (u, v, p, q) \in X,$$  \hfill (3.7)

That is

$$\begin{align*}
-p &= \lambda u, \\
-q &= \lambda v, \\
M(\nu)(-\Delta)^n u + \beta (-\Delta)^n p &= \lambda p, \\
M(\nu)(-\Delta)^n v + \beta (-\Delta)^n q &= \lambda q.
\end{align*}$$  \hfill (3.8)

The first two equations in (3.8) are brought into the last two equations in (3.8) respectively, we get

$$\begin{align*}
\lambda^2 u + M(\nu)(-\Delta)^n u - \beta \lambda (-\Delta)^n u &= 0, \\
\lambda^2 v + M(\nu)(-\Delta)^n v - \beta \lambda (-\Delta)^n v &= 0, \\
u|_{\partial \Omega} &= 0, \frac{\partial^i u}{\partial \eta^i}|_{\partial \Omega} = 0, i = 1, 2, \ldots, m - 1, \\
v|_{\partial \Omega} &= 0, \frac{\partial^i v}{\partial \eta^i}|_{\partial \Omega} = 0, i = 1, 2, \ldots, m - 1.
\end{align*}$$  \hfill (3.9)

Let $u_k$ and $v_k$ replace $u$ and $v$ in (3.9) respectively. And then taking $u_k$ and $v_k$ inner product respectively, we obtain

$$\lambda^2 \left\| u_k \right\|^2 + M(\nu_k) \left\| \nabla^n u_k \right\|^2 - \beta \lambda \left\| \nabla^n u_k \right\|^2 = 0. \hfill (3.10)$$

$$\lambda^2 \left\| v_k \right\|^2 + M(\nu_k) \left\| \nabla^n v_k \right\|^2 - \beta \lambda \left\| \nabla^n v_k \right\|^2 = 0. \hfill (3.11)$$

Summing up (3.10) and (3.11), we get

$$\lambda^2 \left( \left\| u_k \right\|^2 + \left\| v_k \right\|^2 \right) + M(\nu_k) \left( \left\| \nabla^n u_k \right\|^2 + \left\| \nabla^n v_k \right\|^2 \right) - \beta \lambda \left( \left\| \nabla^n u_k \right\|^2 + \left\| \nabla^n v_k \right\|^2 \right) = 0. \hfill (3.12)$$

When (3.12) is considered an a yuan quadratic equation on $\lambda$, we can get

$$\lambda_k^\pm = \frac{\beta \mu_k \pm \sqrt{\beta^2 \mu_k^2 - 4 \mu_k M(\nu_k)}}{2},$$  \hfill (3.13)

where $\mu_k$ is the eigenvalue of $(-\Delta)^n$ in $H_0^m$, then $\mu_k = c_0 k^m$. if $\beta^2 \mu_k \geq 4M(\nu_k)$, that is $\beta^2 \mu_k \geq 4m$, the eigenvalues of $H$ are all positive and real numbers, the corresponding eigenfunction have the form $U_k^\pm = (u_k, v_k, -\lambda_k^\pm u_k, -\lambda_k^\pm v_k)$. For (3.13) and future reference, we observe that for all $k \geq 1$,

$$\left\| \nabla^n u_k \right\|^2 + \left\| \nabla^n v_k \right\|^2 = \mu_k \left\| u_k \right\|^2 + \left\| v_k \right\|^2 = 1, \left\| \nabla^n u_k \right\|^2 + \left\| \nabla^n v_k \right\|^2 = \frac{1}{\mu_k}. \hfill (3.14)$$

**Lemma 3.1** $g_i : H_0^m \times H_0^m \rightarrow H_0^m \times H_0^m, i = 1, 2$ is uniformly bounded and globally Lipschitz continuous.
\textbf{Proof.} \(\forall u_1, u_2, v_1, v_2 \in H^w_0(\Omega)\), we get

\[
\|g_i(u_1, v_1) - g_i(u_2, v_2)\|_{W^{1, w}_0} \leq \|g_i(\xi, v_1)\|_w \|u_1 - u_2\|_{W^{1, w}_0} + \|g_i(u_2, \eta)\|_w \|v_1 - v_2\|_{W^{1, w}_0},
\]

(3.15)

where \(\xi = \theta u_1 + (1 - \theta) u_2, \eta = \theta v_1 + (1 - \theta) v_2, 0 < \theta < 1\). According to (H1), we can obtain

\[
\|g_i(u_1, v_1) - g_i(u_2, v_2)\|_{W^{1, w}_0} \leq C_1 \|u_1 - u_2\|_{W^{1, w}_0} + C_2 \|v_1 - v_2\|_{W^{1, w}_0} \leq \frac{1}{2} \left(\|u_1 - u_2\|_{W^{1, w}_0} + \|v_1 - v_2\|_{W^{1, w}_0}\right).
\]

(3.16)

\textbf{Theorem 3.1} If \(\beta^2 \mu_i \geq 4m_i\) holds, \(l/2\) is maximum Lipschitz constant of \(g_i(u, v), (i = 1, 2)\), and if \(N_i \in N\) is sufficiently large such that when \(N \geq N_i\), the following inequality holds:

\[
(\mu_{N+1} - \mu_N) \left(\frac{\beta}{2} - \frac{1}{2} \sqrt{\beta^2 \mu_i - 4m_i}\right) \geq \frac{4\sqrt{2}l}{\beta^2 \mu_i - 4m_i} + 1, \quad (3.17)
\]

Then the operator \(H\) satisfies the spectral gap condition of (2.6).

\textbf{Proof.} When \(\beta^2 \mu_i \geq 4m_i\), all the eigenvalues of \(H\) are real and positive, and we can easily know that both sequences \(\{\lambda_{j^+}\}_{j \leq 1}\) and \(\{\lambda_{j^-}\}_{j \leq 1}\) are increasing.

The whole process of proof is divided into four steps.

\textbf{Step 1.} Since \(\lambda_{j^-}\) is arranged in nondecreasing order. According to Lemma 2.1, given \(N\) such that \(\lambda_{N^-}\) and \(\lambda_{N+1^-}\) are consecutive, we separate the eigenvalue of \(H\) as

\[
\sigma_1 = \left\{\lambda_{j^-}, \lambda_{j^+}\right\} \max \left\{\lambda_{j^-}, \lambda_{j^+}\right\} \leq \lambda_N, \quad (3.18)
\]

\[
\sigma_2 = \left\{\lambda_{j^-}, \lambda_{j^+}\right\} \lambda_j \leq \lambda_N \leq \min \left\{\lambda_{j^-}, \lambda_{j^+}\right\}, \quad (3.19)
\]

\textbf{Step 2.} We make decomposition of \(X\)

\[
X_1 = \text{span}\left\{U_{j^-}, U_{j^+} \left| \lambda_{j^-}, \lambda_{j^+} \in \sigma_i\right.\right\}. \quad (3.20)
\]

\[
X_2 = \text{span}\left\{U_{j^-}, U_{j^+} \left| \lambda_{j^-}, \lambda_{j^+} \in \sigma_2\right.\right\}. \quad (3.21)
\]

In order to make these two subspaces orthogonal and satisfy spectral inequality (2.6) \(\Lambda_1 = \lambda_N, \Lambda_2 = \lambda_{N+1}\), we further decompose

\[
X_2 = X_c \oplus X_r, \quad (3.22)
\]

with

\[
X_c = \text{span}\left\{U_{j^-} \left| \lambda_{j^-} \leq \lambda_N < \lambda_{j^+}\right.\right\}, \quad (3.23)
\]

\[
X_r = \text{span}\left\{U_{j^+} \left| \lambda_{j^-} \leq \lambda_N \leq \lambda_{j^+}\right.\right\}. \quad (3.24)
\]

And let \(X_N = X_1 \oplus X_c\). Next, we stipulate an eigenvalue scale product of \(X\) such that \(X_1\) and \(X_2\) are orthogonal, therefore we need to introduce two functions:

Let \(\Phi : X_N \rightarrow R, \psi : X_r \rightarrow R\).
\[
\Phi(U, V) = \beta(\nabla^n u, \nabla^n y) + 2\beta(\nabla^n z, \nabla^n v) + 2\beta(\nabla^n p, \nabla^n y) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v) + 4(\nabla^n q, \nabla^n q) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v),
\]

(3.25)

\[
\psi(U, V) = (\nabla^n u, \nabla^n y) + 2(\nabla^n z, \nabla^n v) + 2(\nabla^n p, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v),
\]

(3.26)

where \( U = (u, v, p, q) \) \( V = (y, z, z_z) \) \( \nabla^n u, \nabla^n y \) \( \nabla^n v \) \( \nabla^n z \) 

Let \( U = (u, v, p, q) \) \( X_N \) \( \Phi(U, V) \)

\[
= \beta(\nabla^n u, \nabla^n u) + 2\beta(\nabla^n p, \nabla^n u) + 2\beta(\nabla^n q, \nabla^n q) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v) = \beta(\nabla^n u, \nabla^n u) + 2\beta(\nabla^n p, \nabla^n u) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v) = \beta^2(\nabla^n u, \nabla^n u) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v) = \beta^2(\nabla^n u, \nabla^n u) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v)
\]

(3.27)

\[
\psi(U, V) = (\nabla^n u, \nabla^n y) + 2(\nabla^n z, \nabla^n v) + 2(\nabla^n p, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v) = \beta(\nabla^n u, \nabla^n u) + 2\beta(\nabla^n p, \nabla^n u) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v)
\]

(3.28)

Since \( \beta^2 \mu > 4m, m \leq M(s) \leq m \), we can know \( \Phi(U, U) \geq 0 \). Therefore, for all \( U \in X_N \), analogously, for all \( U \in X_N \), we can get

\[
\psi(U, U) = (\nabla^n u, \nabla^n u) + 2(\nabla^n z, \nabla^n v) + 2(\nabla^n p, \nabla^n u) + 2(\nabla^n q, \nabla^n v) + 2(\nabla^n p, \nabla^n u) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v) = \beta(\nabla^n u, \nabla^n u) + 2\beta(\nabla^n p, \nabla^n u) + 4\beta(\nabla^n q, \nabla^n q) + 4(\nabla^n q, \nabla^n z) - 4M(\nu)(\nabla^n u, \nabla^n u) + 2\beta^2 - \beta)(\nabla^n u, \nabla^n v)
\]

(3.29)
From above, we know that for all $\mathbf{R} \in X$, then $\psi(U, U) \geq 0$ holds. So, we define a scale product with $\Phi$ and $\psi$ in $X$

$$\langle (U, V) \rangle_X = \Phi(P_\alpha U, P_\alpha V) + \psi(P_\beta U, P_\beta V),$$

where $P_\alpha$, $P_\beta$ are respectively the projection: $X \rightarrow X$, $X \rightarrow X$.

In the inner product of $X$ in (3.29), $X_1$ and $X_2$ are orthogonal. In fact, we need prove that $X_1$ and $X_2$ are orthogonal.

$$\langle (U_1^*, U^*) \rangle_X = \Phi(U_1^*, U^*)$$

$$= \beta\langle \nabla u_1, \nabla u_2 \rangle + \beta\langle -\lambda_1^\gamma \nabla u_1, \nabla u_2 \rangle + \beta\langle \nabla v_1, \nabla v_2 \rangle$$

$$+ 2\beta\langle -\lambda_2^\gamma \nabla v_1, \nabla v_2 \rangle + 2\beta\langle -\lambda_2^\gamma \nabla u_1, \nabla u_2 \rangle + 4\langle -\lambda_1^\gamma \nabla u_1, -\lambda_2^\gamma \nabla u_2 \rangle$$

$$+ 2\beta\langle -\lambda_1^\gamma \nabla v_1, \nabla v_2 \rangle + 4\langle -\lambda_1^\gamma \nabla u_1, -\lambda_2^\gamma \nabla v_2 \rangle - 4M\left(\frac{\sqrt{\mu}}{\mu}\right)(\nabla u_1, u_2)$$

$$+ (2\beta^2 - \beta)(\nabla v_1, v_2) + (2\beta^2 - \beta)(\nabla v_2, v_1)$$

$$= \beta\langle \nabla u_1, \nabla u_1 \rangle + \beta\langle \nabla v_1, \nabla v_1 \rangle - 2\beta(\lambda_1^\gamma + \lambda_2^\gamma)\|u_2\|^2$$

$$- 2\beta(\lambda_1^\gamma + \lambda_2^\gamma)\|v_1\|^2 + 4\lambda_1^\gamma \lambda_2^\gamma \left(\langle \nabla u_1, \nabla u_1 \rangle + \langle \nabla v_1, \nabla v_1 \rangle\right) - 4M\left(\frac{\sqrt{\mu}}{\mu}\right)\|u_2\|^2$$

$$+ (2\beta^2 - \beta)(\nabla u_1, v_2) + (2\beta^2 - \beta)(\nabla v_1, v_2)$$

$$= 2\beta^2 \mu_1 - 2\beta(\lambda_1^\gamma + \lambda_2^\gamma) + 4\lambda_1^\gamma \lambda_2^\gamma \frac{1}{\mu_1} - 4M\left(\frac{\sqrt{\mu}}{\mu}\right) = 0.$$  

According to $\|u_1\|^2 = 1$, $\|v_1\|^2 = \mu_j$,

$$\langle \nabla u_1, \nabla v_1 \rangle = \frac{1}{\mu_j}$$

and $\lambda_1^\gamma + \lambda_2^\gamma = \beta\mu_j$, $\lambda_1^\gamma \lambda_2^\gamma = M\left(\frac{\sqrt{\mu}}{\mu}\right)\mu_j$, we can get the above results.

**Step 3.** Next, we estimate the Lipschitz constant $I_{\beta}$ of $F$.

$$F(U) = (0, 0, f_1(x) - g_i(u, v), f_2(x) - g_i(u, v))^T.$$  

$g_i : H^m \times H^m \rightarrow H^m \times H^m, (i = 1, 2)$ are globally Lipschitz continuous with maximum Lipschitz constant $\frac{l}{2}$ of $g_i$ from (3.27), (3.28), for arbitrarily $U = (u, v, p, q) \in X$, we have.

Let $P_1 : X \rightarrow X_1$, $P_2 : X \rightarrow X_2$ are the orthogonal projection.

If $U = (u, v, p, q) \in X$, $U_1 = (u_1, v_1, p_1, q_1) = P_1 U$ , $U_2 = (u_2, v_2, p_2, q_2) = P_2 U$ hold, then $P_1 u = u_1, P_1 v = v_1, P_2 u = u_2, P_2 v = v_2$.

$$\|U\|^2 = \Phi(P_1 U, P_2 U) + \Psi(P_1 U, P_2 U)$$

$$\geq \left(\beta^2 \mu_1 - 4M(v_1)\right)(\|u_1\|^2 + \|v_1\|^2) + \left(2\beta^2 \mu_1 - 4M(v_2)\right)(\|u_2\|^2 + \|v_2\|^2)$$

$$\geq \left(\beta^2 \mu_1 - 4m_1\right)(\|u_1\|^2 + \|v_1\|^2).$$  

Let $U = (u_1, v_1, u_1, v_1), V = (u_2, v_2, u_2, v_2)$, then
\[ \left\| F(U) - F(V) \right\|_X = \left\| g_1(u_1, v_1) - g_1(u_2, v_2) \right\|_{\mathcal{L}^g_{g_1}} + \left\| g_2(u_1, v_1) - g_2(u_2, v_2) \right\|_{\mathcal{L}^g_{g_2}} \leq \frac{f}{2} \left( \left\| u_1 - u_2 \right\|_{\mathcal{L}^g_g} + \left\| v_1 - v_2 \right\|_{\mathcal{L}^g_g} \right) + \frac{f}{2} \left( \left\| v_1 - v_2 \right\|_{\mathcal{L}^g_g} + \left\| u_1 - u_2 \right\|_{\mathcal{L}^g_g} \right) \]
\[ = f \left( \left\| u_1 - u_2 \right\|_{\mathcal{L}^g_g} + \left\| v_1 - v_2 \right\|_{\mathcal{L}^g_g} \right) \leq \frac{\sqrt{2l}}{\sqrt{\beta^2 \mu_l - 4m_1}} \left\| U - V \right\|_X . \]

Therefore \( l_F \leq \frac{\sqrt{2l}}{\sqrt{\beta^2 \mu_l - 4m_1}} . \)

**Step 4.** Now, we need to prove the spectral gap condition (2.6) holds.

From the above mentioned \( \Lambda_1 = \lambda_N^- \) and \( \Lambda_2 = \lambda_{N+1}^- \), we can get

\[ \Lambda_2 - \Lambda_1 = \lambda_{N+1}^- - \lambda_N^- = \beta \left( \mu_{N+1} - \mu_N \right) + \frac{1}{2} \left( \sqrt{R(N)} - \sqrt{R(N+1)} \right) , \] 
(3.34)

where \( R(N) = \beta^2 \mu_N - 4m_N \).

We determine \( N_1 > 0 \) such that for all \( N \geq N_1 \), let

\[ R_1(N) = 1 - \frac{\beta^2}{\sqrt{\beta^2 \mu_l - 4m_1}} \frac{4m_1}{\mu_N \left( \beta^2 \mu_l - 4m_1 \right)} , \] 
(3.35)

we can get

\[ \sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2 \mu_l - 4m_1} \left( \mu_{N+1} - \mu_N \right) = \sqrt{\beta^2 \mu_l - 4m_1} \left( \mu_{N+1} R_1(N+1) - \mu_N R_1(N) \right) . \]
(3.36)

According to the previous hypothesis \( \varepsilon < m \leq M \leq m = \frac{\beta^2 \mu_l - 1}{4} \), we can know

\[ \lim_{N \to +\infty} \left( \sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2 \mu_l - 4m_1} \left( \mu_{N+1} - \mu_N \right) \right) = 0 . \] 
(3.37)

Then, combining (3.33), (3.34), (3.17) and (3.37), we obtain

\[ \Lambda_2 - \Lambda_1 > \left( \mu_{N+1} - \mu_N \right) \left( \frac{\beta}{2} - \frac{1}{2} \sqrt{\beta^2 \mu_l - 4m_1} \right) - 1 \]
\[ \geq \frac{4 \sqrt{2l}}{\sqrt{\beta^2 \mu_l - 4m_1}} \geq 4l_F . \] 
(3.38)

The proof is completed.

**Theorem 3.2.** [8] Under the condition of Theorem 3.1, the initial boundary value problem (1.1)-(1.6) admits an inertial manifold \( \mu \) in \( X \) of the form

\[ \mu = \text{graph}(\rho) = \{ \omega + \rho(\zeta) : \zeta \in X_1 \} , \]
(3.39)

where \( X_1, X_2 \) are as in (3.20), (3.21) and \( \rho : X_1 \to X_2 \) is a Lipschitz continuous function.

**4. Conclusion**

In this paper, we prove the existence of the inertial manifolds for a class of high-
er-order coupled Kirchhoff-type equations. In the process of research, we take advantage of Hadamard’s graph to get the equivalent form of the original equations and then use spectral gap condition. Based on some of the work above, we prove the existence of the inertial manifolds of the system. For this problem, we will study the exponential attractors, blow-up, random attractors and so on.

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**References**


