Journal of Applied Mathematics and Physics, 2018, 6, 1055-1064
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Inertial Manifolds for a Class of Higher-Order Coupled Kirchhoff-Type Equations

Guoguang Lin, Sanmei Yang

Department of Mathematics, Yunnan University, Kunming, China
Email: gglin@ynu.edu.cn, 1934211371@qq.com

How to cite this paper: Lin, G.G. and Yang, S.M. (2018) The Inertial Manifolds for a Class of Higher-Order Coupled Kirchhoff-Type Equations. *Journal of Applied Mathematics and Physics*, **6**, 1055-1064.
https://doi.org/10.4236/jamp.2018.65091

Received: April 7, 2018
Accepted: May 27, 2018
Published: May 30, 2018

Abstract

In this paper, we mainly deal with a class of higher-order coupled Kirchhoff-type equations. At first, we take advantage of Hadamard's graph to get the equivalent form of the original equations. Then, the inertial manifolds are proved by using spectral gap condition. The main result we gained is that the inertial manifolds are established under the proper assumptions of $M(s)$ and $g_i(u,v), i=1,2$.

Keywords

Higher-Order Coupled Kirchhoff-Type Equations, Inertial Manifold, Hadamard's Graph, Spectral Gap Condition

1. Introduction

This paper mainly deals with existence of inertial manifolds for a class of higher-order coupled Kirchhoff-type equations:

$$u_{tt}+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v)=f_1(x), \tag{1.1}$$

$$v_{tt}+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x), \tag{1.2}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, \tag{1.3}$$

$$v(x,0)=v_0(x),v_t(x,0)=v_1(x),x\in\Omega, \tag{1.4}$$

$$u\big|_{\partial\Omega}=0,\frac{\partial^i u}{\partial\mu^i}\Big|_{\partial\Omega}=0,i=1,2,3,\cdots,m-1, \tag{1.5}$$

$$v\big|_{\partial\Omega}=0,\frac{\partial^i v}{\partial v^i}\Big|_{\partial\Omega}=0,i=1,2,3,\cdots,m-1, \tag{1.6}$$

where $m > 1$ is an integer constant, Ω is a bounded domain of R^n with a smooth Dirichlet boundary $\partial\Omega$, $u_0(x), u_1(x), v_0(x), v_1(x)$ are the initial value. μ_i and v_i are the unit outward normal on $\partial\Omega$, $M(s)$ is a nonnegative C^1 function, $(-\Delta)^m u_t$ and $(-\Delta)^m v_t$ are strongly damping, $g_1(u,v)$ and $g_2(u,v)$ are nonlinear source terms, $f_1(x)$ and $f_2(x)$ are given forcing functions.

It is significant to establish inertial manifolds for the study of the long-time behavior of infinite dimensional dynamical systems, because it is an important bridge between infinite-dimensional dynamic system and finite-dimensional dynamical system.

To better carry out our work, let's recall some results regarding wave equations.

Jingzhu Wu and Guoguang Lin [1] studied the following two-dimensional strong damping Boussinesq equation while $\alpha > 2$:

$$u_{tt} - \alpha\Delta u_t - \Delta u + u^{2k+1} = f(x,y), (x,y) \in \Omega, \tag{1.7}$$

$$u(x,y,0) = u_0(x,y), (x,y) \in \Omega, \tag{1.8}$$

$$u(x,y,t) = u(x+\pi,y,t) = u(x,y+\pi,t) = 0, (x,y) \in \Omega, \tag{1.9}$$

where $\Omega = (0,\pi)\times(0,\pi) \subset R\times R, t > 0$. They obtained result that is existence of inertial manifolds.

Guigui Xu, Libo Wang and Guoguang Lin [2] investigated the strongly damped wave equation:

$$u_{tt} - \alpha\Delta u + \beta\Delta^2 u - \gamma\Delta u_t + g(u) = f(x,t), (x,t) \in \Omega\times R^+, \tag{1.10}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.11}$$

$$u\big|_{\partial\Omega} = 0, \Delta u\big|_{\partial\Omega} = 0, (x,t) \in \partial\Omega\times R^+. \tag{1.12}$$

They gave some assumptions for the nonlinearity term $g(u)$ to satisfy the following inequalities:

(A1) $\liminf_{|s|\to\infty} \frac{G(s)}{s^2} \ge 0, s \in R, G(s) = \int_0^s g(r)\,dr$.

(A2) There is positive constant C_1 such that

$\liminf_{|s|\to\infty} \frac{sg(s) - C_1 G(s)}{s^2} \ge 0, s \in R$.

According to the above assumptions, they proved the inertial manifolds by using the Hadamard's graph transformation method.

Ruijin Lou, Penhui Lv, Guoguang Lin [3] considered a class of generalized nonlinear Kirchhoff-Sine-Gordon equation:

$$u_{tt} - \beta\Delta u_t + \alpha u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + g(\sin u) = f(x), \tag{1.13}$$

$$u(x,t) = 0, x \in \Omega, t \ge 0, \tag{1.14}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \tag{1.15}$$

Under some reasonable assumptions, they obtained some results that are squeezing property of the nonlinear semigroup associated with this equation and the existence of exponential attractors and inertial manifolds.

Lin Chen, Wei Wang and Guoguang Lin [4] studied higher-order Kirchhoff-type equation with nonlinear strong dissipation in n dimensional space:

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\|\nabla u\|^2\right)(-\Delta)^m u+g(u)=f(x),x\in\Omega,t>0,m>1, \quad (1.16)$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,m-1,x\in\partial\Omega,t>0, \quad (1.17)$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x). \quad (1.18)$$

For the above equation, they made some suitable assumptions about $\phi(s)$ and $g(u)$ to get existence of exponential attractors and inertial manifolds.

In this article, we first take advantage of Hadamard's graph to transform equations (1.1)-(1.2) into an equivalent one-order system of form. Then, we proved the inertial manifolds by using spectral gap condition.

2. Preliminaries

We denote the some simple symbols, $\|\cdot\|$ and (,) stand for norm and inner product of H and $H^m=H^m(\Omega)$, $H_0^m(\Omega)=H^m(\Omega)\cap H_0^1(\Omega)$, $H_0^{2m}(\Omega)=H^{2m}(\Omega)\cap H_0^1(\Omega)$, $H=L^2(\Omega)$, $f_i=f_i(x),(i=1,2)$, $\upsilon=\|\nabla u\|^2+\|\nabla v\|^2$, $\upsilon_k=\|\nabla u_k\|^2+\|\nabla v_k\|^2,(k=1,2,\cdots)$, $\|\cdot\|=\|\cdot\|_{L^2(\Omega)}$, $X=H_0^m(\Omega)\times H_0^m(\Omega)\times L^2(\Omega)\times L^2(\Omega)$, $\|\cdot\|_\infty=\|\cdot\|_{L^\infty(\Omega)}$. $c_i,(i=1,2,\cdots)$ are various positive constants.

Next, we give some assumptions needed for problem (1.1)-(1.6).

$$\text{(H1)}\quad g_i(u,v),(i=1,2),M(s)\in C^1(R). \quad (2.1)$$

$$\text{(H2)}\quad \varepsilon\le m_0\le M(s)\le m_1=\frac{\beta^2\mu_1-1}{4}. \quad (2.2)$$

Then, we give the basic concepts below.

Definition 2.1. [6] An inertial manifold μ is a finite-dimensional manifold enjoying the following three properties:

1) μ is Lipschitz,

2) μ is positively invariant for the semi-group $\{S(t)\}_{t\ge0}$, *i.e.* $S(t)\mu\subset\mu,\forall t\ge0$,

(3) μ attracts exponentially all the orbits of the solution.

Definition 2.2. [6] Let $A:X\to X$ be an operator and assume that $F\in C_b(X,X)$ satisfies the Lipschitz condition:

$$\|F(U)-F(V)\|_X\le l_F\|U-V\|_X,U,V\in X, \quad (2.3)$$

where $X=H_0^m(\Omega)\times H_0^m(\Omega)\times L^2(\Omega)\times L^2(\Omega)$. The operator A is said to satisfy the spectral gap condition relative to F, if the point spectrum of the operator A can be divided into two parts σ_1 and σ_2, of which σ_1 is finite, and such that, if

$$\Lambda_1=\sup\{\operatorname{Re}\lambda|\lambda\in\sigma_1\},\Lambda_2=\sup\{\operatorname{Re}\lambda|\lambda\in\sigma_2\}, \quad (2.4)$$

and

$$X_i = span\left\{\omega_j \middle| j \in \sigma_i\right\}, i = 1,2. \tag{2.5}$$

Then

$$\Lambda_2 - \Lambda_1 > 4l_F . \tag{2.6}$$

And the orthogonal decomposition

$$X = X_1 \oplus X_2, \tag{2.7}$$

holds with continuous orthogonal projections $P_1 : X \to X_1$ and $P_1 : X \to X_2$.

Lemma 2.1. [7] Let the eigenvalues $\lambda_j^{\pm}, j \ge 1$ be arranged in nondecreasing order, for all $m \in N$, there is $N \ge m$ such that λ_N^- and λ_{N+1}^- are consecutive.

3. Inertial Manifold

In this section, we use the Hadamard's graph transformation method to prove the existence of inertial manifolds of problem (1.1)-(1.6) when N is sufficiently large.

Equations (1.1)-(1.2) are equivalent to the following one order evolution equation:

$$U_t + HU = F(U), \tag{3.1}$$

where $U = (u, v, p, q), p = u_t, q = v_t$,

$$H = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \\ M(\upsilon)(-\Delta)^m & 0 & \beta(-\Delta)^m & 0 \\ 0 & M(\upsilon)(-\Delta)^m & 0 & \beta(-\Delta)^m \end{pmatrix}, \tag{3.2}$$

$$F(U) = \begin{pmatrix} 0 \\ 0 \\ f_1(x) - g_1(u,v) \\ f_2(x) - g_2(u,v) \end{pmatrix}, \tag{3.3}$$

$$D(H) = \left\{(u, v, p, q) \in H^{2m} \times H^{2m} \times H^m \times H^m\right\}. \tag{3.4}$$

In X, we denote the usual graph norm, which is introduced by the scalar product as the following form

$$\begin{aligned}(U,V)_X &= \left(M(s)\cdot\nabla^m u, \nabla^m \overline{y}_1\right) + \left(M(s)\cdot\nabla^m v, \nabla^m \overline{y}_2\right) \\ &\quad + (\overline{z}_1, p) + (\overline{z}_2, q)\end{aligned}, \tag{3.5}$$

where $U = (u, v, p, q), V = (y_1, y_2, z_1, z_2)$. $\overline{y}_1, \overline{y}_2, \overline{z}_1, \overline{z}_2$ denote the conjugation of y_1, y_2, z_1, z_2 respectively, $u, v, y_1, y_2 \in H_0^{2m}(\Omega), p, q, z_1, z_2 \in H_0^m(\Omega)$.

For $U \in D(H)$, we have

$$\begin{aligned}(HU,U)_X &= \left(-M(\upsilon)\nabla^m p, \nabla^m \overline{u}\right) + \left(-M(\upsilon)\nabla^m q, \nabla^m \overline{v}\right) \\ &\quad + \left(\overline{p}, M(\upsilon)(-\Delta)^m u + \beta(-\Delta)^m p\right) \\ &\quad + \left(\overline{q}, M(\upsilon)(-\Delta)^m v + \beta(-\Delta)^m q\right) \\ &= \beta\left\|\nabla^m p\right\|^2 + \beta\left\|\nabla^m q\right\|^2 \ge 0.\end{aligned} \tag{3.6}$$

Therefore, the operator H in (3.2) is monotone, and $\left(HU,U\right)_X$ is a nonnegative and real number.

To obtain the eigenvalues of H, we consider the following eigenvalue equation:

$$HU=\lambda U, U=\left(u,v,p,q\right)\in X\,, \tag{3.7}$$

That is

$$\begin{cases} -p=\lambda u, \\ -q=\lambda v, \\ M\left(\upsilon\right)\left(-\Delta\right)^m u+\beta\left(-\Delta\right)^m p=\lambda p. \\ M\left(\upsilon\right)\left(-\Delta\right)^m v+\beta\left(-\Delta\right)^m q=\lambda q. \end{cases} \tag{3.8}$$

The first two equations in (3.8) are brought into the last two equations in (3.8) respectively, we get

$$\begin{cases} \lambda^2 u+M\left(\upsilon\right)\left(-\Delta\right)^m u-\beta\lambda\left(-\Delta\right)^m u=0, \\ \lambda^2 v+M\left(\upsilon\right)\left(-\Delta\right)^m v-\beta\lambda\left(-\Delta\right)^m v=0, \\ u\big|_{\partial\Omega}=0, \dfrac{\partial^i u}{\partial\mu^i}\big|_{\partial\Omega}=0, i=1,2,3,\cdots,m-1, \\ v\big|_{\partial\Omega}=0, \dfrac{\partial^i v}{\partial v^i}\big|_{\partial\Omega}=0, i=1,2,3,\cdots,m-1. \end{cases} \tag{3.9}$$

Let u_k and v_k replace u and v in (3.9) respectively. And then taking u_k and v_k inner product respectively, we obtain

$$\lambda^2\left\|u_k\right\|^2+M\left(\upsilon_k\right)\left\|\nabla^m u_k\right\|^2-\beta\lambda\left\|\nabla^m u_k\right\|^2=0\,. \tag{3.10}$$

$$\lambda^2\left\|v_k\right\|^2+M\left(\upsilon_k\right)\left\|\nabla^m v_k\right\|^2-\beta\lambda\left\|\nabla^m v_k\right\|^2=0\,. \tag{3.11}$$

Summing up (3.10) and (3.11), we get

$$\lambda^2\left(\left\|u_k\right\|^2+\left\|v_k\right\|^2\right)+M\left(\sqrt[m]{\mu_k}\right)\left(\left\|\nabla^m u_k\right\|^2+\left\|\nabla^m v_k\right\|^2\right)-\beta\lambda\left(\left\|\nabla^m u_k\right\|^2+\left\|\nabla^m v_k\right\|^2\right)=0\,. \tag{3.12}$$

When (3.12) is considered an a yuan quadratic equation on λ, we can get

$$\lambda_k^{\pm}=\frac{\beta\mu_k\pm\sqrt{\beta^2\mu_k^2-4\mu_k M\left(\sqrt[m]{\mu_k}\right)}}{2}, \tag{3.13}$$

where μ_k is the eigenvalue of $\left(-\Delta\right)^m$ in H_0^{2m}, then $\mu_k=c_0 k^{\frac{m}{n}}$. if $\beta^2\mu_k\ge 4M\left(\sqrt[m]{\mu_k}\right)$, that is $\beta^2\mu_k\ge 4m_1$, the eigenvalues of H are all positive and real numbers, the corresponding eigenfunction have the form $U_k^{\pm}=\left(u_k,v_k,-\lambda_k^{\pm}u_k,-\lambda_k^{\pm}v_k\right)$. For (3.13) and future reference, we observe that for all $k\ge 1$,

$$\left\|\nabla^m u_k\right\|^2+\left\|\nabla^m v_k\right\|^2=\mu_k, \left\|u_k\right\|^2+\left\|v_k\right\|^2=1, \left\|\nabla^{-m}u_k\right\|^2+\left\|\nabla^{-m}v_k\right\|^2=\frac{1}{\mu_k}. \tag{3.14}$$

Lemma 3.1 $g_i: H_0^m\times H_0^m\to H_0^m\times H_0^m, i=1,2$ is uniformly bounded and globally Lipschitz continuous.

Proof. $\forall u_1,u_2,v_1,v_2 \in H_0^m(\Omega)$, we get

$$\left\|g_i(u_1,v_1)-g_i(u_2,v_2)\right\|_{H_0^m\times H_0^m} \le \left\|g_{iu}(\xi,v_1)\right\|_\infty \left\|u_1-u_2\right\|_{H_0^m} + \left\|g_{iv}(u_2,\eta)\right\|_\infty \left\|v_1-v_2\right\|_{H_0^m}, \tag{3.15}$$

where $\xi=\theta u_1+(1-\theta)u_2, \eta=\theta v_1+(1-\theta)v_2, 0<\theta<1$. According to (H1), we can obtain

$$\begin{aligned}&\left\|g_i(u_1,v_1)-g_i(u_2,v_2)\right\|_{H_0^m\times H_0^m}\\ &\le C_1\left\|u_1-u_2\right\|_{H_0^m}+C_2\left\|v_1-v_2\right\|_{H_0^m}\le\frac{l}{2}\left(\left\|u_1-u_2\right\|_{H_0^m}+\left\|v_1-v_2\right\|_{H_0^m}\right)\end{aligned}. \tag{3.16}$$

Theorem 3.1 If $\beta^2\mu_k \ge 4m_1$ holds, $l/2$ is maximum Lipschitz constant of $g_i(u,v),(i=1,2)$, and if $N_1 \in N$ is sufficiently large such that when $N \ge N_1$, the following inequality holds:

$$(\mu_{N+1}-\mu_N)\left(\frac{\beta}{2}-\frac{1}{2}\sqrt{\beta^2\mu_k-4m_1}\right)\ge\frac{4\sqrt{2}l}{\sqrt{\beta^2\mu_k-4m_1}}+1, \tag{3.17}$$

Then the operator H satisfies the spectral gap condition of (2.6).

Proof. When $\beta^2\mu_k \ge 4m_1$, all the eigenvalues of H are real and positive, and we can easily know that both sequences $\{\lambda_k^-\}_{k\ge1}$ and $\{\lambda_k^+\}_{k\ge1}$ are increasing.

The whole process of proof is divided into four steps.

Step 1. Since $\lambda_k^\pm$ is arranged in nondecreasing order. According to Lemma 2.1, given N such that λ_N^- and λ_{N+1}^- are consecutive, we separate the eigenvalue of H as

$$\sigma_1=\left\{\lambda_j^-,\lambda_k^+ \,\middle|\, \max\left\{\lambda_j^-,\lambda_k^+\right\}\le\lambda_N^-\right\}, \tag{3.18}$$

$$\sigma_2=\left\{\lambda_j^-,\lambda_k^\pm \,\middle|\, \lambda_j^-\le\lambda_N^-\le\min\left\{\lambda_j^-,\lambda_k^\pm\right\}\right\}, \tag{3.19}$$

Step 2. We make decomposition of X

$$X_1=span\left\{U_j^-,U_k^+ \,\middle|\, \lambda_j^-,\lambda_k^+\in\sigma_1\right\}. \tag{3.20}$$

$$X_2=span\left\{U_j^+,U_k^\pm \,\middle|\, \lambda_j^-,\lambda_k^+\in\sigma_2\right\}. \tag{3.21}$$

In order to make these two subspaces orthogonal and satisfy spectral inequality (2.6) $\Lambda_1=\lambda_N^-,\Lambda_2=\lambda_{N+1}^-$, we further decompose

$$X_2=X_c\oplus X_R, \tag{3.22}$$

with

$$X_c=span\left\{U_j^+ \,\middle|\, \lambda_j^-\le\lambda_N^-<\lambda_j^+\right\}, \tag{3.23}$$

$$X_R=span\left\{U_R^\pm \,\middle|\, \lambda_N^-\le\lambda_k^\pm\right\}. \tag{3.24}$$

And let $X_N=X_1\oplus X_c$. Next, we stipulate an eigenvalue scale product of X such that X_1 and X_2 are orthogonal, therefore we need to introduce two functions:

Let $\Phi: X_N\to R, \psi: X_R\to R$.

$$\begin{aligned}\Phi(U,V)=&\beta\left(\nabla^m u,\nabla^m \overline{y}_1\right)+2\beta\left(\nabla^{-m}z_1,\nabla^m u\right)+2\beta\left(\nabla^m v,\nabla^m \overline{y}_2\right)\\&+2\beta\left(\nabla^{-m}z_2,\nabla^m v\right)+2\beta\left(\nabla^{-m}p,\nabla^m \overline{y}_1\right)+4\left(\nabla^{-m}p,\nabla^{-m}z_1\right)\\&+2\beta\left(\nabla^{-m}q,\nabla^m \overline{y}_2\right)+4\left(\nabla^{-m}q,\nabla^{-m}z_2\right)-4M(\upsilon)\left(\overline{u},y_1\right)\\&+\left(2\beta^2-\beta\right)\left(\nabla^m\overline{u},\nabla^m y_1\right)-4M(\upsilon)\left(\overline{v},y_2\right)+\left(2\beta^2-\beta\right)\left(\nabla^m\overline{v},\nabla^m y_2\right),\end{aligned} \tag{3.25}$$

$$\begin{aligned}\psi(U,V)=&\left(\nabla^m u,\nabla^m y_1\right)+\left(\nabla^m v,\nabla^m y_2\right)+2\left(\nabla^{-m}z_1,\nabla^m u\right)+2\left(\nabla^{-m}z_2,\nabla^m v\right)\\&+\left(\nabla^{-m}z_1,\nabla^{-m}p\right)+\left(\nabla^{-m}z_2,\nabla^{-m}q\right)-4M(\upsilon)\left(\overline{u},y_1\right)\\&+2\beta^2\left(\nabla^m\overline{u},\nabla^m y_1\right)-4M(\upsilon)\left(\overline{v},y_2\right)+2\beta^2\left(\nabla^m\overline{v},\nabla^m y_2\right),\end{aligned} \tag{3.26}$$

where $U=(u,v,p,q),V=(y_1,y_2,z_1,z_2)$. $\overline{y}_1,\overline{y}_2,\overline{z}_1,\overline{z}_2$ respectively are the conjugation of y_1,y_2,z_1,z_2.

Let $U=(u,v,p,q)\in X_N$, then

$$\begin{aligned}&\Phi(U,U)\\&=\beta\left(\nabla^m u,\nabla^m\overline{u}\right)+2\beta\left(\nabla^{-m}p,\nabla^m u\right)+\beta\left(\nabla^m v,\nabla^m\overline{v}\right)+2\beta\left(\nabla^{-m}q,\nabla^m v\right)\\&\quad+2\beta\left(\nabla^{-m}p,\nabla^m\overline{u}\right)+4\left(\nabla^{-m}p,\nabla^{-m}p\right)+2\beta\left(\nabla^{-m}q,\nabla^m\overline{v}\right)\\&\quad+4\left(\nabla^{-m}q,\nabla^{-m}q\right)-4M(\upsilon)\left(\overline{u},u\right)+\left(2\beta^2-\beta\right)\left(\nabla^m\overline{u},\nabla^m u\right)\\&\quad-4M(\upsilon)\left(\overline{v},v\right)+\left(2\beta^2-\beta\right)\left(\nabla^m\overline{v},\nabla^m v\right)\\&=\beta\left\|\nabla^m u\right\|^2-2\left\|\nabla^{-m}p\right\|^2-\frac{\beta^2}{2}\left\|\nabla^m u\right\|^2+\beta\left\|\nabla^m v\right\|^2-2\left\|\nabla^{-m}q\right\|^2-\frac{\beta^2}{2}\left\|\nabla^m v\right\|^2\\&\quad-2\left\|\nabla^{-m}p\right\|^2-\frac{\beta^2}{2}\left\|\nabla^m\overline{u}\right\|^2+4\left\|\nabla^{-m}p\right\|^2-2\left\|\nabla^{-m}q\right\|^2-\frac{\beta^2}{2}\left\|\nabla^m\overline{v}\right\|^2+4\left\|\nabla^{-m}q\right\|^2\\&\quad-4M(\upsilon)\left\|u\right\|^2+\left(2\beta^2-\beta\right)\left\|\nabla^m u\right\|^2-4M(\upsilon)\left\|v\right\|^2+\left(2\beta^2-\beta\right)\left\|\nabla^m v\right\|^2\\&\geq\beta^2\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)-4M(\upsilon)\left(\left\|u\right\|^2+\left\|v\right\|^2\right)\\&\geq\left(\beta^2\mu_1-4M(\upsilon)\right)\left(\left\|u\right\|^2+\left\|v\right\|^2\right)\\&\geq\left(\beta^2\mu_1-4m_1\right)\left(\left\|u\right\|^2+\left\|v\right\|^2\right).\end{aligned} \tag{3.27}$$

Since $\beta^2\mu_1>4m_1,m_0\leq M(s)\leq m_1$ holds, we can know $\Phi(U,U)\geq 0$. Therefore, for all $U\in X_N$, analogously, for all $U\in X_R$, we can get

$$\begin{aligned}&\psi(U,U)\\&=\left(\nabla^m u,\nabla^m u\right)+\left(\nabla^m v,\nabla^m v\right)+2\left(\nabla^{-m}p,\nabla^m u\right)+2\left(\nabla^{-m}q,\nabla^m v\right)\\&\quad+\left(\nabla^{-m}p,\nabla^{-m}p\right)+\left(\nabla^{-m}q,\nabla^{-m}q\right)-4M(\upsilon)\left(\overline{u},u\right)\\&\quad+2\beta^2\left(\nabla^m\overline{u},\nabla^m u\right)-4M(\upsilon)\left(\overline{v},v\right)+2\beta^2\left(\nabla^m\overline{v},\nabla^m v\right)\\&=\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2+2\left(\nabla^{-m}p,\nabla^m u\right)+2\left(\nabla^{-m}q,\nabla^m v\right)+\left\|\nabla^{-m}p\right\|^2\\&\quad+\left\|\nabla^{-m}q\right\|^2-4M(\upsilon)\left\|u\right\|^2+2\beta^2\left\|\nabla^m u\right\|^2-4M(\upsilon)\left\|v\right\|^2+2\beta^2\left\|\nabla^m v\right\|^2\\&\geq\left(2\beta^2\mu_1-4M(\upsilon)\right)\left(\left\|u\right\|^2+\left\|v\right\|^2\right)\\&\geq\left(2\beta^2\mu_1-4m_1\right)\left(\left\|u\right\|^2+\left\|v\right\|\right)^2\geq 0.\end{aligned} \tag{3.28}$$

From above, we know that for all $U \in X_R$, then $\psi(U,U) \geq 0$ holds. So, we define a scale product with Φ and ψ in X.

$$\langle\langle U,V \rangle\rangle_X = \Phi(P_N U, P_N V) + \psi(P_R U, P_R V), \tag{3.29}$$

where P_N, P_R are respectively the projection: $X \to X_N$, $X \to X_R$.

In the inner product of X in (3.29), X_1 and X_2 are orthogonal. In fact, we need prove that X_1 and X_c are orthogonal.

$$\begin{aligned}
&\langle\langle U_j^-, U_j^+ \rangle\rangle_X = \Phi(U_j^-, U_j^+) \\
&= \beta(\nabla^m u_j, \nabla^m \bar{u}_j) + 2\beta(-\lambda_j^+ \nabla^{-m} u_j, \nabla^m u_j) + \beta(\nabla^m v_j, \nabla^m \bar{v}_j) \\
&\quad + 2\beta(-\lambda_j^+ \nabla^{-m} v_j, \nabla^m v_j) + 2\beta(-\lambda_j^- \nabla^{-m} u_j, \nabla^m \bar{u}_j) + 4(-\lambda_j^+ \nabla^{-m} u_j, -\lambda_j^- \nabla^{-m} u_j) \\
&\quad + 2\beta(-\lambda_j^- \nabla^{-m} v_j, \nabla^{-m} \bar{v}_j) + 4(-\lambda_j^+ \nabla^{-m} v_j, -\lambda_j^- \nabla^{-m} v_j) - 4M(\sqrt[m]{\mu_j})(\bar{u}_j, u_j) \\
&\quad + (2\beta^2 - \beta)(\nabla^m \bar{u}_j, \nabla^m u_j) - 4M(\sqrt[m]{\mu_j})(\bar{v}_j, v_j) + (2\beta^2 - \beta)(\nabla^m \bar{v}_j, \nabla^m v_j) \\
&= \beta\|\nabla^m u_j\|^2 + \beta\|\nabla^m v_j\|^2 - 2\beta(\lambda_j^+ + \lambda_j^-)\|u_j\|^2 \\
&\quad - 2\beta(\lambda_j^+ + \lambda_j^-)\|v_j\|^2 + 4\lambda_j^- \lambda_j^+ \left(\|\nabla^{-m} u_j\|^2 + \|\nabla^{-m} v_j\|^2\right) - 4M(\sqrt[m]{\mu_j})\|u_j\|^2 \\
&\quad + (2\beta^2 - \beta)\|\nabla^m u_j\|^2 - 4M(\sqrt[m]{\mu_j})\|v_j\|^2 + (2\beta^2 - \beta)\|\nabla^m v_j\|^2 \\
&= 2\beta^2 \mu_j - 2\beta(\lambda_j^+ + \lambda_j^-) + 4\lambda_j^- \lambda_j^+ \frac{1}{\mu_j} - 4M(\sqrt[m]{\mu_j}) = 0.
\end{aligned} \tag{3.30}$$

According to $\|u_j\|^2 + \|v_j\|^2 = 1$, $\|\nabla^m u_j\|^2 + \|\nabla^m v_j\|^2 = \mu_j$, $\|\nabla^{-m} u_j\|^2 + \|\nabla^{-m} v_j\|^2 = \frac{1}{\mu_j}$ and $\lambda_j^+ + \lambda_j^- = \beta\mu_j$, $\lambda_j^+ \lambda_j^- = M(\sqrt[m]{\mu_j})\mu_j$, we can get the above results.

Step 3. Next, we estimate the Lipschitz constant l_F of F,

$$F(U) = \left(0, 0, f_1(x) - g_1(u,v), f_2(x) - g_2(u,v)\right)^{\mathrm{T}}. \tag{3.31}$$

$g_i : H^m \times H^m \to H^m \times H^m, (i = 1,2)$ are globally Lipschitz continuous with maximum Lipschitz constant $\frac{l}{2}$ of g_i from (3.27), (3.28), for arbitrarily $U = (u, v, p, q) \in X$, we have.

Let $P_1 : X \to X_1, P_2 : X \to X_2$ are the orthogonal projection.

If $U = (u, v, p, q) \in X$, $U_1 = (u_1, v_1, p_1, q_1) = P_1 U$, $U_2 = (u_2, v_2, p_2, q_2) = P_2 U$ hold, then $P_1 u = u_1, P_1 v = v_1, P_2 u = u_2, P_2 v = v_2$.

$$\begin{aligned}
\|U\|_X^2 &= \Phi(P_1 U, P_1 U) + \Psi(P_2 U, P_2 U) \\
&\geq \left(\beta^2 \mu_1 - 4M(\upsilon_1)\right)\left(\|u_1\|^2 + \|v_1\|^2\right) + \left(2\beta^2 \mu_1 - 4M(\upsilon_2)\right)\left(\|u_2\|^2 + \|v_2\|^2\right) \\
&\geq \left(\beta^2 \mu_1 - 4m_1\right)\left(\|u\|^2 + \|v\|^2\right).
\end{aligned} \tag{3.32}$$

Let $U = (u_1, v_1, \bar{u}_1, \bar{v}_1), V = (u_2, v_2, \bar{u}_2, \bar{v}_2)$, then

$$\begin{aligned}&\left\|F(U)-F(V)\right\|_X\\&=\left\|g_1(u_1,v_1)-g_1(u_2,v_2)\right\|_{H_0^m\times H_0^m}+\left\|g_2(u_1,v_1)-g_2(u_2,v_2)\right\|_{H_0^m\times H_0^m}\\&\le\frac{l}{2}\left(\left\|u_1-u_2\right\|_{H_0^m}+\left\|v_1-v_2\right\|_{H_0^m}\right)+\frac{l}{2}\left(\left\|u_1-u_2\right\|_{H_0^m}+\left\|v_1-v_2\right\|_{H_0^m}\right)\\&=l\left(\left\|u_1-u_2\right\|_{H_0^m}+\left\|v_1-v_2\right\|_{H_0^m}\right)\le\frac{\sqrt{2}l}{\sqrt{\beta^2\mu_1-4m_1}}\left\|U-V\right\|_X.\end{aligned}\tag{3.33}$$

Therefore $l_F\le\frac{\sqrt{2}l}{\sqrt{\beta^2\mu_1-4m_1}}$.

Step 4. Now, we need prove the spectral gap condition (2.6) holds.

From the above mentioned $\Lambda_1=\lambda_N^-$ and $\Lambda_2=\lambda_{N+1}^-$, we can get

$$\Lambda_2-\Lambda_1=\lambda_{N+1}^--\lambda_N^-=\frac{\beta}{2}\left(\mu_{N+1}-\mu_N\right)+\frac{1}{2}\left(\sqrt{R(N)}-\sqrt{R(N+1)}\right),\tag{3.34}$$

where $R(N)=\beta^2\mu_N^2-4\mu_N M\left(\sqrt[m]{\mu_N}\right)$.

We determine $N_1>0$ such that for all $N\ge N_1$, let

$$R_1(N)=1-\sqrt{\frac{\beta^2}{\beta^2\mu_1-4m_1}-\frac{4m_1}{\mu_N\left(\beta^2\mu_1-4m_1\right)}},\tag{3.35}$$

we can get

$$\begin{aligned}&\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4m_1}\left(\mu_{N+1}-\mu_N\right)\\&=\sqrt{\beta^2\mu_1-4m_1}\left(\mu_{N+1}R_1(N+1)-\mu_N R_1(N)\right).\end{aligned}\tag{3.36}$$

According to the previous hypothesis $\varepsilon<m_0\le M(s)\le m_1=\frac{\beta^2\mu_1-1}{4}$, we can know

$$\lim_{N\to\infty}\left(\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4m_1}\left(\mu_{N+1}-\mu_N\right)\right)=0.\tag{3.37}$$

Then, combining (3.33), (3.34), (3.17) and (3.37), we obtain

$$\begin{aligned}\Lambda_2-\Lambda_1&>\left(\mu_{N+1}-\mu_N\right)\left(\frac{\beta}{2}-\frac{1}{2}\sqrt{\beta^2\mu_1-4m_1}\right)-1\\&\ge\frac{4\sqrt{2}l}{\sqrt{\beta^2\mu_1-4m_1}}\ge 4l_F.\end{aligned}\tag{3.38}$$

The proof is completed.

Theorem 3.2. [8] Under the condition of Theorem 3.1, the initial boundary value problem (1.1)-(1.6) admits an inertial manifold μ in X of the form

$$\mu=graph(\rho)=\left\{\omega+\rho(\zeta):\zeta\in X_1\right\},\tag{3.39}$$

where X_1, X_2 are as in (3.20), (3.21) and $\rho:X_1\to X_2$ is a Lipschitz continuous function.

4. Conclusion

In this paper, we prove the existence of the inertial manifolds for a class of higher-order coupled Kirchhoff-type equations. In the process of research, we take advantage of Hadamard's graph to get the equivalent form of the original equations and then use spectral gap condition. Based on some of the work above, we prove the existence of the inertial manifolds of the system. For this problem, we will study the exponential attractors, blow-up, random attractors and so on.

Acknowledgements

We express our heartful thanks to the anonymous reader for his/her careful reading of this paper. We hope that we can obtain valuable comments and advices. These contributions vastly improved the paper and making the paper better.

References

[1] Wu, J.Z. and Lin, G.G. (2010) An Inertial Manifold of the Two-Dimensional Strongly Damped Boussinesq Equation. *Journal of Yunnan University* (*Natural Science Edition*), **32**, 119-224.

[2] Xu, G.G., Wang, L.B. and Lin, G.G. (2014) Inertial Manifold for a Class of the Retarded Nonlinear Wave Equations. *Mathematica Applicata*, **27**, 887-891.

[3] Lou, R.J., Lv, P.H. and Lin, G.G. (2016) Exponential Attractors and Inertial Manifolds for a Class of Generalized Nonlinear Kirchhoff-Sine-Gordon Equation. *Journal of Advances in Mathematics*, **12**, 6361-6375.

[4] Chen, L., Wang, W. and Lin, G.G. (2016) Exponential Attractors and Inertial Manifolds for the Higher-Order Nonlinear Kirchhof-Type Equation. *International Journal of Modern Communication Technologies & Research*, **4**, 6-12.

[5] Lin, G.G. (2011) Nonlinear Evolution Equation. Yunnan University Press, Kunming.

[6] Robinson, J.C. (2001) Infinite Dimensional Dynamical System. Cambridge University Press, London.

[7] Zheng, S.M. and Milani, A. (2004) Exponential Attractors and Inertial Manifold for Singular Perturbations of the Cahn-Hilliard Equations. *Nonlinear Analysis*, **57**, 843-877. https://doi.org/10.1016/j.na.2004.03.023

[8] Tmam, R. (1988) Infinite Dimensional Dynamical Systems in Mechanics and Physics. Springer Verlag, New York. https://doi.org/10.1007/978-1-4684-0313-8