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On a Generalization of Einstein's $E = mc^2$

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Abstract

In order to maintain momentum conservation in collisions which undergo Lorentz transformation Einstein had to modify the Newtonian definition of momentum to relativistic domains. This resulted in his famous mass-energy equivalence relation $E = mc^2$. We suggest here that Lorentz invariance offers a more general form of relativistic momentum which would give a more general expression for kinetic and total energy. We further suggest methods to test the validity of this generalized relativistic mechanics.

Keywords

Special Theory of Relativity, Lorentz Invariance, Relativistic Momentum, Relativistic Kinetic Energy, Hyperbolic Trigonometric Functions

1. Introduction

The equivalence of mass and energy given by the relation:

$$E = mc^2 \tag{1}$$

is the most famous contribution of Einstein [1] [2] [3] [4] and indeed synonymous with his name. Equation (1) is a direct consequence of the definition of relativistic momentum that Einstein discovered which was needed for momentum conservation. It is much later that Einstein with the help of Minkowski, recognized that Equation (1) is tied up with Lorentz invariance. Among the many admirers of Einstein, Planck [5], liked this relation but felt this equation was just a "first approximation". It is this remark that we explore further in this paper to determine if this relation is an approximation to a more general relation that is consistent with Lorentz invariance.

In order to determine a more general version of the mass-energy relation, we study the factors that led to the development of this relation. In the subsequent section we will develop a generalization of the mass-energy relation, consistent with Lorentz invariance, whose first approximation will result in the familiar mass-energy equivalence. Finally we conclude by suggesting how experiments can test whether the expressions for the generalized kinetic energy and relativistic momentum are correct.

2. Development of the Energy Relation

Einstein noticed that the Lorentz transformation for velocity, being nonlinear, will not respect the linearly additive conservation of momentum in a collision between particles if the momentum $\boldsymbol{p}$ is defined as in Newtonian mechanics as:

$$\boldsymbol{p} = m_o \boldsymbol{v}, \tag{2}$$

where m_o is mass (inertia) of a particle moving with a velocity $\boldsymbol{v}$. He noticed that conservation of momentum will be at stake under Lorentz transformation if we stick to the Newtonian definition for momentum. The stroke of his genius was to define the relativistic momentum $\boldsymbol{p}_r$ as:

$$\boldsymbol{p}_r = \gamma m_o \boldsymbol{v}, \tag{3}$$

where γ is the Lorentz factor given by:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}, \beta = \frac{v}{c}. \tag{4}$$

This directly results in the expression for the kinetic energy T as:

$$T = \int_0^v \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{p} = \int_0^v v \mathrm{d}p = (\gamma - 1) m_o c^2. \tag{5}$$

The velocity independent term in Equation (5), *i.e.*, the constant of integration prompted Einstein [6] [7] [8] to write:

$$E = T + m_o c^2 = \gamma m_o c^2, \tag{6}$$

where E is the total energy and the second term $m_o c^2$ is termed the rest energy. Equation (6) is what Einstein termed as energy-mass equivalence.

The success of this expression for kinetic energy as given in Equation (5) lies in the observation that it leads to in the approximation at low velocities:

$$T = (\gamma - 1) m_o c^2 = m_o c^2 \left(1 + \beta^2/2 + \cdots - 1\right) = \frac{1}{2} m_o v^2. \tag{7}$$

familiar in non-relativistic Newtonian mechanics. Einstein considered this as the main validation for the expression in Equation (5). Equation (3) and Equation (6) lead us to the important relation: for the energy:

$$E^2 = c^2 \boldsymbol{p} \cdot \boldsymbol{p} + m_o^2 c^4. \tag{8}$$

This relation guarantees Lorentz invariance and is a direct consequence of the simple identity:

$$\gamma^2 - \gamma^2 \beta^2 = 1. \tag{9}$$

This identity is akin to the properties of hyperbolic trigonometric cosh and sinh functions. This is the key used to generalize the definition of relativistic momentum in the next section. As has been mentioned by many authors [7] [9], Einstein never made any reference in any of his research publications or letters to Equation (8) that establishes Lorentz invariance of the mass m_o. Although there were many attempts to prove or justify Equation (5), ii is primarily the guess of the genius. The evidence for this comes from the fact that Einstein himself wrote subsequently several papers to do this but was not satisfied by any of them. For him, this was required for the conservation of momentum which is crucial for mechanics.

3. General Formulation for Energy

We attempt here to see if there is a more general expression for the relativistic momentum for which Equation (3) is just an approximation and analyze its consequences. Specifically we define the momentum as:

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{u}_v m_o c \sinh^{-1}(\beta\gamma) \\ \boldsymbol{p} &= \boldsymbol{u}_v m_o c \cosh^{-1}\gamma \\ \boldsymbol{p} &= \boldsymbol{u}_v m_o c \tanh^{-1}\beta \\ \boldsymbol{p} &= \boldsymbol{u}_v m_o c \log\left(\gamma(1+\beta)\right) \end{aligned} \tag{10}$$

where $\boldsymbol{u}_v$ denotes the unit vector in the direction of $\boldsymbol{v}$ and v denotes $\|\boldsymbol{v}\|_2$ or alternatively

$$\begin{aligned} &\sinh\left(\frac{p}{m_o c}\right) = \beta\gamma \\ &\cosh\left(\frac{p}{m_o c}\right) = \gamma \\ &\tanh\left(\frac{p}{m_o c}\right) = \beta = \frac{v}{c} \\ &\frac{m_o c^2 \cosh\left(\frac{p}{m_o c}\right)}{m_o c^2 \sinh\left(\frac{p}{m_o c}\right)} = \frac{c}{v} \end{aligned} \tag{11}$$

With this definition, the expression for the kinetic energy becomes:

$$K = \int_0^v \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{p} = \int_0^v v\mathrm{d}p = m_o c^2 \int_0^\beta \frac{\beta \mathrm{d}\beta}{1-\beta^2} = m_o c^2 \log\gamma. \tag{12}$$

Naively as a first approximation we get:

$$K \approx m_o c^2 (\gamma - 1) = T, \tag{13}$$

which is the usual expression given in Equation (5). More precisely, since $\gamma \ge 1$:

$$K = -m_o c^2 \log\sqrt{1-\beta^2}, \tag{14}$$

which for smaller values of β yields:

$$K \approx m_o c^2 \left(1 - \sqrt{1-\beta^2}\right). \tag{15}$$

This in particular appears as the kinetic energy of a relativistic particle in the classical Lagrangian. From Equation (10), we obtain:

$$\left[m_o c^2 \cosh\left(\frac{p}{m_o c}\right)\right]^2 - \left[m_o c^2 \sinh\left(\frac{p}{m_o c}\right)\right]^2 = m_o^2 c^4 \tag{16}$$

This further implies that:

$$\left(\gamma m_o c^2\right)^2 - \left(\gamma\beta m_o c^2\right)^2 = \left(\gamma m_o c^2\right)^2 - \left(\gamma m_o vc\right)^2 = m_o^2 c^4. \tag{17}$$

Thus:

$$\gamma m_o c^2 = m_o c^2 \mathrm{e}^{\log\gamma} = m_o c^2 \mathrm{e}^{\frac{K}{m_o c^2}} \approx m_o c^2\left(1 + \frac{K}{m_o c^2}\right) = K + m_o c^2 = E$$

$$\gamma m_o vc = m_o c^2 \sinh\left(\frac{p}{m_o c}\right) \tag{18}$$

For low values of p this becomes:

$$\gamma m_o vc \approx m_o c^2 \frac{p}{m_o c} = cp. \tag{19}$$

Then we get the expression for low values of p:

$$E^2 - c^2 \boldsymbol{p}\cdot\boldsymbol{p} = m_o^2 c^4,$$

which is Equation (6) as a first approximation. The close connection of the Lorentz transform to hyperbolic functions has been known from the early days of special relativity. We emphasize that in the generalization in Equation (17), the first term is E^2 and the second term is $c^2 \boldsymbol{p}\cdot\boldsymbol{p}$ for Einstein. This is only an approximation to the more general expression, where the first term in the identity:

$$\left(m_o c^2 \cosh\left(\frac{p}{m_o c}\right)\right)^2 = \left(m_o c^2 \mathrm{e}^{\frac{T}{m_o c^2}}\right)^2 \tag{20}$$

and the second term is given by $\left(m_o c^2 \sinh\left(\frac{p}{m_o c}\right)\right)^2$.

4. Conclusion

A new definition for relativistic momentum that is consistent with Lorentz invariance is suggested. As is known, the relativistic formulas for momentum and energy are not amenable to direct tests of high precision [8] [10]. The one thing that has been tested experimentally is that kinetic energy becomes infinite as the speed approaches the speed of light. This is again true with the generalized expression given in Equation (12). The other test is on the precision of the magnetic moment of the electron in a magnetic field. It is not clear whether these experiments can rule out the generalizations made in this paper. However, there is absolutely no change in this generalization that the energy at rest is equal to mc^2.

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