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Dynamic Inequalities for Convex Functions Harmonized on Time Scales

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Abstract

We present here some general fractional Schlömilch's type and Rogers-Hölder's type dynamic inequalities for convex functions harmonized on time scales. First we present general fractional Schlömilch's type dynamic inequalities and generalize it for convex functions of several variables by using Bernoulli's inequality, generalized Jensen's inequality and Fubini's theorem on diamond-α calculus. To conclude our main results, we present general fractional Rogers-Hölder's type dynamic inequalities for convex functions by using general fractional Schlömilch's type dynamic inequality on diamond-α calculus for $p_i > 1$ with $\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

Keywords

Delta, Nabla and Diamond-α Time Scales, Fractional Integral Inequalities

1. Introduction

In the following, we present a result proved by Mitrinović and Pečarić as given in [1] and ([2], p. 235).

Theorem 1. Let $g_i \in G(f_i, k)$ for $(i = 1, 2)$ be a class, where $f_i(x)$ for $(i = 1, 2)$ are continuous functions and $f_2(x) > 0$ implies $g_2(x) > 0$ for every $x \in [a, b]$ and $g_i : [a, b] \to \mathbb{R}$ are represented by

$$g_i(x) := \int_a^b k(x, y) f_i(y) \mathrm{d}y, \quad \forall x \in [a, b], \ i = 1, 2,$$

where $k(x, y)$ is nonnegative arbitrary kernel. Consider $w(x) \geq 0$ for every $x \in [a, b]$. Let $F : \mathbb{R}_0^+ = [0, \infty) \to \mathbb{R}$ be a convex and increasing function, then the following inequality holds

$$\int_a^b w(x) F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) \mathrm{d}x \le \int_a^b F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) s(y) \mathrm{d}y, \tag{1}$$

where,

$$s(y) := f_2(y) \int_a^b \frac{w(x) k(x,y)}{g_2(x)} \mathrm{d}x, \quad \forall y \in [a,b], \quad g_2(x) \ne 0.$$

Next we present a result on diamond-α calculus, as given in [3].

Theorem 2. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two time scales, and $a,b \in \mathbb{T}_1$; $c,d \in \mathbb{T}_2$; $k(x,y)$ is a kernel function with $x \in [a,b]_{\mathbb{T}_1}$, $y \in [c,d]_{\mathbb{T}_2}$; k is continuous function from $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ into $\mathbb{R}_0^+ = [0,\infty)$. Consider

$$K(x) := \int_c^d k(x,y) \lozenge_\alpha y, \ \forall x \in [a,b]_{\mathbb{T}_1}.$$

We assume that $K(x) > 0$, $\forall x \in [a,b]_{\mathbb{T}_1}$. Consider $f : [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ continuous, and the $\lozenge_\alpha$-integral operator function

$$g(x) := \int_c^d k(x,y) f(y) \lozenge_\alpha y, \ \forall x \in [a,b]_{\mathbb{T}_1}.$$

Consider also the weight function $w : [a,b]_{T_1} \to \mathbb{R}_0^+$, which is continuous. Define further the function $s(y) := \int_a^b \frac{w(x) k(x,y)}{K(x)} \lozenge_\alpha x, \forall y \in [c,d]_{T_2}$. Let I denote any of $(0,\infty)$ or $[0,\infty)$, and $F : I \to \mathbb{R}$ be a convex and increasing function. In particular, we assume that

$$|f|\left([c,d]_{T_2}\right) \subseteq I.$$

Then

$$\int_a^b w(x) F\left(\frac{|g(x)|}{K(x)}\right) \lozenge_\alpha x \le \int_c^d s(y) F\left(|f(y)|\right) \lozenge_\alpha y. \tag{2}$$

We extend these results on time scale calculus. In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a,b \in \mathbb{T}$ with $a < b$ and an interval $[a,b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adapted from [4] [5] [6].

Time scale calculus was initiated by Stefan Hilger as given in [7]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. It is denoted by $\mathbb{T}$. For $t \in \mathbb{T}$, *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0,\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, \infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called *dense*. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the derivative f^{Δ} is defined as follows. Let $t \in \mathbb{T}^k$, if there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a neighborhood U of t, such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \le \epsilon \left| \sigma(t) - s \right|,$$

for all $s \in U$, then f is said to be *delta differentiable* at t, and $f^{\Delta}(t)$ is called the *delta derivative* of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*), if it is continuous at each right-dense point and there exists a finite left limit in every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [4] [5] [6].

Definition 1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$, then the *delta integral* of f is defined by

$$\int_a^b f(t) \Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [4] [5] [6] [8].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}_k = \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that for any $\epsilon > 0$, there exists a neighborhood V of t, such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \le \epsilon \left| \rho(t) - s \right|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is *left-dense continuous* (*ld-continuous*), provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [4] [5] [6] [8].

Definition 2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a *nabla antiderivative* of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$, then the *nabla integral* of g is defined by

$$\int_a^b g(t) \nabla t = G(b) - G(a).$$

Now we present short introduction of diamond-α derivative as given in [4] [9].

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, *diamond-α dynamic derivative* $f^{\Diamond_\alpha}(t)$ is defined by

$$f^{\Diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0,1)$.

Theorem 3. [9]: Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then

1) $f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$(f \pm g)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t) \pm g^{\Diamond_\alpha}(t).$$

2) $fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$(fg)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t) g(t) + \alpha f^{\sigma}(t) g^{\Delta}(t) + (1-\alpha) f^{\rho}(t) g^{\nabla}(t).$$

3) For $g(t) g^{\sigma}(t) g^{\rho}(t) \ne 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\Diamond_\alpha}(t) = \frac{f^{\Diamond_\alpha}(t) g^{\sigma}(t) g^{\rho}(t) - \alpha f^{\sigma}(t) g^{\rho}(t) g^{\Delta}(t) - (1-\alpha) f^{\rho}(t) g^{\sigma}(t) g^{\nabla}(t)}{g(t) g^{\sigma}(t) g^{\rho}(t)}.$$

Theorem 4. [9]: Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \Diamond_\alpha s = \alpha \int_a^t h(s) \Delta s + (1-\alpha) \int_a^t h(s) \nabla s, \quad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 5. [9]: Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\Diamond_\alpha$-integrable functions on $[a,b]_{\mathbb{T}}$, then

1) $\int_a^t [f(s) \pm g(s)] \Diamond_\alpha s = \int_a^t f(s) \Diamond_\alpha s \pm \int_a^t g(s) \Diamond_\alpha s$;

2) $\int_a^t c f(s) \Diamond_\alpha s = c \int_a^t f(s) \Diamond_\alpha s$;

3) $\int_a^t f(s) \Diamond_\alpha s = -\int_t^a f(s) \Diamond_\alpha s$;

4) $\int_a^t f(s) \Diamond_\alpha s = \int_a^b f(s) \Diamond_\alpha s + \int_b^t f(s) \Diamond_\alpha s$;

5) $\int_a^a f(s) \Diamond_\alpha s = 0$.

We need the following results.

Theorem 6. [4]: Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a,b]_{\mathbb{T}}, (c,d))$ and $h \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |h(s)| \Diamond_\alpha s > 0$. If $F \in C((c,d), \mathbb{R})$ is convex, then *generalized Jensen's inequality is*

$$F\left(\frac{\int_a^b |h(s)| g(s) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}\right) \le \frac{\int_a^b |h(s)| F(g(s)) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}. \tag{3}$$

If F is strictly convex, then the inequality $\le$ can be replaced by $<$.

Theorem 7. [3] [10]: Let $a,b \in \mathbb{T}$. Let $f_i \in C\left([a,b]_{\mathbb{T}}, \mathbb{R}\right)$, $i = 1, \cdots, n$ are $\diamond_\alpha$-integrable functions and $p_i > 1$ such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then

$$\int_a^b \prod_{i=1}^{n} |f_i(t)| \diamond_\alpha t \le \prod_{i=1}^{n} \left(\int_a^b |f_i(t)|^{p_i} \diamond_\alpha t\right)^{\frac{1}{p_i}}, \tag{4}$$

which is *generalized Rogers-Hölder's Inequality*.

Definition 3. [11]: A function $f : \mathbb{T} \to \mathbb{R}$ is called *convex* on $I_{\mathbb{T}} = I \cap \mathbb{T}$, where I is an interval of $\mathbb{R}$ (open or closed), if

$$f(\lambda t + (1-\lambda)s) \le \lambda f(t) + (1-\lambda) f(s), \tag{5}$$

for all $t, s \in I_{\mathbb{T}}$ and all $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in I_{\mathbb{T}}$.

The function f is strictly convex on $I_{\mathbb{T}}$ if (5) is strict for distinct $t, s \in I_{\mathbb{T}}$ and $\lambda \in (0,1)$.

The function f is concave (respectively, strictly concave) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

3. Main Results

First we present $\diamond_\alpha$-integral general fractional Schlömilch's type inequalities on time scales, which is an extension of Schlömilch's inequality given in [12].

Theorem 8. Let $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ be two time scales; $k(x,y) : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to \mathbb{R}_0^+$ is continuous kernel function with $x \in [a,b]_{\mathbb{T}_1}$ and $y \in [c,d]_{\mathbb{T}_2}$. Let $\diamond_\alpha$-integral operator functions $g_i : [a,b]_{\mathbb{T}_1} \to \mathbb{R}$ belonging to a class $G(f_i, k)$ for $(i = 1, 2)$ are represented by

$$g_i(x) := \int_c^d k(x,y) f_i(y) \diamond_\alpha y,$$

where $f_i : [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ are continuous functions. Continuous weight function is defined by $w : [a,b]_{\mathbb{T}_1} \to \mathbb{R}_0^+$ with $\int_a^b w(x) \diamond_\alpha x = 1$. Define $s(y) := f_2(y) \int_a^b w(x) \frac{k(x,y)}{g_2(x)} \diamond_\alpha x$ and $\forall y \in [c,d]_{\mathbb{T}_2}$, where $f_2(y) > 0$ implies $g_2(x) > 0$. Let $F : \mathbb{R}_0^+ = [0,\infty) \to \mathbb{R}_0^+$ be a convex and increasing function.

If $\eta_2 \ge \eta_1 \ge 1$, then the following inequality holds

$$\left(\int_a^b w(x) F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)^{\eta_1} \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \le \left(\int_c^d F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)^{\eta_2} s(y) \diamond_\alpha y\right)^{\frac{1}{\eta_2}}. \tag{6}$$

Proof. In order to prove this Theorem, we need Bernoulli's inequality, that is, if $x > 0$, then

$$px+1-p\le x^{p},\quad \text{if } p\ge 1.$$

Since $\eta_2\ge\eta_1\ge 1$, we have $\frac{\eta_2}{\eta_1}\ge 1$. Thus, by Bernoulli's inequality, we have

$$\int_a^b w(x)\left(\frac{F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)}{\int_a^b w(x)F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)\Diamond_\alpha x}\right)^{\frac{\eta_2}{\eta_1}}\Diamond_\alpha x$$

$$\ge\int_a^b w(x)\left(\frac{\frac{\eta_2}{\eta_1}F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)}{\int_a^b w(x)F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)\Diamond_\alpha x}+1-\frac{\eta_2}{\eta_1}\right)\Diamond_\alpha x=1,$$

that is,

$$\int_a^b w(x)F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)^{\frac{\eta_2}{\eta_1}}\Diamond_\alpha x\ge\left(\int_a^b w(x)F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)\Diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}.$$

Let F be replaced by F^{η_1} and taking power $\frac{1}{\eta_2}>0$, we get

$$\left(\int_a^b w(x)F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)^{\eta_1}\Diamond_\alpha x\right)^{\frac{1}{\eta_1}}$$

$$\le\left(\int_a^b w(x)F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)^{\eta_2}\Diamond_\alpha x\right)^{\frac{1}{\eta_2}}$$

$$=\left(\int_a^b w(x)F\left(\left|\frac{\int_c^d k(x,y)f_1(y)\Diamond_\alpha y}{g_2(x)}\right|\right)^{\eta_2}\Diamond_\alpha x\right)^{\frac{1}{\eta_2}}$$

$$=\left(\int_a^b w(x)F\left(\left|\frac{1}{g_2(x)}\int_c^d k(x,y)f_2(y)\frac{f_1(y)}{f_2(y)}\Diamond_\alpha y\right|\right)^{\eta_2}\Diamond_\alpha x\right)^{\frac{1}{\eta_2}}$$

$$\le\left(\int_a^b w(x)\left(\frac{1}{g_2(x)}\int_c^d k(x,y)f_2(y)F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)^{\eta_2}\Diamond_\alpha y\right)\Diamond_\alpha x\right)^{\frac{1}{\eta_2}}$$

$$=\left(\int_c^d F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)^{\eta_2}\left(f_2(y)\int_a^b w(x)\frac{k(x,y)}{g_2(x)}\Diamond_\alpha x\right)\Diamond_\alpha y\right)^{\frac{1}{\eta_2}}$$

$$=\left(\int_c^d F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)^{\eta_2}s(y)\Diamond_\alpha y\right)^{\frac{1}{\eta_2}},$$

where we used the generalized Jensen's inequality and Fubini's theorem.

This proves the claim. □

Remark. If we set $\eta_1 = \eta_2 = 1$ and $F:[0,\infty) \to \mathbb{R}$ be a convex and increasing function, then (6) takes the form

$$\int_a^b w(x) F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) \Diamond_\alpha x \le \int_c^d F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) s(y) \Diamond_\alpha y. \tag{7}$$

If $[a,b]_{\mathbb{T}_1} = [c,d]_{\mathbb{T}_2}$, where $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then (7) takes the form of (1).

Corollary 1. If $\eta_1 = \eta_2 = 1$, $F:[0,\infty) \to \mathbb{R}$ be a convex and increasing function and $\alpha = 1$, then delta version form of (6) is

$$\int_a^b w(x) F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) \Delta x \le \int_c^d F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) s(y) \Delta y. \tag{8}$$

If $\eta_1 = \eta_2 = 1$, $F:[0,\infty) \to \mathbb{R}$ be a convex and increasing function and $\alpha = 0$, then nabla version form of (6) is

$$\int_a^b w(x) F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) \nabla x \le \int_c^d F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) s(y) \nabla y. \tag{9}$$

Remark. Now we take that F is not necessarily increasing and is taken from $(0,\infty)$ into $\mathbb{R}_0^+$ and $\dfrac{f_1(y)}{f_2(y)}$ has fixed and strict sign, then according to Theorem 8, we get

$$\left(\int_a^b w(x) F\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)^{\eta_1} \Diamond_\alpha x\right)^{\frac{1}{\eta_1}} \le \left(\int_c^d F\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)^{\eta_2} s(y) \Diamond_\alpha y\right)^{\frac{1}{\eta_2}}.$$

Corollary 2. If we apply for $F(x) = x^p$, $p > 1$, then (6) takes the form

$$\left(\int_a^b w(x) \left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)^{p\eta_1} \Diamond_\alpha x\right)^{\frac{1}{\eta_1}} \le \left(\int_c^d \left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)^{p\eta_2} s(y) \Diamond_\alpha y\right)^{\frac{1}{\eta_2}}. \tag{10}$$

Corollary 3. If we apply for $F(x) = \mathrm{e}^x, x \ge 0$, then (6) takes the form

$$\left(\int_a^b w(x) e^{\eta_1\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right)} \Diamond_\alpha x\right)^{\frac{1}{\eta_1}} \le \left(\int_c^d \mathrm{e}^{\eta_2\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)} s(y) \Diamond_\alpha y\right)^{\frac{1}{\eta_2}}. \tag{11}$$

Corollary 4. If $\eta_1 = \eta_2 = 1$, $F:(0,\infty) \to \mathbb{R}$ be a convex and not necessarily increasing function, $\dfrac{f_1(y)}{f_2(y)}$ has fixed and strict sign and we apply for $F(x) = -\ln x, x > 0$, then (6) takes the form

$$\int_a^b w(x) \ln\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) \Diamond_\alpha x \ge \int_c^d \ln\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) s(y) \Diamond_\alpha y. \tag{12}$$

Remark. If we set $f_2(y) = 1$, $g_1(x) = g(x)$, $f_1(y) = f(y)$, $\eta_1 = \eta_2 = 1$ and $F:[0,\infty) \to \mathbb{R}$ be a convex and increasing function, then

$$g_2(x) = \int_c^d k(x,y) \lozenge_\alpha y = K(x), \quad \forall x \in [a,b]_{\mathbb{T}_1}.$$

We assume that $K(x) > 0$, and define

$$s(y) := \int_a^b \frac{w(x)k(x,y)}{K(x)} \lozenge_\alpha x, \ \forall y \in [c,d]_{T_2}.$$

Then (6) takes the form of (2), as proved in [3].

Corollary 5. If we take $\mathbb{T}_1 = q^{\mathbb{N}_0}$, $q > 1$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\mathbb{T}_2 = \mathbb{R}$.

Then

$$\int_{q^m}^{q^n} f(x) \lozenge_\alpha x = (q-1) \sum_{i=m}^{n-1} q^i \left[\alpha f(q^i) + (1-\alpha) f(q^{i+1}) \right],$$

for $[a,b]_{\mathbb{T}_1} = [q^m, q^n]_{q^{\mathbb{N}_0}}$, $m < n$, where $m, n \in \mathbb{N}_0$.

And

$$\int_c^d f(y) \lozenge_\alpha y = \int_c^d f(y) \mathrm{d}y.$$

When $\eta_1 = \eta_2 = 1$ and $F : [0,\infty) \to \mathbb{R}$ be a convex and increasing function, then (6) can be written as

$$(q-1) \sum_{i=m}^{n-1} q^i \left[\alpha w(q^i) F\left(\left| \frac{g_1(q^i)}{g_2(q^i)} \right| \right) + (1-\alpha) w(q^{i+1}) F\left(\left| \frac{g_1(q^{i+1})}{g_2(q^{i+1})} \right| \right) \right]$$
$$\leq \int_c^d F\left(\left| \frac{f_1(y)}{f_2(y)} \right| \right) s(y) \mathrm{d}y.$$

We can generalize Theorem 8 for convex functions of several variables on time scales in the upcoming theorem.

Theorem 9. Let $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ be two time scales; $k(x,y) : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to \mathbb{R}_0^+$ is continuous kernel function with $x \in [a,b]_{\mathbb{T}_1}$ and $y \in [c,d]_{\mathbb{T}_2}$. Let $\lozenge_\alpha$-integral operator functions $g_i : [a,b]_{\mathbb{T}_1} \to \mathbb{R}$ belonging to a class $G(f_i, k)$ for $(i = 1,2,3)$ are represented by

$$g_i(x) := \int_c^d k(x,y) f_i(y) \lozenge_\alpha y,$$

where $f_i : [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ are continuous functions. Continuous weight function is defined by $w : [a,b]_{\mathbb{T}_1} \to \mathbb{R}_0^+$ with $\int_a^b w(x) \lozenge_\alpha x = 1$. Define $s(y) := f_2(y) \int_a^b w(x) \frac{k(x,y)}{g_2(x)} \lozenge_\alpha x$ and $\forall y \in [c,d]_{\mathbb{T}_2}$, where $f_2(y) > 0$ implies $g_2(x) > 0$. Let $F : \mathbb{R}_0^+ \times \mathbb{R}_0^+ = [0,\infty) \times [0,\infty) \to \mathbb{R}_0^+$ be a convex and increasing function.

If $\eta_2 \geq \eta_1 \geq 1$, then the following inequality holds

$$\left(\int_a^b w(x) F\left(\left| \frac{g_1(x)}{g_2(x)} \right|, \left| \frac{g_3(x)}{g_2(x)} \right| \right)^{\eta_1} \lozenge_\alpha x \right)^{\frac{1}{\eta_1}} \leq \left(\int_c^d F\left(\left| \frac{f_1(y)}{f_2(y)} \right|, \left| \frac{f_3(y)}{f_2(y)} \right| \right)^{\eta_2} s(y) \lozenge_\alpha y \right)^{\frac{1}{\eta_2}}. \quad (13)$$

Proof. Proof is similar to Theorem 8. □

Remark. If we set $\eta_1 = \eta_2 = 1$, $F:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a convex and increasing function and $[a,b]_{\mathbb{T}_1} = [c,d]_{\mathbb{T}_2}$, where $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then (13) reduces to

$$\int_a^b w(x) F\left(\left|\frac{g_1(x)}{g_2(x)}\right|, \left|\frac{g_3(x)}{g_2(x)}\right|\right) \mathrm{d}x \le \int_a^b F\left(\left|\frac{f_1(y)}{f_2(y)}\right|, \left|\frac{f_3(y)}{f_2(y)}\right|\right) s(y)\mathrm{d}y,$$

as given in ([2], p. 236).

Now we present $\Diamond_\alpha$-integral general fractional Rogers-Holder's type inequalities.

Upcoming result is an application of general fractional Schlömilch's type dynamic inequality.

Theorem 10. Let $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ be two time scales; $k_i(x,y):[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}\to\mathbb{R}_0^+$ for $i=1,\cdots,n\in\mathbb{N}$ are continuous kernel functions with $x\in[a,b]_{\mathbb{T}_1}$ and $y\in[c,d]_{\mathbb{T}_2}$. Let $\Diamond_\alpha$-integral operator functions $f_i, g_i:[a,b]_{\mathbb{T}_1}\to\mathbb{R}$ for $i=1,\cdots,n\in\mathbb{N}$ are represented by

$$f_i(x) := \int_c^d k_i(x,y) u_i(y) \Diamond_\alpha y,$$

and

$$g_i(x) := \int_c^d k_i(x,y) v_i(y) \Diamond_\alpha y,$$

where $u_i, v_i:[c,d]_{\mathbb{T}_2}\to\mathbb{R}$ are continuous functions for $i=1,\cdots,n\in\mathbb{N}$. Continuous weight function is defined by $w:[a,b]_{\mathbb{T}_1}\to\mathbb{R}_0^+$ with $\int_a^b w(x)\Diamond_\alpha x = 1$. Define $s_i(y) := v_i(y)\int_a^b w(x)\frac{k_i(x,y)}{g_i(x)}\Diamond_\alpha x$, and $\forall y\in[c,d]_{\mathbb{T}_2}$ for $i=1,\cdots,n\in\mathbb{N}$, where $v_i(y)>0$ implies $g_i(x)>0$ for $i=1,\cdots,n\in\mathbb{N}$. Let $F_i:\mathbb{R}_0^+\to\mathbb{R}_0^+$ for $i=1,\cdots,n\in\mathbb{N}$ are convex and increasing functions.

If $p_i>1$ with $\sum_{i=1}^n \frac{1}{p_i}<1$. Then the following inequality holds

$$\int_a^b w(x)\prod_{i=1}^n F_i\left(\left|\frac{f_i(x)}{g_i(x)}\right|\right)\Diamond_\alpha x \le \prod_{i=1}^n\left(\int_c^d F_i\left(\left|\frac{u_i(y)}{v_i(y)}\right|\right)^{p_i} s_i(y)\Diamond_\alpha y\right)^{\frac{1}{p_i}}. \tag{14}$$

Proof. Let $\gamma := \sum_{i=1}^n \frac{1}{p_i} < 1$ and $\zeta_i := \gamma p_i < p_i$ for $i=1,\cdots,n$. Then $\sum_{i=1}^n \frac{1}{\zeta_i} = 1$, where $\zeta_i>1$ for $i=1,\cdots,n$. We use here generalized Rogers-Hölder's inequality, Schlömilch's inequality, generalized Jensen's inequality and Fubini's theorem, as

$$\begin{aligned}
&\int_a^b w(x)\prod_{i=1}^n F_i\left(\left|\frac{f_i(x)}{g_i(x)}\right|\right)\Diamond_\alpha x \\
&= \int_a^b \prod_{i=1}^n\left(w(x)^{\frac{1}{\zeta_i}} F_i\left(\left|\frac{f_i(x)}{g_i(x)}\right|\right)\right)\Diamond_\alpha x \\
&\le \prod_{i=1}^n\left(\int_a^b w(x) F_i\left(\left|\frac{f_i(x)}{g_i(x)}\right|\right)^{\zeta_i}\Diamond_\alpha x\right)^{\frac{1}{\zeta_i}}
\end{aligned}$$

$$\le \prod_{i=1}^{n}\left(\int_a^b w(x)F_i\left(\left|\frac{f_i(x)}{g_i(x)}\right|\right)^{p_i}\Diamond_\alpha x\right)^{\frac{1}{p_i}}$$

$$=\prod_{i=1}^{n}\left(\int_a^b w(x)F_i\left(\left|\frac{\int_c^d k_i(x,y)u_i(y)\Diamond_\alpha y}{g_i(x)}\right|\right)^{p_i}\Diamond_\alpha x\right)^{\frac{1}{p_i}}$$

$$=\prod_{i=1}^{n}\left(\int_a^b w(x)F_i\left(\left|\frac{1}{g_i(x)}\int_c^d k_i(x,y)v_i(y)\frac{u_i(y)}{v_i(y)}\Diamond_\alpha y\right|\right)^{p_i}\Diamond_\alpha x\right)^{\frac{1}{p_i}}$$

$$\le \prod_{i=1}^{n}\left(\int_a^b w(x)\left(\frac{1}{g_i(x)}\int_c^d k_i(x,y)v_i(y)F_i\left(\left|\frac{u_i(y)}{v_i(y)}\right|\right)^{p_i}\Diamond_\alpha y\right)\Diamond_\alpha x\right)^{\frac{1}{p_i}}$$

$$=\prod_{i=1}^{n}\left(\int_c^d F_i\left(\left|\frac{u_i(y)}{v_i(y)}\right|\right)^{p_i}\left(v_i(y)\int_a^b\frac{w(x)k_i(x,y)}{g_i(x)}\Diamond_\alpha x\right)\Diamond_\alpha y\right)^{\frac{1}{p_i}}$$

$$=\prod_{i=1}^{n}\left(\int_c^d F_i\left(\left|\frac{u_i(y)}{v_i(y)}\right|\right)^{p_i}s_i(y)\Diamond_\alpha y\right)^{\frac{1}{p_i}}.$$

This proves the claim. □

Corollary 6. If we apply for $F_i(x)=x^{\xi_i}$, $x\ge 0$, $i=1,\cdots,n$ and let $\xi_i\ge 1$, $i=1,\cdots,n$. Then (14) takes the form

$$\int_a^b w(x)\prod_{i=1}^{n}\left(\left|\frac{f_i(x)}{g_i(x)}\right|\right)^{\xi_i}\Diamond_\alpha x\le\prod_{i=1}^{n}\left(\int_c^d\left(\left|\frac{u_i(y)}{v_i(y)}\right|\right)^{\xi_i p_i}s_i(y)\Diamond_\alpha y\right)^{\frac{1}{p_i}}. \tag{15}$$

4. Conclusion and Future Work

The study of dynamic inequalities on time scales has a lot of scope. This research article is devoted to some general fractional Schlömilch's type and Rogers-Hölder's type dynamic inequalities for convex functions harmonized on diamond-α calculus and their delta and nabla versions are similar cases. Similarly, in future, we can present such inequalities by using Riemann-Liouville type fractional integrals and fractional derivatives on time scales. It will also be very interesting to present such inequalities on quantum calculus.

References

[1] Mitrinović, D.S. and Pečarić, J.E. (1988) Generalizations of Two Inequalities of Godunova and Levin. *Bulletin of the Polish Academy of Sciences*, **36**, 645-648.

[2] Pečarić, J.E., Proschan, F. and Tong, Y.L. (1992) Convex Functions, Partial Orderings, and Statistical Applications, vol. 187 of Mathematics in Science and Engineering. Academic Press, Boston.

[3] Anastassiou, G.A. (2012) Integral Operator Inequalities on Time Scales. *International Journal of Difference Equations*, **7**, 111-137.

[4] Agarwal, R.P., O'Regan, D. and Saker, S. (2014) Dynamic Inequalities on Time Scales. Springer International Publishing, Cham, Switzerland. https://doi.org/10.1007/978-3-319-11002-8

[5] Bohner, M. and Peterson, A. (2001) Dynamic Equations on Time Scales. Birkhäuser Boston. https://doi.org/10.1007/978-1-4612-0201-1

[6] Bohner, M. and Peterson, A. (2003) Advances in Dynamic Equations on Time Scales. Birkhäuser Boston, Boston. https://doi.org/10.1007/978-0-8176-8230-9

[7] Hilger, S. (1988) Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. PhD Thesis, Universität Würzburg, Würzburg.

[8] Anderson, D., Bullock, J., Erbe, L., Peterson, A. and Tran, H. (2003) Nabla Dynamic Equations on Time Scales. *Pan-American Mathematical Journal*, **13**, 1-48.

[9] Sheng, Q., Fadag, M., Henderson, J. and Davis, J.M. (2006) An Exploration of Combined Dynamic Derivatives on Time Scales and Their Applications. *Nonlinear Analysis*: *Real World Applications*, **7**, 395-413. https://doi.org/10.1016/j.nonrwa.2005.03.008

[10] Chen, G.S., Huang, F.L. and Liao, L.F. (2014) Generalizations of Hölder Inequality and Some Related Results on Time Scales. *Journal of Inequalities and Applications*, **2014**, 207. https://doi.org/10.1186/1029-242X-2014-207

[11] Dinu, C. (2008) Convex Functions on Time Scales, Annals of the University of Craiova. *Mathematics and Computer Science Series*, **35**, 87-96.

[12] Hardy, G.H., Littlewood, J.E. and Pölya, G. (1952) Inequalities. 2nd Edition, Cambridge University Press, Cambridge.