

On the Conservative Finite Difference Scheme for the Generalized Novikov Equation

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Abstract

In this paper, we investigate a numerical method for the generalized Novikov equation. We propose a conservative finite difference scheme and use Brouwer fixed point theorem to obtain the existence of the solution of the corresponding difference equation. We also prove the convergence and stability of the solution by using the discrete energy method. Moreover, we obtain the truncation error of the difference scheme which is $R_j^n = O(\tau^2 + h^2)$.

Keywords

Generalized Novikov Equation, Finite Difference Scheme, Conservation Law, Stability, Convergence

1. Introduction

The Camassa-Holm (CH) equation

$$m_t + 2u_x m + um_x = 0 \quad (1.1)$$

was derived by Camassa and Holm [1] as a model for the unidirectional propagation of the shallow water waves over a flat bottom [2] [3], where $m = u - u_{xx}$. It was found earlier by Fuchssteiner and Fokas [4] using the recursion operator of the KdV equation. More interestingly, the CH equation could also be obtained by the tri-Hamiltonian duality approach from the bi-Hamiltonian structure of the KdV equation [5] [6]. So the CH equation could be regarded as a dual system of the KdV equation. The CH equation had attracted much attention because of its nice features: existence of peaked solitons [1] [7] (which has a discontinuous first derivative at the spike), complete integrability [1] [4], nice geometric formulations [8] [9] [10] and wave breaking phenomena [11] [12] [13] [14] etc. It was noticed that the peaked solitons were not allowed by the well-known in-

tegrable equations such as the KdV equation, the mKdV equation, and the Schrodinger equation etc. The stability of peakons of the CH Equation (1.1) was proved in [15] [16]. A similar integrable equation with quadratic nonlinearities was the Degasperis-Procesi (DP) equation [17], which took the form

$$m_t + 3u_x m + u m_x = 0 \quad (1.2)$$

It was regarded as a model for nonlinear shallow water dynamics and could be derived from the governing equation for water waves [2]. Analogous to the CH equation, the DP equation also possessed an infinite number of conservation laws, bi-Hamiltonian structure, peaked solitons etc. Moreover, it admitted shocked peaked solitons [18]. Its integrability, existence of peaked solitons, stability of peaked solitons, and wave breaking phenomena were studied extensively in [19] [20] [21] [22].

Notice that both the CH and DP equations had the quadratic nonlinearities and H^1 conservation laws. So it was of great interest to search for such type of equations with higher-order nonlinearities. To the best of our knowledge, two integrable equations with cubic nonlinearities had been proposed. One was a modified CH equation

$$m_t + \left((u^2 - u_x^2) m \right)_x = 0 \quad (1.3)$$

which was obtained by the tri-Hamiltonian duality approach from the bi-Hamiltonian structure of the mKdV equation [5] [6]. Its well-posedness, blow-up, wave breaking, peaked solitons, and their stability were studied in recent works [23] [24] [25] [26]. The mCH Equation (1.3) exhibits new features of peaked solitons, wave-breaking mechanism, and blow-up criteria. The mCH equation could also be obtained from an invariant non-stretching planar curve flow in Euclidean geometry [24]. So it was regarded as an Euclidean version of the CH equation in this sense. The other one was the so-called Novikov equation

$$m_t + 3uu_x m + u^2 m_x = 0 \quad (1.4)$$

which was obtained by Novikov [27] in the symmetry classification of such type equations. The integrability, existence of peaked solitons and their stability, global well-posedness and wave breaking phenomena of Equation (1.4) were discussed in [28] [29] [30] [31]. Applying the tri-Hamiltonian duality approach to the Gardner equation

$$u_t + u_{xxx} + a_1 u^2 u_x + a_2 u u_x = 0$$

they deduced the following generalized CH equation with the cubic and quadratic nonlinearities [32]

$$m_t + a_1 \left((u^2 - u_x^2) m \right)_x + a_2 (2u_x m + u m_x) = 0 \quad (1.5)$$

This equation admitted Lax-pair and peaked solitons [33].

There are few studies on numerical solutions of the generalized Novikov equation. Therefore in this paper, we will construct a finite difference scheme for the initial boundary problem of equations as follows,

$$m_t + a_1(3uu_x m + u^2 m_x) + a_2 u_x m + a_3 u m_x = 0, \tag{1.6}$$

$$m = u - u_{xx}, 0 < t \leq T, 0 \leq x \leq L$$

$$u(0, t) = u(L, t), 0 < t \leq T \tag{1.7}$$

$$u(0, x) = u_0(x), 0 < x \leq L \tag{1.8}$$

where $u(x, t)$ is a function of time t and a single spatial variable x , and a_1, a_2, a_3 are constants. It is clear that Equation (1.6) is reduced respectively to the Novikov Equation (1.4), the CH Equation (1.1) and the DP Equation (1.2), when $a_1 = 1, a_2 = 0, a_3 = 0$; $a_1 = 0, a_2 = 2, a_3 = 1$; $a_1 = 0, a_2 = 3, a_3 = 1$. Clearly, Equation (1.6) is a linear combination of the Novikov equation, the CH equation, and the DP equation.

We shall give an energy conservative finite scheme for Equations (1.6)-(1.8) and obtain the module estimation. Moreover, we prove the convergence and stability of the finite scheme by use of discrete energy method.

2. Preliminaries

For convenience, we denote $\Omega = \{(x, t) | 0 \leq x \leq L, 0 < t \leq T\}$ in the following section. Let N, J be any positive integers and $h = \frac{L}{J+1}, \tau = \frac{T}{N}, x_j = jh$ for $j = 0, 1, \dots, J+1$. Denote $t^n = n\tau, u_j^n = u(x_j, t^n), u^n = (u_0^n, \dots, u_{J+1}^n)$ for $n = 0, \dots, N$ and $Z_h^0 = \{u = (u_0, \dots, u_{J+1}) | u_0 = u_{J+1} = 0\}$.

For simplicity, we introduce some notations as follows:

$$u_x^n = \frac{u_{j+1}^n - u_j^n}{h}, u_x^n = \frac{u_j^n - u_{j-1}^n}{h}, u_x^n = \frac{u_{j+1}^n - u_{j-1}^n}{2h},$$

$$u_t^n = \frac{u_j^{n+1} - u_j^n}{\tau}, u_t^n = \frac{u_j^n - u_j^{n-1}}{\tau}, u_t^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau},$$

$$\bar{u} = \frac{u_j^{n+1} + u_j^{n-1}}{2}, u^n = u_j^n \tag{2.1}$$

For $u, v \in Z_h^0$, we define a discrete inner product and the discrete L^2 -norm $\|\cdot\|_h$ as

$$(u, v)_h = h \sum_{j=1}^J u_j v_j, \|u\|_h = \left(h \sum_{j=1}^J u_j^2 \right)^{\frac{1}{2}} \tag{2.2}$$

In order to obtain the module estimation, investigate the convergence and stability of the finite difference scheme, we need introduce two lemmas as follows,

Lemma 2.1. (Discrete Sobolev inequality [34]) There exist constants c_1, c_2 satisfying

$$\|u^n\|_\infty \leq c_1 \|u^n\| + c_2 \|u_x^n\| \tag{2.3}$$

Lemma 2.2. (Discrete Gronwall inequality [34]) Suppose that there exist negative functions $\omega(k), \rho(k)$, where $\rho(k)$ is decreasing. For any k and

$c > 0$, if

$$\omega(k) \leq \rho(k) + c\tau \sum_{l=0}^{k-1} \omega(l) \tag{2.4}$$

then

$$\omega(k) \leq \rho(k) e^{c\tau k} \tag{2.5}$$

3. A Energy Conservative Finite Scheme

The Equation (1.6) can be rewritten in the following form:

$$u_t - u_{xxt} + a_1 (3uu_x(u - u_{xx}) + u^2(u_x - u_{xxx})) + a_2 u_x(u - u_{xx}) + a_3 u(u_x - u_{xxx}) = 0$$

which can be expressed as

$$u_t - u_{xxt} + 4a_1 u^2 u_x + (a_2 + a_3) uu_x = 3a_1 uu_x u_{xx} + a_1 u^2 u_{xxx} + a_2 u_x u_{xx} + a_3 uu_{xxx}$$

Firstly, we construct an energy conservative finite scheme for the problem (1.6)-(1.8) as follows:

$$\begin{aligned} & u_t^n - u_{t\bar{x}\bar{x}}^n + a_1 \left(\left(\left(u^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} + \left(u^{n+\frac{1}{2}} \right)^2 u_x^{n+\frac{1}{2}} \right) + \frac{a_2 + a_3}{3} \left(\left(\left(u^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} + u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} \right) \\ &= a_1 \left(\left(\left(u^{n+\frac{1}{2}} \right)^2 u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right) + a_3 \left(\left(u^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right) \\ &+ (a_2 - 2a_3) \left(\left(\left(u_x^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} - u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right) \end{aligned} \tag{3.1}$$

$$u_j^0 = u_0(x_j), \quad 0 \leq j \leq J+1 \tag{3.2}$$

and

$$u_0^n = u_{J+1}^n = 0, \quad 0 \leq n \leq N \tag{3.3}$$

Lemma 3.1. The difference scheme satisfies discrete conservative law as follows:

$$E^n = \|u^n\|^2 + \|u_x^n\|^2 = E^{n-1} = \dots = E^0 \tag{3.4}$$

Proof. Computing the inner product of the difference scheme with $2u^{n+\frac{1}{2}}$, we have

$$\begin{aligned} \left(u_t, 2u^{n+\frac{1}{2}} \right) &= h \sum_{j=1}^J u_t \cdot \left(2u^{n+\frac{1}{2}} \right) = \frac{h}{\tau} \sum_{j=1}^J (u_j^{n+1} - u_j^n) (u_j^{n+1} + u_j^n) \\ &= \frac{1}{\tau} \left(\|u^{n+1}\|^2 - \|u^n\|^2 \right) \\ \left(u_{t\bar{x}\bar{x}}, 2u^{n+\frac{1}{2}} \right) &= h \sum_{j=1}^J u_{t\bar{x}\bar{x}} \cdot \left(2u^{n+\frac{1}{2}} \right) = -\frac{h}{\tau} \sum_{j=1}^J (u_j^{n+1} - u_j^n)_x (u_j^{n+1} + u_j^n)_x \\ &= -\frac{1}{\tau} \left(\|u_x^{n+1}\|^2 - \|u_x^n\|^2 \right) \end{aligned}$$

$$\begin{aligned}
 & \left(a_1 \left(\left(\left(u^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} + \left(u^{n+\frac{1}{2}} \right)^2 u_x^{n+\frac{1}{2}} \right), 2u^{n+\frac{1}{2}} \right) \\
 &= 2a_1 h \sum_{j=1}^J \left(\left(\left(u_j^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} + \left(u_j^{n+\frac{1}{2}} \right)^2 u_{xj}^{n+\frac{1}{2}} \cdot u_j^{n+\frac{1}{2}} \right) \\
 &= 2a_1 h \sum_{j=1}^J \left(\left(\left(u_j^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} - \left(\left(u_j^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} u_j^{n+\frac{1}{2}} \right) \\
 &= 0 \\
 & \left(\frac{a_2 + a_3}{3} \left(\left(\left(u^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} + u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} \right), 2u^{n+\frac{1}{2}} \right) \\
 &= \frac{2(a_2 + a_3)}{3} h \sum_{j=1}^J \left(\left(\left(u_j^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} u_{xj}^{n+\frac{1}{2}} \cdot u_j^{n+\frac{1}{2}} \right) \\
 &= \frac{2(a_2 + a_3)}{3} h \sum_{j=1}^J \left(\left(\left(u_j^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} - \left(\left(u_j^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} u_j^{n+\frac{1}{2}} \right) \\
 &= 0 \\
 & \left(a_1 \left(\left(\left(u^{n+\frac{1}{2}} \right)^2 u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right), 2u^{n+\frac{1}{2}} \right) \\
 &= 2a_1 h \sum_{j=1}^J \left(\left(\left(u_j^{n+\frac{1}{2}} \right)^2 u_{xxj}^{n+\frac{1}{2}} \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} u_{xj}^{n+\frac{1}{2}} u_{xxj}^{n+\frac{1}{2}} \cdot u_j^{n+\frac{1}{2}} \right) \\
 &= 2a_1 h \sum_{j=1}^J \left(\left(\left(u_j^{n+\frac{1}{2}} \right)^2 u_{xxj}^{n+\frac{1}{2}} \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} - \left(\left(u_j^{n+\frac{1}{2}} \right)^2 u_{xxj}^{n+\frac{1}{2}} \right)_{\bar{x}} u_j^{n+\frac{1}{2}} \right) \\
 &= 0 \\
 & \left(a_3 \left(\left(\left(u^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right), 2u^{n+\frac{1}{2}} \right) \\
 &= 2a_3 h \sum_{j=1}^J \left(\left(u_j^{n+\frac{1}{2}} u_{xxj}^{n+\frac{1}{2}} \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} + u_{xj}^{n+\frac{1}{2}} u_{xxj}^{n+\frac{1}{2}} \cdot u_j^{n+\frac{1}{2}} \right) \\
 &= 2a_3 h \sum_{j=1}^J \left(\left(u_j^{n+\frac{1}{2}} u_{xxj}^{n+\frac{1}{2}} \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}} \left(u_j^{n+\frac{1}{2}} u_{xxj}^{n+\frac{1}{2}} \right)_{\bar{x}} \right) \\
 &= 0 \\
 & \left((a_2 - 2a_3) \left(\left(\left(u_x^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} - u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right), 2u^{n+\frac{1}{2}} \right) \\
 &= 2(a_2 - 2a_3) h \sum_{j=1}^J \left(\left(\left(u_{xj}^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} \cdot u_j^{n+\frac{1}{2}} - u_{xj}^{n+\frac{1}{2}} u_{xxj}^{n+\frac{1}{2}} \cdot u_j^{n+\frac{1}{2}} \right) \\
 &= 2(a_2 - 2a_3) h \sum_{j=1}^J \left(- \left(u_{xj}^{n+\frac{1}{2}} \right)^2 \cdot u_j^{n+\frac{1}{2}} + u_{xj}^{n+\frac{1}{2}} u_{xxj}^{n+\frac{1}{2}} \cdot u_j^{n+\frac{1}{2}} \right) \\
 &= 0
 \end{aligned}$$

From the above formulas, we can simplify formula as

$$\frac{1}{\tau} \left(\|u^{n+1}\|^2 - \|u^n\|^2 \right) + \frac{1}{\tau} \left(\|u_x^{n+1}\|^2 - \|u_x^n\|^2 \right) = 0 \tag{3.5}$$

then

$$\|u^{n+1}\|^2 + \|u_x^{n+1}\|^2 = \|u^n\|^2 + \|u_x^n\|^2 \tag{3.6}$$

So the difference scheme satisfies the discrete conservative law as follows

$$E^n = \|u^n\|^2 + \|u_x^n\|^2 = E^{n-1} = \dots = E^0 \tag{3.7}$$

This completes the proof of Lemma 3.1.

Theorem 3.2. The solution of the difference scheme (3.1)-(3.3) satisfies:

$$\|u^n\| \leq C, \quad \|u_x^n\| \leq C, \quad \|u^n\|_{\infty} \leq C \tag{3.8}$$

Proof. From Lemma 3.1, we know $\|u^n\| \leq C$ and $\|u_x^n\| \leq C$. Then by Lemma 2.1, we obtain $\|u^n\|_{\infty} \leq C$. This completes the proof of Theorem 3.2.

Theorem 3.3. There exists $u^n \in Z_h^0$ satisfying the difference scheme (3.1)-(3.3).

Proof. We shall use Mathematical Induction to prove Theorem 3.3.

When $n=1$, it is known that there exists u^1 satisfying the difference scheme from initial condition. Next, we need to prove the case of $n > 1$.

Assume that there exists u^n satisfies the difference scheme when $n < N$, then we need to prove that there exists u^{n+1} satisfies the difference scheme.

Define a operator $\omega(v)$ in Z_h^0 as follows:

$$\begin{aligned} \omega(v) = & 2v - 2u^n - (2v_{x\bar{x}} - 2u_{x\bar{x}}^n) + a_1\tau \left((v^3)_{\bar{x}} + v^2v_x \right) \\ & + \frac{a_2 + a_3}{3} \tau \left((v^2)_{\bar{x}} + vv_x \right) - a_1\tau \left((v^2v_{xx})_{\bar{x}} + vv_xv_{xx} \right) \\ & - a_3\tau \left((vv_{xx})_{\bar{x}} + v_xv_{xx} \right) - (a_2 - 2a_3) \tau \left((v_x)_{\bar{x}}^2 - v_xv_{x\bar{x}} \right) \end{aligned} \tag{3.9}$$

It is clear that ω is continuous.

Computing the inner product of the operator $\omega(v)$, we obtain

$$\begin{aligned} (2v, v) &= 2\|v\|^2 \\ (2v_{x\bar{x}}, v) &= -2(v_x, v_x) = -2\|v_x\|^2 \\ \left((v^3)_{\bar{x}} + v^2v_x, v \right) &= h \sum_{j=1}^J \left((v^3)_{\bar{x}} \cdot v + v^2v_x \cdot v \right) = 0 \\ \left((v^2)_{\bar{x}} + vv_x, v \right) &= h \sum_{j=1}^J \left((v^2)_{\bar{x}} \cdot v + vv_x \cdot v \right) = 0 \\ \left((v^2v_{xx})_{\bar{x}} + vv_xv_{xx}, v \right) &= h \sum_{j=1}^J \left((v^2v_{xx})_{\bar{x}} \cdot v + vv_xv_{xx} \cdot v \right) = 0 \\ \left((vv_{xx})_{\bar{x}} + v_xv_{xx}, v \right) &= h \sum_{j=1}^J \left((vv_{xx})_{\bar{x}} \cdot v + v_xv_{xx} \cdot v \right) = 0 \end{aligned}$$

$$\left(\left((v_x)^2 \right)_{\bar{x}} - v_x v_{x\bar{x}}, v \right) = h \sum_{j=1}^J \left(-(v_x)^2 \cdot v_x + v_x v_x \cdot v_x \right) = 0$$

By using Cauchy-Schwartz inequality, we obtain

$$(2u^n, v) \leq 2 \|u^n\| \cdot \|v\| \tag{3.10}$$

$$(2u_{x\bar{x}}^n, v) = -2(u_x^n, v_x) \geq -2 \|u_x^n\| \cdot \|v_x\| \tag{3.11}$$

From above discussions, we get

$$\begin{aligned} (\omega(v), v) &\geq 2 \|v\|^2 - 2 \|u^n\| \cdot \|v\| + 2 \|v_x\|^2 - 2 \|u_x^n\| \cdot \|v_x\| \\ &\geq \|v\|^2 - \|u^n\|^2 - \|u_x^n\|^2 \end{aligned} \tag{3.12}$$

For $\forall v \in Z_h^0$, then

$$\|v\|^2 = \|u^n\|^2 + \|u_x^n\|^2 + 1 \tag{3.13}$$

So we get $(\omega(v), v) > 0$. By Brouwer fixed point theorem, there exists $v^* \in H$ such that $\omega(v^*) = 0$ and $\|v^*\| \leq \partial$.

Let $u^{n+1} = 2v^* - u^n$, it is easy to verify that u^{n+1} satisfies the difference scheme for the problem.

So we complete the proof of Theorem 3.3.

4. Convergence and Stability of the Difference Scheme

In order to investigate the convergence and stability of the difference scheme, we need to obtain the truncation error of the scheme.

Theorem 4.1. If the solution $u(x, t)$ of equation is sufficiently regular, then the truncation error of the difference scheme is $R_j^n = O(\tau^2 + h^2)$.

Proof. Firstly, we can use the Taylor expansion of u_j^{n+1} , u_j^n , u_{j+1}^{n+1} , u_{j+1}^n , u_{j-1}^{n+1} , u_{j-1}^n at the point $\left(x_j, t_{\frac{n+1}{2}} \right)$. Secondly, from the above Taylor expansions we reorganize difference equation at point $\left(x_j, t_{\frac{n+1}{2}} \right)$, then

$$\begin{aligned} \left(x_j, t_{\frac{n+1}{2}} \right) u_t^n &= \frac{u_j^{n+1} - u_j^n}{\tau} \\ &= \frac{1}{\tau} \left(\left(u_j^{n+\frac{1}{2}} + \frac{\tau}{2} \left(\frac{\partial u}{\partial t} \right)_j^{n+\frac{1}{2}} + \frac{\tau^2}{8} \left(\frac{\partial^2 u}{\partial t^2} \right)_j^{n+\frac{1}{2}} + \frac{\tau^3}{48} \left(\frac{\partial^3 u}{\partial t^3} \right)_j^{n+\frac{1}{2}} + \dots \right) \right. \\ &\quad \left. - \left(u_j^{n-\frac{1}{2}} - \frac{\tau}{2} \left(\frac{\partial u}{\partial t} \right)_j^{n-\frac{1}{2}} + \frac{\tau^2}{8} \left(\frac{\partial^2 u}{\partial t^2} \right)_j^{n-\frac{1}{2}} + \dots \right) \right) \\ &= \left(\frac{\partial u}{\partial t} \right)_j^{n+\frac{1}{2}} + \frac{\tau^3}{24} \left(\frac{\partial^3 u}{\partial t^3} \right)_j^{n+\frac{1}{2}} + \dots \end{aligned}$$

$$\begin{aligned}
 u_{xx}^n &= \frac{(u_{xx})_j^{n+1} - (u_{xx})_j^n}{\tau} = \frac{((u_x)_{j+1}^{n+1} - (u_x)_j^{n+1}) - ((u_x)_{j+1}^n - (u_x)_j^n)}{\tau h} \\
 &= \frac{1}{h^2} \left(\left(\left(\frac{\partial u}{\partial t} \right)_{j+1}^{n+\frac{1}{2}} - 2 \left(\frac{\partial u}{\partial t} \right)_j^{n+\frac{1}{2}} + \left(\frac{\partial u}{\partial t} \right)_{j-1}^{n+\frac{1}{2}} \right) \right. \\
 &\quad \left. + \left(\frac{\tau^2}{24} \left(\frac{\partial^3 u}{\partial t^3} \right)_{j+1}^{n+\frac{1}{2}} - \frac{\tau^2}{12} \left(\frac{\partial^3 u}{\partial t^3} \right)_j^{n+\frac{1}{2}} + \frac{\tau^2}{24} \left(\frac{\partial^3 u}{\partial t^3} \right)_{j-1}^{n+\frac{1}{2}} \right) + \dots \right) \\
 &= \left(\frac{\partial^3 u}{\partial t \partial x^2} \right)_j^{n+\frac{1}{2}} + \dots \\
 \left(\left(u^{\frac{n+1}{2}} \right)_{\bar{x}}^3 \right) &= \frac{\left(\left(u^{\frac{n+1}{2}} \right)_j^3 \right) - \left(\left(u^{\frac{n+1}{2}} \right)_{j-1}^3 \right)}{h} \\
 &= 3 \left(u^{\frac{n+1}{2}} \right)^2 \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} - \frac{h}{2} \left(3 \left(u^{\frac{n+1}{2}} \right)^2 \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + 6 u^{\frac{n+1}{2}} \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + \dots \right) \\
 \left(u^{\frac{n+1}{2}} \right)^2 u_x^{n+\frac{1}{2}} &= \left(u^{\frac{n+1}{2}} \right)^2 \cdot \frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{h} \\
 &= \left(u^{\frac{n+1}{2}} \right)^2 \cdot \left(\left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + \frac{h}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \frac{h^2}{6} \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \right) \\
 \left(\left(u^{\frac{n+1}{2}} \right)^2 \right)_{\bar{x}} &= \frac{\left(\left(u^{\frac{n+1}{2}} \right)_j^2 \right) - \left(\left(u^{\frac{n+1}{2}} \right)_{j-1}^2 \right)}{h} \\
 &= 2 u^{\frac{n+1}{2}} \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} - \frac{h}{2} \left(2 u^{\frac{n+1}{2}} \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + 2 \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + \dots \right) \\
 u^{\frac{n+1}{2}} u_x^{n+\frac{1}{2}} &= u^{\frac{n+1}{2}} \cdot \frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{h} \\
 &= u^{\frac{n+1}{2}} \cdot \left(\left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + \frac{h}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \frac{h^2}{6} \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \right) \\
 u^{\frac{n+1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} &= u^{\frac{n+1}{2}} \cdot \frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{h} \cdot \frac{u_{j+2}^{n+\frac{1}{2}} - 2u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}}}{h^2} \\
 &= u^{\frac{n+1}{2}} \cdot \left(\left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + \frac{h}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \dots \right) \cdot \left(\left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + h \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \right)
 \end{aligned}$$

$$\begin{aligned}
 \left(\left(u^{n+\frac{1}{2}} \right)_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} &= \frac{\left(u_j^{n+\frac{1}{2}} \right)^2 u_{xj}^{n+\frac{1}{2}} - \left(u_{j-1}^{n+\frac{1}{2}} \right)^2 u_{xx(j-1)}^{n+\frac{1}{2}}}{h} \\
 &= 2u^{n+\frac{1}{2}} \cdot \left(\left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} \cdot \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + h \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \right) \\
 u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} &= \frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{h} \cdot \frac{u_{j+2}^{n+\frac{1}{2}} - 2u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}}}{h^2} \\
 &= \left(\left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + \frac{h}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \dots \right) \cdot \left(\left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + h \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \right) \\
 \left(u^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} &= \frac{u_j^{n+\frac{1}{2}} u_{xj}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}} u_{xx(j-1)}^{n+\frac{1}{2}}}{h} \\
 &= \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} \cdot \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + h \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \\
 \left(\left(u_x^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} &= \frac{u_{xj}^{n+\frac{1}{2}} u_{xj}^{n+\frac{1}{2}} - u_{x(j-1)}^{n+\frac{1}{2}} u_{x(j-1)}^{n+\frac{1}{2}}}{h} \\
 &= \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} \cdot \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + h \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \\
 u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} &= \frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{h} \cdot \frac{u_{j+1}^{n+\frac{1}{2}} - 2u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}}{h^2} \\
 &= \left(\left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + \frac{h}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \dots \right) \cdot \left(\left(\frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + h \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} + \dots \right)
 \end{aligned}$$

From the above expansions, we obtain the linear part of Equation (3.1)

$u_t^n - u_{t\bar{x}\bar{x}}^n$ at the point $\left(x_j, t_{n+\frac{1}{2}} \right)$ satisfying

$$u_t^n - u_{t\bar{x}\bar{x}}^n = \left(\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} \right) \Big|_{\left(x_j, t_{n+\frac{1}{2}} \right)} + O(\tau^2 + h^2)$$

On the other hand, the nonlinear parts at the point $\left(x_j, t_{n+\frac{1}{2}} \right)$ satisfying

$$\begin{aligned}
 a_1 \left(\left(\left(u^{n+\frac{1}{2}} \right)_{\bar{x}}^3 \right) + \left(u^{n+\frac{1}{2}} \right)_x u_x^{n+\frac{1}{2}} \right) &= 4a_1 u^2 \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + O(\tau^2 + h^2) \\
 \frac{a_2 + a_3}{3} \left(\left(\left(u^{n+\frac{1}{2}} \right)_{\bar{x}}^2 \right) + u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} \right) &= (a_2 + a_3) u \left(\frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} + O(\tau^2 + h^2)
 \end{aligned}$$

$$\begin{aligned}
 & a_1 \left(\left(\left(u^{n+\frac{1}{2}} \right)^2 u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right) \\
 &= a_1 \left(3u \left(\frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + 2u \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} \right) + O(\tau^2 + h^2) \\
 & a_3 \left(\left(u^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right) = a_3 \left(2 \left(\frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} \right) + O(\tau^2 + h^2) \\
 & (a_2 - 2a_3) \left(\left(u_x^{n+\frac{1}{2}} \right)_{\bar{x}} - u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right) \\
 &= (a_2 - 2a_3) \left(2 \left(\frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2} \right)_j^{n+\frac{1}{2}} + \left(\frac{\partial^3 u}{\partial x^3} \right)_j^{n+\frac{1}{2}} \right) + O(\tau^2 + h^2)
 \end{aligned}$$

Then we can easily obtain that truncation error of scheme (3.1) is $R_j^n = O(\tau^2 + h^2)$.

Theorem 4.2. The solution of the difference scheme (3.1)-(3.3) approaches to the solution of the original differential Equations (1.6)-(1.8) in L_∞ , and the corresponding truncation error is $O(\tau^2 + h^2)$.

Proof. Assume that u_j^n is the solution of the difference scheme (3.1)-(3.3), U_j^n is the solution of the original differential Equations (1.6)-(1.8). We can easily have $e_j^n = U_j^n - u_j^n$.

For the difference scheme (3.1),

$$\begin{aligned}
 R^n &= U_i^n - U_{i\bar{x}}^n + a_1 \left(\left(\left(U^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} + \left(U^{n+\frac{1}{2}} \right)^2 U_x^{n+\frac{1}{2}} \right) \\
 &+ \frac{a_2 + a_3}{3} \left(\left(\left(U^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} + U^{n+\frac{1}{2}} U_x^{n+\frac{1}{2}} \right) - a_1 \left(\left(\left(U^{n+\frac{1}{2}} \right)^2 U_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + U^{n+\frac{1}{2}} U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} \right) \quad (4.1) \\
 &- a_3 \left(\left(U^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} \right) - (a_2 - 2a_3) \left(\left(U_x^{n+\frac{1}{2}} \right)_{\bar{x}} \right)^2 - U_x^{n+\frac{1}{2}} U_{x\bar{x}}^{n+\frac{1}{2}}
 \end{aligned}$$

then (4.1) minus (3.1), yields

$$R^n = e_i^n - e_{i\bar{x}}^n + I_1 + I_2 + I_3 + I_4 + I_5 \tag{4.2}$$

where

$$\begin{aligned}
 I_1 &= a_1 \left(\left(\left(U^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} + \left(U^{n+\frac{1}{2}} \right)^2 U_x^{n+\frac{1}{2}} - \left(\left(u^{n+\frac{1}{2}} \right)^3 \right)_{\bar{x}} - \left(u^{n+\frac{1}{2}} \right)^2 u_x^{n+\frac{1}{2}} \right) \\
 I_2 &= \frac{a_2 + a_3}{3} \left(\left(\left(U^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} + U^{n+\frac{1}{2}} U_x^{n+\frac{1}{2}} - \left(\left(u^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} - u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} \right)
 \end{aligned}$$

$$\begin{aligned}
 I_3 &= a_1 \left(\left(\left(u_x^{n+\frac{1}{2}} \right)^2 u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u_x^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} - \left(\left(U_x^{n+\frac{1}{2}} \right)^2 U_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} - U_x^{n+\frac{1}{2}} U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} \right) \\
 I_4 &= a_3 \left(\left(u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} + u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} - \left(U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} \right)_{\bar{x}} - U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} \right) \\
 I_5 &= (a_2 - 2a_3) \left(\left(\left(u_x^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} - u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} - \left(\left(U_x^{n+\frac{1}{2}} \right)^2 \right)_{\bar{x}} + U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} \right)
 \end{aligned}$$

Taking the inner product of (4.2) with $2e^{n+\frac{1}{2}}$, we obtain

$$\begin{aligned}
 \left(R^n, 2e^{n+\frac{1}{2}} \right) &= \left(e_t^n, 2e^{n+\frac{1}{2}} \right) - \left(e_{tx\bar{x}}^n, 2e^{n+\frac{1}{2}} \right) + \left(I_1, 2e^{n+\frac{1}{2}} \right) + \left(I_2, 2e^{n+\frac{1}{2}} \right) \\
 &\quad + \left(I_3, 2e^{n+\frac{1}{2}} \right) + \left(I_4, 2e^{n+\frac{1}{2}} \right) + \left(I_5, 2e^{n+\frac{1}{2}} \right)
 \end{aligned}$$

By computing $\left(e_t^n, 2e^{n+\frac{1}{2}} \right)$, $\left(e_{tx\bar{x}}^n, 2e^{n+\frac{1}{2}} \right)$, $\left(I_1, 2e^{n+\frac{1}{2}} \right)$, $\left(I_2, 2e^{n+\frac{1}{2}} \right)$, $\left(I_3, 2e^{n+\frac{1}{2}} \right)$, $\left(I_4, 2e^{n+\frac{1}{2}} \right)$ and $\left(I_5, 2e^{n+\frac{1}{2}} \right)$ respectively, and from Lemma 2.3 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
 \left(e_t^n, 2e^{n+\frac{1}{2}} \right) &= h \sum_{j=0}^J e_y \cdot 2e_j^{n+\frac{1}{2}} = \frac{1}{\tau} \left(\|e^{n+1}\|^2 - \|e^n\|^2 \right) \\
 \left(e_{tx\bar{x}}^n, 2e^{n+\frac{1}{2}} \right) &= h \sum_{j=0}^J e_{tx\bar{y}} \cdot 2e_j^{n+\frac{1}{2}} = \frac{1}{\tau} \left(\|e_x^{n+1}\|^2 - \|e_x^n\|^2 \right) \\
 \left(I_1, 2e^{n+\frac{1}{2}} \right) &\leq C \left(\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 \right) \\
 \left(I_2, 2e^{n+\frac{1}{2}} \right) &\leq C \left(\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 \right) \\
 \left(I_3, 2e^{n+\frac{1}{2}} \right) &\leq C \left(\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 \right) \\
 \left(I_4, 2e^{n+\frac{1}{2}} \right) &\leq C \left(\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 \right) \\
 \left(I_5, 2e^{n+\frac{1}{2}} \right) &\leq C \left(\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 \right)
 \end{aligned}$$

Therefore,

$$\begin{aligned} \left(R^n, 2e^{n+\frac{1}{2}} \right) &= \left(e_t^n, 2e^{n+\frac{1}{2}} \right) - \left(e_{\alpha\bar{\alpha}}^n, 2e^{n+\frac{1}{2}} \right) + \left(I_1, 2e^{n+\frac{1}{2}} \right) + \left(I_2, 2e^{n+\frac{1}{2}} \right) \\ &\quad + \left(I_3, 2e^{n+\frac{1}{2}} \right) + \left(I_4, 2e^{n+\frac{1}{2}} \right) + \left(I_5, 2e^{n+\frac{1}{2}} \right) \\ &= \frac{1}{\tau} \left(\|e^{n+1}\|^2 - \|e^n\|^2 \right) - \frac{1}{\tau} \left(\|e_x^{n+1}\|^2 - \|e_x^n\|^2 \right) + \left(I_1, 2e^{n+\frac{1}{2}} \right) + \left(I_2, 2e^{n+\frac{1}{2}} \right) \\ &\quad + \left(I_3, 2e^{n+\frac{1}{2}} \right) + \left(I_4, 2e^{n+\frac{1}{2}} \right) + \left(I_5, 2e^{n+\frac{1}{2}} \right) \end{aligned}$$

Taking Schwartz inequality, it follows that

$$\left(R^n, 2e^{n+\frac{1}{2}} \right) \leq 2\|R^n\| \cdot \left\| e^{n+\frac{1}{2}} \right\| \leq \|R^n\|^2 + 2\left(\|e^{n+1}\|^2 + \|e^n\|^2 \right)$$

From the above discussion, we have

$$\begin{aligned} &\|e^{n+1}\|^2 - \|e^n\|^2 + \|e_x^{n+1}\|^2 - \|e_x^n\|^2 \\ &\leq C\tau\|R^n\|^2 + C\tau\left(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 \right) \end{aligned} \tag{4.3}$$

Let $\varphi_n = \|e^n\|^2 + \|e_x^n\|^2$, then (4.3) becomes to

$$\varphi_{n+1} - \varphi_n \leq C\tau(\varphi_{n+1} + \varphi_n) + C\tau\|R^n\|^2$$

Hence we have

$$\begin{aligned} \varphi_n - \varphi_{n-1} &\leq C\tau(\varphi_n + \varphi_{n-1}) + C\tau\|R^{n-1}\|^2 \\ \varphi_{n-1} - \varphi_{n-2} &\leq C\tau(\varphi_{n-1} + \varphi_{n-2}) + C\tau\|R^{n-2}\|^2 \\ &\vdots \\ \varphi_1 - \varphi_0 &\leq C\tau(\varphi_1 + \varphi_0) + C\tau\|R^0\|^2 \end{aligned}$$

From the above inequalities, one has

$$\varphi_n \leq \varphi_0 + C\tau \sum_{i=0}^{n-1} \varphi_i + C\tau \sum_{i=0}^{n-1} \|R^i\|^2$$

where $k \sum_{i=0}^{n-1} \|R^i\|^2 \leq n\tau \max_{0 \leq i \leq n-1} \|R^i\|^2 \leq TO(\tau^2 + h^2)^2$.

Since $\varphi_0 = 0$, then $\varphi_n \leq C\tau \sum_{i=0}^{n-1} \varphi_i + CTO(\tau^2 + h^2)^2$. From discrete Gronwall inequality, $\varphi_n \leq O(\tau^2 + h^2)^2$, that is

$$\|e_x^n\|^2 + \|e^n\|^2 \leq O(\tau^2 + h^2)^2$$

Then we have

$$\|e^n\| \leq O(\tau^2 + h^2), \|e_x^n\| \leq O(\tau^2 + h^2)$$

From Theorem 3.2, we assert

$$\|e_x^n\|_{\infty} \leq O(\tau^2 + h^2)$$

We complete the proof of Theorem 4.2.

Theorem 4.3. The solution of the difference scheme (3.1)-(3.3) is stable in L_∞ .

Proof. Assume that u_j^n is the solution of the difference scheme (3.1)-(3.3), U_j^n is the solution of original differential Equations (1.6)-(1.8). We can easily obtain that $e_j^n = U_j^n - u_j^n$.

Then from Theorem 4.2, the following inequality holds true

$$\|e^n\|^2 \leq C \|U_0 - u_0\|^2$$

Thus, we complete the proof of Theorem 4.3.

5. Conclusion

In this paper, we give a difference scheme for the generalized Novikov equation. In Section 2, we give some preparation knowledge. In Section 3, we propose a conservative finite difference scheme for the generalized Novikov equation and use Brouwer fixed point theorem to obtain the existence of the solution for the corresponding difference equation. In Section 4, we prove the convergence and stability of the solution by using the discrete energy method.

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