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On the Bifurcations of Dean Flow between Porous Horizontal Cylinders with a Radial Flow and a Radial Temperature Gradient

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Abstract

The stability of a heat-conducting flow due to the pumping of a fluid around the annulus of horizontal porous cylinders is studied. The basic flow is under the action of radial flow and a radial temperature gradient. The objects of investigations are different regimes and bifurcations which may arise in this flow.

Keywords

Dean Flow, Porous Cylinders, Transitions, Instability, Bifurcations

1. Introduction

Due to the technological and engineering importance, the investigation of the stability of curved flows has been a source of interest for several years.

Experimental studies of main flow in curved pipes carried out by Eustice [1] [2] pointed out the secondary flow formation that analytically confirmed by Dean [3]. His stability analysis has shown that flow between two horizontal cylinders becomes unstable for small axisymmetric disturbances when the Dean number exceeds the critical value. Above this value, there appear pairs of counter-rotating streamwise-oriented vortices known as Dean vortex flows. Later on, this theoretical analysis has been confirmed by W. Reid [4], G. Hammerlin [5] and experimentally by D. Browster *et al.* [6].

The stability investigation of Dean flow was continued by many authors (see [7]-[13]).

Important results were obtained also in the study of Dean flow stability of complex fluids. [14] and [15] present experimental and numerical analysis of the

behavior of physiological flow in a curved pipe. The numerical study of magnetohydrodynamic viscous steady biofluid flow through a curved pipe with circular cross section can be found in [16]. [17] is devoted to the analytical study of conductive fluids in highly conductive curved pipes. The influence of porous channel with a radial flow and the effect of a radial temperature gradient on the stability of Dean flow have been considered in [18] [19]. The Dean flow stability of a heat-conducting fluid between horizontal porous cylinders with a radial temperature gradient and a radial flow has been studied in [20] [21].

In these papers, investigations are mainly carried out in the linear approximation, which allows one to judge the first loss of stability of main flow. It was shown that the first instability of the basic steady flow can be not only a monotonic instability leading to the Dean vortices but also can be oscillatory instability resulting in a time-periodic secondary regime.

For practical purposes, it is important to consider the transitions to more complex regimes, the appearance of which precedes the development of higher instability and leads to turbulence.

For Dean flow stability, the numerical and experimental investigations of transitions to turbulence were carried out, for instance, in [22] [23] [24] [25]. In [22], the linear stability of axisymmetric Dean vortex flow to non-axisymmetric perturbations is considered. Two different nonaxisymmetric flows, undulating and twisting Dean vortex flow, are found to be associated with the two transitions. In [23], the stability of the Dean-type flow is studied experimentally and compared with instabilities and transitions in the Taylor-Couette flow. In [24], the effect of curvature on laminar channel flow is studied experimentally. It is shown that two types of secondary instabilities, with distinctly different frequencies, are at hand simultaneously, and their spatial distribution and growth are determined. In [25], numerical study of transitions that occur with increasing Reynolds number in a curved chanel is performed.

Our aim is to study possible transitions to complex modes in a heat-coducting flow between horizontal porous cylinders with radial flow and radial temperature gradient. The linear analysis of that problem for small axisymmetric perturbations has been performed in [20] [21]. Investigating the transitions to more complex modes, we are interested in the structures of transitions to turbulence in this flow, depending on the parameters of the problem. Based on the nonlinear analysis, we can predict the existence of quasiperiodic oscillations and chaotic regimes for certain small absolute values of parameters of the problem.

2. Formulation of the Problem

We consider the steady flow in an annulus between the fixed horizontal heated cylinders to be maintained by a constant azimuthal pressure gradient in the presence of a radial flow through the porous cylinders and of a radial temperature gradient. We denote the radii and temperature of the inner and outer cylinders by R_1, R_2 and T_1, T_2, respectively. A constant flow through the annulus is
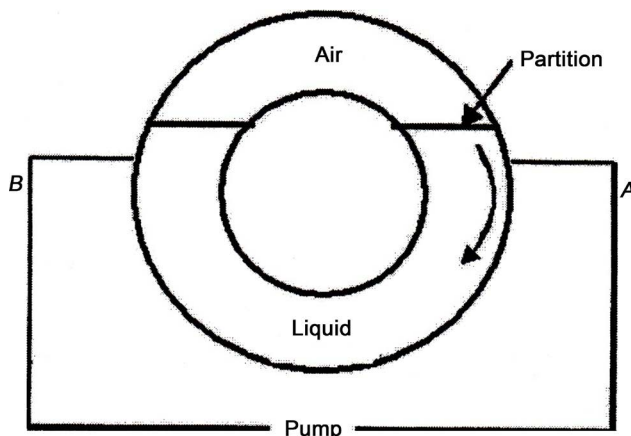
Figure 1. Horizontal concentric cylinder system.

maintained by a suitable pumping circuit. It is assumed that the fluid inflow through one cylinder is equal to the fluid outflow through the other. The system that will be considered here is shown in **Figure 1**. Point A denotes the delivery side, while B denotes the suction side of the pump. The annulus is partially filled with the liquid. There is a partition in the annulus at the air-liquid interface to maintain the azimuthal pressure gradient. The radial flow through the porous cylinders may be either outward or inward.

In the cylindrical coordinates (r, φ, z) with the z-axis coinciding with that of the cylinders, the Navier-Stokes equations as wall as those for heat transfer, continuity and state are the form [26]:

$$
\begin{aligned}
&\frac{\partial v_r'}{\partial t} + (V', \nabla) v_r' - \frac{v_\varphi'^2}{r} + \frac{1}{\rho'} \frac{\partial \Pi'}{\partial r} = \nu \left(\Delta v_r' - \frac{v_r'}{r^2} - \frac{2}{r^2} \frac{\partial v_\varphi'}{\partial \varphi} \right), \\
&\frac{\partial v_\varphi'}{\partial t} + (V', \nabla) v_\varphi' + \frac{v_\varphi' v_r'}{r} + \frac{1}{r \rho'} \frac{\partial \Pi'}{\partial \varphi} = \nu \left(\Delta v_\varphi' - \frac{v_\varphi'}{r^2} + \frac{2}{r^2} \frac{\partial v_r'}{\partial \varphi} \right), \\
&\frac{\partial v_z'}{\partial t} + (V', \nabla) v_{z'} + \frac{1}{\rho'} \frac{\partial \Pi'}{\partial z} = \nu \Delta v_{z'}, \quad \frac{\partial T'}{\partial t} + (V', \nabla) T' = \chi \Delta T', \\
&\frac{\partial \rho'}{\partial t} + div(\rho' V') = 0, \quad \rho' = \rho_0 \left(1 - \beta (T' - T_0) \right),
\end{aligned}
\tag{1}
$$

where $V' = \{v_r', v_\varphi', v_z'\}$ is the velocity vector; T' is temperature; Π' is pressure; ρ' is the fluid density; t is time; ν, χ, β are, respectively, the coefficients of kinematic viscosity, thermal diffusivity and thermal expansion, ρ_0; and T_0 denote constant density and temperature, respectively.

Operators Δ and ∇ in the cylindrical coordinates are of the form

$$
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla = \left\{ \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z} \right\}.
$$

Equation (1) are considered under the condition

$$
\int_0^{2\pi} \int_1^{\frac{R_2}{R_1}} \rho' v_z' r \, dr \, d\varphi = 0,
\tag{2}
$$

which ensures the absence of the fluid discharge through the cavity cross-section

of the cylinders.

Let us assume that on the surface of the cylinders the following boundary conditions

$$v'_r = U_0, \quad v'_\varphi = 0, \quad v'_z = 0, \quad T' = T_1 \ (r = R_1),$$
$$v'_r = \frac{U_0}{R}, \quad v'_\varphi = 0, \quad v'_z = 0, \quad T' = T_2 \ (r = R_2),$$

(3)

are fulfilled, where $R = \frac{R_2}{R_1}$.

Problem (1)-(3) possesses a group of symmetries $\mathbf{G} = SO(2) * O(2)$ which are invariant with respect to rotations L_φ^δ near the cylinder axis, to shifts L_φ^h along that axis, and also to inversion J. These factors act on the vector-function $F(v_r, v_\varphi, v_z, T)$ according to the rules

$$\left(L_\varphi^\delta F\right)(t, r, \varphi, z) = F(t, r, \varphi + \delta, z),$$

$$(L_z^h F)(t, r, \varphi, z) = F(t, r, \varphi, z + h),$$

(4)

$$(JF)(t, r, \varphi, z) = \left\{ v_r(t, r, \varphi, -z), v_\varphi(t, r, \varphi, -z), -v_z(t, r, \varphi, -z), T(t, r, \varphi, -z) \right\}$$

for any real δ and h.

As is easily seen, problem (1)-(3) admits an exact solution

$$V_0 = \left\{ u_0(r), v_0(r), 0 \right\}, \quad T_0 = c_1 + c_2 r^{\varkappa P_r},$$

$$u_0(r) = \frac{R_1 U_0}{r}, \quad v_0(r) = \begin{cases} \dfrac{K}{\varkappa}\left(ar^{\varkappa+1} + \dfrac{b}{r} - r \right), & \varkappa \neq -2, \\[2ex] \dfrac{K}{2}\left(\dfrac{a_1 \ln r + b_1}{r} \right), & \varkappa = -2, \end{cases}$$

(5)

$$\frac{\partial \Pi_0}{\partial r} = \frac{\rho\left(u_0^2 + v_0^2\right)}{r},$$

where

$$K = \frac{1}{2\rho\nu}\left(\frac{\partial \Pi_0}{\partial \theta}\right)_0, \quad a = \frac{R^2 - 1}{\left(R^{\varkappa+2} - 1\right)R_1^\varkappa}, \quad a_1 = \frac{R_1^2\left(R^2 - 1\right)}{\ln R},$$

$$b = \frac{R_2^2\left(R^\varkappa - 1\right)}{R^{\varkappa+2} - 1}, \quad b_1 = -\frac{R_1^2 \ln R_2 - R_2^2 \ln R_1}{\ln R},$$

$$c_1 = \frac{T_1 R^{\mathrm{Pr}\varkappa} - T_2}{R^{\varkappa \mathrm{Pr}} - 1}, \quad c_2 = \frac{T_2 - T_1}{R_1^{\varkappa \mathrm{Pr}} - 1}$$

and $\varkappa = \dfrac{U_0 R_1}{\nu}$ is the radial Reynolds number, $P_r = \dfrac{\nu}{\chi}$ is the Prandtl number.

The radial flow is inward for $\varkappa < 0$ (converging flow) and outward for $\varkappa > 0$ (diverging flow).

The flow with the velocity vector V_0, temperature T_0 and pressure Π_0 will be called the Dean stationary flow of a heat-conducting fluid with a radial temperature gradient and a radial flow.

3. Nonlinear System for Perturbations

A solution of problem (1)-(3) is sought in the form

$$V' = V_0 + V\left(v_r, v_\varphi, v_z\right), \quad T' = T_0 + T, \quad \Pi' = \Pi_0 + \Pi. \tag{6}$$

Taking into account that the main stationary flow is a rotating shear flow, we denote rotation shear S by $\dfrac{V_m}{d}$, where V_m is an average velocity in the azimuthal direction, $d = R_2 - R_1$ is a gap width between cylinders. Introducing dimensionless variables for time, velocity, temperature and pressure by S, R_2, SR_2, $T_2 - T_1$, $\nu\rho' S$ in the system (1), for the vector-functions $F = \{v_r, v_\varphi, v_z, T\}$ and $F_1 = \{u_r, u_\varphi, u_z, T_1\}$, we obtain the following nonlinear problem of finding perturbations V, T and Π:

$$\frac{\partial F}{\partial t} + NF - \frac{1}{\mathrm{Re}}MF + \frac{1}{\mathrm{Re}}\nabla_1\Pi = -\mathcal{L}(F, F),$$
$$\left(\nabla_1, rF\right) = 0, \quad F\big|_{r=1,R} = 0, \tag{7}$$

where

$$MF = \left\{ \Delta_1 v_r - \frac{1-\varkappa}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}, \Delta_1 v_\varphi - \frac{1+\varkappa}{r^2}v_\varphi + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}, \Delta_1 v_z, \frac{1}{\mathrm{Pr}}\Delta_1 T \right\},$$

$$NF = \omega_1 \frac{\partial F}{\partial \varphi} + \left\{ \mathrm{Ra}\,\omega_2 T - 2\omega_1 v_\varphi, -g_1 v_r, 0, \frac{g_2}{\mathrm{Pr}}v_r \right\},$$

$$\mathcal{L}(F, F_1) = \left\{ (F, \nabla_1)u_r - \frac{v_\varphi u_\varphi}{r}, (F, \nabla_1)u_\varphi + \frac{v_r u_\varphi}{r}, (F, \nabla_1)u_z, (F, \nabla_1)T_1 \right\},$$

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1-\varkappa}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla_1 = \left\{ \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}, 0 \right\},$$

$$\mathrm{Re} = \frac{V_m d}{\nu} \text{ is the azimuthal Reynolds number,}$$

$$\mathrm{Ra} = \frac{\beta(T_2 - T_1)}{2} \text{ is the Rayleigh number,}$$

$$\omega_1 = \frac{V_m g(r)}{r}, \quad \omega_2 = \omega_1^2 r, \quad V_m = \frac{K}{\varkappa}\cdot\frac{d}{R_2}D4,$$

$$g(r) = \frac{d}{R_2}\frac{D1 r^{\varkappa+2} + D2 - r^2}{rD4},$$

$$g_1(r) = \frac{\mathrm{d}v_0}{\mathrm{d}r} + \frac{v_0}{r} = \frac{K}{\varkappa}\cdot\frac{d}{R_2}\frac{D1(\varkappa+2)r^\varkappa - 2}{D4},$$

$$g_2(r) = \frac{\varkappa\,\mathrm{Pr}^2\cdot R^{\varkappa\mathrm{Pr}}}{R^{\varkappa\mathrm{Pr}} - 1}r^{\varkappa\mathrm{Pr}} - 1,$$

$$D1 = \frac{(R^2 - 1)R^\varkappa}{R^{\varkappa+2} - 1}, \quad D2 = 1 - D1,$$

$$D3 = \frac{R^2 - 1}{R^2}, \quad D4 = D2\ln R - \frac{\varkappa D3}{2(\varkappa+2)}.$$

Note, that $D_e = \text{Re}\left(\dfrac{d}{R_2}\right)^{\frac{1}{2}}$ is the Dean number.

Problem (7) is written in terms of the Boussinesq approximation, which is based on the assumption that the thermal expansion coefficient is small [27]. In the sequel it will always be assumed that the velocity, temperature and pressure components are periodic with respect to z and φ with the known periods $2\pi/\alpha$ and $2\pi/m$, respectively. Thus the solution of the problem depends on the 7 dimensional parameters: azimuthal and radial Reynolds numbers, Rayleigh and Prandtl numbers, the ratio of the radii of cylinders R, axial and azimuthal wave numbers α, m, respectively.

4. Neutral Curves

The theoretical and experimental studies have shown that after the loss of stability of Dean flow there occured secondary modes via either axisymmetric or nonaxisymmetric disturbances as Dean vortices and oscillatory modes in the form of traveling waves. To study the transition to complex regimes of special attention are the points of intersection of neutral curves, corresponding to the two above-mentioned kinds of the secondary flows, since at these points with a high probability may appear various regimes, including the complex one.

Let $(\text{Ra}_0, \text{Re}_0)$ be the point lying on the plane of parameters (Ra, Re) and corresponding to the intersection of the neutral curves corresponding to the monotonic ($m = 0$) axisymmetric and oscillatory nonaxisymmetric loss of stability of main flow (3). In this paper we will explore the complex modes that may appear in the vicinity of intersection points of neutral curves corresponding to the vortex and azimuthal wave bifurcations. Under the definite values of parameters of the problem, the neutral curves may be nonintersecting that indicates that under the corresponding values of parameters of the problem we cannot expect the appearance of complex regimes.

To construct neutral curves, we assume that the perturbations V, temperature T and pressure Π are infinitely small. Neglecting in (7) the nonlinear terms and seeking for a solution of that problem in the case of axisymmetric monotone perturbations, we obtain the following spectral problem

$$(M - \text{Re}\, N)\Phi_0 = \nabla_1 p_0, \quad (\nabla_1, r\Phi_0) = 0, \quad \Phi_0\big|_{r=1,R} = 0, \tag{8}$$

where

$$\Phi_0 = \{u_0(r), v_0(r), iw_0(r), \tau_0(r)\}e^{i\alpha z}, \quad p_0 = q_0(r)e^{i\alpha z}. \tag{9}$$

By the substitution of (9) into (8), after dividing the variables in (8) we obtain the spectral problem for the ordinary differential equations

$$\left(L - \frac{1-\varkappa}{r^2} - \alpha^2\right)u_0 = \frac{\text{d}q_0}{\text{d}r} - 2\,\text{Re}\,\omega_1 v_0 + \text{Re}\,\text{Ra}\,\omega_2 \tau_0,$$

$$\left(L - \frac{1+\varkappa}{r^2} - \alpha^2\right)v_0 = -\text{Re}\,g_1 u_0,$$

$$\left(L-\alpha^2\right)w_0 = \alpha q_0, \quad \left(L-\alpha^2\right)\tau_0 = \operatorname{Re} g_2 u_0, \quad \frac{du_0}{dr} + \frac{u_0}{r} - \alpha w_0 = 0,$$

$$u_0 = v_0 = w_0 = \tau_0 = 0 \quad (r = 1, R),$$

(10)

where $L = \dfrac{d^2}{dr^2} + \dfrac{1-\varkappa}{r}\dfrac{d}{dr}$.

The neutral curves corresponding to the bifurcation of azimuthal waves origination are found by solving the spectral problem

$$\left(M - \operatorname{Re} N - ic\operatorname{Re}\right)\Phi_1 = \nabla_1 p_1, \quad \left(\nabla_1, r\Phi_1\right) = 0, \quad \Phi_1\big|_{r=1,R} = 0.$$

(11)

A solution of problem (11) is sought in the form

$$\Phi_1 = \left\{u_1(r), v_1(r), w_1(r), \tau_1(r)\right\}e^{-i(m\varphi+\alpha z)}, \quad p_1 = q_1(r)e^{-i(m\varphi+\alpha z)},$$

(12)

where c is the unknown frequency of autooscillations, m is the azimuthal wave number. Substitution of (12) into (11) after dividing the variables in (11), we get the spectral problem for the ordinary differential equations:

$$\left[L_1 - \frac{1-\varkappa}{r^2} - \alpha^2 - i\operatorname{Re}\left(c - m\omega_1\right)\right]u_1 = \frac{dq_1}{dr} - 2\operatorname{Re}\omega_1 v_1 + \operatorname{Re}\operatorname{Ra}\omega_2 \tau_1 - \frac{2im}{r^2}v_1,$$

$$\left[L_1 - \frac{1+\varkappa}{r^2} - \alpha^2 - i\operatorname{Re}\left(c - m\omega_1\right)\right]v_1 = -\frac{im}{r}q_1 - \operatorname{Re} g_1(r)u_1 + \frac{2im}{r^2}u_1,$$

$$\left[L_1 - \alpha^2 - i\operatorname{Re}\left(c - m\omega_1\right)\right]w_1 = -i\alpha q_1,$$

(13)

$$\left[L_1 - \alpha^2 - i\operatorname{Re}\operatorname{Pr}\left(c - m\omega_1\right)\right]\tau_1 = \operatorname{Re} g_2 u_1,$$

$$\frac{du_1}{dr} + \frac{u_1}{r} - \frac{im}{r}v - i\alpha w_1 = 0, \quad u_1 = v_1 = w_1 = \tau_1 = 0 \quad (r = 1, R),$$

where

$$L_1 = \frac{d^2}{dr^2} + \frac{1-\varkappa}{r}\frac{d}{dr} - \frac{m^2}{r^2}.$$

Problems of eigenvalues (10) and (13) have been solved by the shooting method for fixed $\varkappa$, α, R, m, Pr, Ra. Thus, for the fixed values of these parameters we established the dependence of the critical value of the azimuthal Reynolds number Re and the neutral mode frequency c corresponding to the bifurcation of vortices and azimuthal waves origination on a Rayleigh number Ra. Further, using the Newton method, we minimize the difference between the obtained critical values of Re. This allows us to calculate with sufficient exactness the values Ra_0, Re_0 and c_0 corresponding to the point of intersection of neutral curves.

The calculations in this paper were performed for the case $R = 2$ (radius of the outer cylinders is two times greater than that of the inner ones), $m = 0,1$, for various values of axial wave number α, $\operatorname{Pr} = 0.71$ and for small absolute values $\varkappa$ ($3 \leq \varkappa \leq 3$). The results of calculations are presented in Table 1.

As our calculations show, these intersections of neutral curves take place especially when temperature of the outer cylinder is higher than that of the inner one $\left(\operatorname{Ra}_0 > 0\right)$ for sufficiently large values of the wave axial number and when

Table 1. The points of intersection of neutral curves.

$\varkappa$	$\alpha = 8$			$\alpha = 5$			$\alpha = 4$		
	Ra_0	Re_0	c_0	Ra_0	Re_0	c_0	Ra_0	Re_0	c_0
3	3.627	20.9	2.795	0.714	68.445	2.473	0.873	87.84	2.49
2	3.14	22.07	2.792	0.667	82.88	2.458	0.809	81.245	2.472
1.5	2.52	24.45	2.7826	0.65	59.86	2.456	0.778	77.421	2.4709
0.5	2.084	26.938	2.763	0.64	53.09	2.473	0.75	68.464	2.492
0.2	1.675	30.2	2.727	0.65	51.488	2.48	0.7524	65.438	2.507
−0.3	-	-	-	0.7	46.25	2.52	0.772	60.164	2.541
−0.5	-	-	-	0.72	44.73	2.525	0.785	58.028	2.558
−1	-	-	-	0.84	39.73	2.593	0.835	52.86	2.606
−1.5	-	-	-	1.036	34.038	2.6678	0.902	48.268	2.6604
−3	-	-	-	1.784	25.479	2.877	1.157	39.24	2.824

the liquid moves through the inner cylinder to the outer one. This indicates that short-wave axially directed perturbations generating vortices and azimuthal waves are interacting, and hence one may expect the appearance of complex regimes. If values of the axial wave numbers decrease, *i.e.*, for the corresponding long-wave perturbations there are also an intersections of neutral curves for the converging flow.

If temperature of the inner cylinder exceeds that of the outer one $\left(\mathrm{Ra}_0 < 0\right)$, neutral curves do not intersect, and thus it is difficult to expect the occurrence of complex modes.

5. The Amplitude System and Transitions

To investigate the secondary flows and the appearance of high instabilities in the flow (3) we use the nonlinear theory of bifurcation of hydrodynamic flows with cylindrical symmetry (see [28]-[33]).

This theory allows one to find various liquid motion regimes existing in the vicinity of the points of intersection of neutral curves corresponding to the two types of secondary instability-vortices and azimuthal waves in the Dean flow. This theory has been applied in a rather wide class of problems, such, for example, as the Couette flow [34], the flow between permeable cylinders [35] [36] [37] [38] [39], a heat-conducting liquid flow between rigid and permeable cylinders [40] [41] [42] [43] [44].

The problems are reduced to the investigation of nonlinear dynamical autonomous systems of the amplitude equation, which is a generalization of Landaus amplitude equation.

The $SO(2)*O(2)$ symmetry enables one to reduce the six-dimensional amplitude system with unknown complex amplitudes η_0, η, η_1 to the four-dimensional motor subsystem for the modules of the amplitudes with free parameters σ, μ (the damping decrements of the monotonic and oscillatory

perturbations, respectively). To the equilibria of this system, which is lying on the invariant subspaces, there correspond the motions of a fluid, which has the concrete physical nature [32]-[44]: main stationary flow; vortex flows, *i.e.*, a secondary stationary axisymmetric flow; purely azimuthal waves, *i.e.*, secondary oscillatory modes; spiral waves, *i.e.*, secondary autooscillatory modes; mixed azimuthal waves, *i.e.*, three frequency regimes; equilibria not lying on the invariant subspaces, *i.e.*, equilibria of a general state, each of them corresponds to a quasiperiodic two-frequence solution of the amplitude system.

As our calculations shows, depending on the parameter values of the problem, the motor subsystem may have no equilibria or may have equilibria of the above mentioned types. It is found, that transition schemes may turn out to be both rather complicated and absolute trivial.

We present here the scheme of equilibria bifurcations of the motor subsystem, which we consider most interesting and allowing us to judge about the transitions characteristic of the considered system of small absolute values of radial Reynolds number $\varkappa$.

Consider in detail the case for $\varkappa = 3$ (dividing flow), $\alpha = 4$ (perturbations are $\dfrac{\pi}{2}$ periodical in the axial direction), $\mathrm{Ra}_0 = 0.873$ (temperature of the outer cylinder is higher than that of the inner one).

For $\sigma < 0$ and $\mu_r < 0$ (μ_r is the real part of damping decrement of oscillatory perturbations μ) the equilibria fail to take place, but for $\mu_r > 0$ of all the equilibria of the motor subsystem there exist only unstable main stationary flow and unstable purely azimuthal waves.

For $\sigma = 0$ and $\mu_r < 0$ there exist only stable main stationary flow,but for $\mu_r > 0$ there are unstable main stationary flow and vortices.

For $\sigma > 0$ and $\mu_r < 0$ the unstable main stationary flow and vortices exists fon any values of free parameters; vortices are stable for $\mu_r < \mu_r^2$ and unstable for $\mu_r > \mu_r^2$.

For $\sigma > 0$, $\mu_r > 0$ the transitions associated with bifurcations are graphically shown in Figure 2.

The single lines show *J*-symmetric equilibria, the double lines indicate *J*-connected pair of equilibria. Stable equilibria are drawn by solid lines and unstable equilibria by dotted lines. The circles are the points at which the motor subsystem cycles bifurcate.

For $\mu_r = 0$ from main flow (3) there simultaneowly bifurcate unstable pure azimuthal waves and unstable *J*-connected pair of spiral waves. Spiral waves are stable for $\mu_r > \mu_r^4$, for $\mu_r = \mu_r^4$ from them bifurcates an unstable connected pair of general equilibria. This pair exists in the range $\mu_r^4 < \mu_r < \mu_r^8$ and disappear for $\mu_r = \mu_r^8$ merging with unstable mixed azimuthal waves. On the other hand at the point $\mu_r = \mu_r^9$ mixed azimuthal waves merging with purely azimuthal waves.

For $\mu_r = \mu_r^3$, of a set of limit symmetric cycles of the motor subsystem in the form of number eight are branching off from mixed azimuthal waves and exists
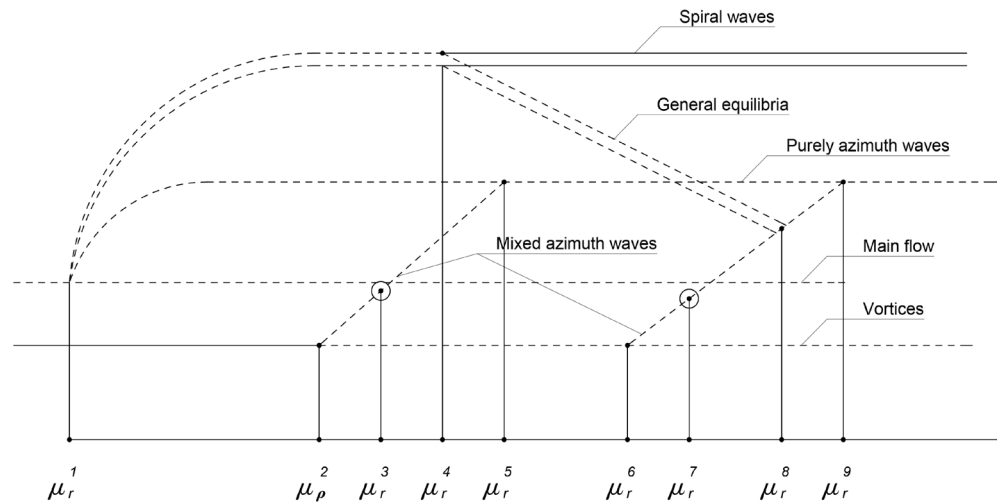
Figure 2. Bifurcation values: $\mu_r^1 = 0$, $\mu_r^2 = 2.75$, $\mu_r^3 = 2.95$, $\mu_r^4 = 3.025$, $\mu_r^5 = 3.265$, $\mu_r^6 = 3.8$, $\mu_r^7 = 4.003$, $\mu_r^8 = 4.05$, $\mu_r^9 = 4.1$.
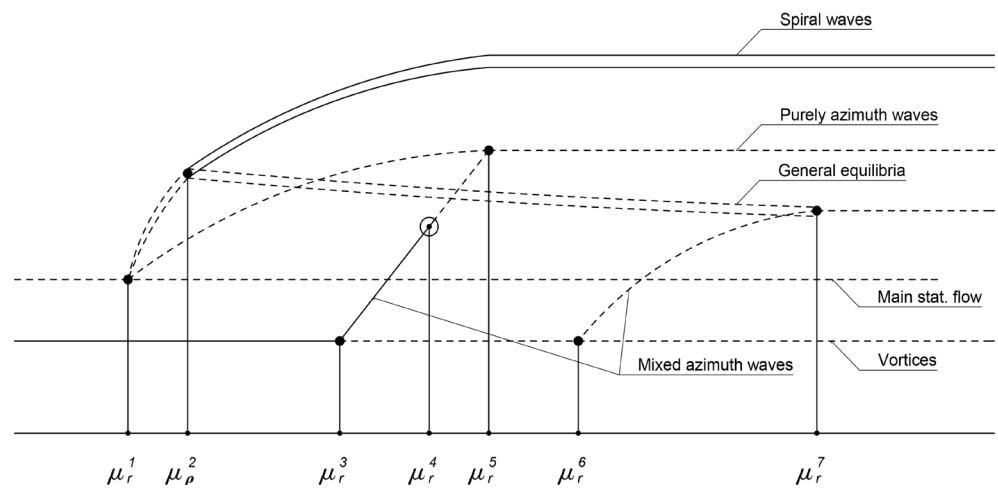
Figure 3. Bifurcation values: $\mu_r^1 = 0$, $\mu_r^2 = 0.8$, $\mu_r^3 = 4.259$, $\mu_r^4 = 4.88$, $\mu_r^5 = 6.3$, $\mu_r^6 = 10.009$, $\mu_r^7 = 15.285$.

in the range $\mu_r^3 < \mu_r < \mu_r^5$. They were calculated by using the software prepared in [34] [42], which enabled us to investigate the evolution, stability and bifurcatios of cycles for various values of the problem. The limit cycles of similar forms as the phase trajectories branching off from the mixed azimuthal waves are repeatedly met in the works (see, for example, [34]-[44]) when considering concrete problems between rotating vertical cylinders. In our case, the cycles are unstable and have no bifurcations. They exists in the small range $4.003 < \mu_r < 4.01$ and merges with the mixed azimuthal waves.

Consider the case $\varkappa = -0.5$ (converging flow), $\alpha = 4$ (perturbations are $\pi/2$ periodical in the axial direction), the Rayleigh number $\mathrm{Ra}_0 = 0.785$ (temperature of the outer cylinder is higher than that of the inner) for $\sigma > 0$. The transitions associated with the bifurcation are graphically shown in **Figure 3**.

In comparison to the case under consideration for diverging flow, the picture of bifurcations for converging flow becomes more complicated.

A characteristic feature of the transitions in this case are the existence of several stable equilibria in the range changes μ_r, for example, $\mu_r^2 < \mu_r < \mu_r^4$. For the values of a free parameter belonging to such ranges, in the experiments, the hysteresis phenomena are observed [42].

In this case, in contrast to the case $\varkappa > 0$ for $\mu_r = \mu_r^4$, from the mixed azimuthal waves branch off the nonsymmetric limit stable cycles similar to the deformed number eight, they lose their stability because of bifurcations of the period duplications at the point $\mu_r = 5.3$ and from it branches off a J-connected pair of stable II—revolution nonsymmetric cycles which exist for $5.3 < \mu_r < 5.7$. A further increase of μ_r leads to the formation of another series of duplications of J-connected pairs of cycles which for $\mu_r = 5.9$ results in to the generation of J-connected pair of chaotic attractors; the form of their phase trajectories differs slightly from that of the trajectories of the generated cycles; changes gradually its form and disappears for $\mu_r = 6.2$. It can be calculated in a very narrow range of variation of the parameter. In the same way, as well as with vertical cylinders (see [44]), we can judge about the appearance of complex regimes.

6. Conclusions

In paper under consideration, we have examined possible transitions to complex modes in a heat-conducting flow produced by pumping a fluid around the annulus of a system of fixed porous horizontal cylinders. The liquid is under the action of a radial flow and a radial temperature gradient.

1) Neutral curves corresponding to vortices and azimuthal waves are calculated.

2) It is established that for diverging flow with small absolute values of radial Reynolds numbers and for sufficiently large values of the wave axial number, the intersections of neutral curves take place especially when temperature of the outer cylinder is higher than that of the inner one $(\mathrm{Ra}_0 > 0)$. This indicates that the short-wave axially directed perturbations generating vortices and azimuthal waves are interacting, and hence one may expect the appearance of complex regimes. If values of the axial wave numbers decrease, *i.e.*, for the corresponding long-wave perturbations there are also intersections of neutral curves for the converging flow.

If temperature of the inner cylinder exceeds that of the outer one $(\mathrm{Ra}_0 < 0)$, neutral curves do not intersect, and thus it is difficult to expect the occurrence of complex modes.

1) Equilibria of a motor subsystem which are lying on the invariant subspaces are calculated. They correspond to stationary or periodic regimes of the fluid motion.

2) Equilibria of a general state are calculated. They corresponds to a quasiperiodic

two-frequency solution of the amplitude system.

3) It is shown that for a converging flow there are several stable equilibria on the range of change μ_r. For the values of a free parameter belonging to such ranges, in the experiments, hysteresis phenomena are observed [42].

4) It is established that for the certain values of the problem, there exist stable or unstable limit cycles of the motor subsystem.

5) It is shown that for converging flow, when temperature of the outer cylinder is higher than that of the inner one, the sequential duplications of a period of stable limit cycles lead to the generation of chaotic attractors and consequently as well as with vertical cylinders [43], we can judge about the appearance of complex regimes.

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