Scientific
Research
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Immersed Interface Method for Fokker-Planck Equation with Discontinuous Drift

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Abstract

The Immersed Interface Method (IIM) is derived to solve the corresponding Fokker-Planck equation of Brownian motion with pure dry friction, which is one of the simplest models of piecewise-smooth stochastic systems. The IIM is capable of treating a discontinuity in the drift of Fokker-Planck equation and it is readily extended to the dry and viscous friction model. Analytic results of the considered model are used to confirm the effectiveness and design accuracy of the scheme.

Keywords

Fokker-Planck Equation, Immersed Interface Method, Brownian Motion, Pure Dry Friction

1. Introduction

In recent years, piecewise-smooth stochastic systems (governed by piecewise-smooth stochastic differential equations) are usually used to describe biological and physical systems. Although for some simple piecewise-linear stochastic differential equations, analytical solutions of the transition probability distribution can be obtained [1] [2], it is difficult to attain analytical expressions for many other cases. Hence, we need to develop some effective numerical methods to deal with the difficulty in order to know more dynamical behaviors of the systems.

In this paper, we attempt to solve numerically a Fokker-Planck equation with discontinuous drift, which results from a so-called Brownian motion with pure dry friction [3]. This dry friction model can be described as the following piecewise linear Langevin equation

$$v'(t) = -\sigma(v(t)) + \xi(t). \tag{1.1}$$

Here $\sigma(v(t))$ denotes the sign of the velocity $v(t)$, representing the dry friction force. $\xi(t)$ is the Gaussian white noise with zero mean and delta correlation $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$ with $D > 0$. The notation $\langle \bullet \rangle$ stands for the average overall possible realizations of the noise, and δ is the Dirac delta function. The transition probability distribution $p(v,t\,|\,v_0,0)$ of (1.1) satisfies the following Fokker-Planck equation [4] [5],

$$\frac{\partial p(v,t\,|\,v_0,0)}{\partial t} = \frac{\partial\left(\sigma(v)p(v,t\,|\,v_0,0)\right)}{\partial v} + D\frac{\partial^2 p(v,t\,|\,v_0,0)}{\partial v^2}. \tag{1.2}$$

The corresponding initial condition is $p(v,0\,|\,v_0,0) = \delta(v-v_0)$ if $v(0) = v_0$ for (1.1).

Since Equation (1.2) has a discontinuous drift $\sigma(v)$, we must deal with it carefully. The IIM is a sharp interface method which can accurately capture discontinuities in the solution and the flux. This method has been used for many problems, such as elliptic interface problems [6], parabolic interface problems [7], moving interface problems [8] and many other applications [9] [10] [11] (see [12] [13] for excellent reviews). To the best of our knowledge, there is no literature about the IIM for solving Fokker-Planck equations with discontinuous drift so far. Hence, our goal is to solve it.

The rest of this paper is organized as follows. In Section 2, we derive the IIM for the Fokker-Planck Equation (1.2). The numerical results are compared with the analytical solutions in Section 3. In addition, the accuracy of the scheme is also obtained. Finally, conclusions are made in Section 4.

2. The Scheme

We set $D = 1$ for convenience. At the discontinuous point $v = 0$, we have the matching condition for the solution,

$$p^+ = p^- \tag{2.1}$$

where $+$ and $-$ stand for the limiting values from the right- and left-hand sides of $v = 0$. Integrating (1.2) across the discontinuity, we find

$$p_v^+ = p_v^- - 2p^- \tag{2.2}$$

and then

$$p_v^- = p_v^+ + 2p^+ \tag{2.3}$$

by replacing p^- with p^+ in (2.2).

It follows from (2.1) that $p_t^+ = p_t^-$, that is

$$\left(\sigma(v)p\right)_v^+ + p_{vv}^+ = \left(\sigma(v)p\right)_v^- + p_{vv}^-$$

according to Equation (1.2). Then using the relations (2.1)-(2.3) we have

$$p_{vv}^+ = p_{vv}^- - 2p_v^- + 2p^- \tag{2.4}$$

and

$$p_{vv}^- = p_{vv}^+ + 2p_v^+ + 2p^+. \tag{2.5}$$

For the numerical scheme, we have first to truncate the computational do-

main $(-\infty,+\infty)$ to a finite domain. Without loss of generality, let us assume the finite domain to be $[-L_v, L_v]$, where L_v is a positive constant. Then we assume the probability vanishes at the boundary, *i.e.*,

$$p(\pm L_v, t \mid v_0, 0) = 0. \qquad (2.6)$$

A uniform grid with step $h = 2L_v / M$ is chosen here, where M is a positive constant.

Therefore, the grid points can be expressed as $v_i = -L_v + (i-1)h$, $i = 1, 2, \cdots, M+1$ with the discontinuous point being between v_j and v_{j+1}, $v_j \le 0 < v_{j+1}$.

We hope to develop finite difference scheme of the form

$$\frac{p_i^{n+1} - p_i^n}{\tau} = \gamma_{i,1} p_{i-1}^n + \gamma_{i,2} p_i^n + \gamma_{i,3} p_{i+1}^n + C_i^n, \quad i = 2, 3, \cdots, M, \qquad (2.7)$$

where τ is the time-step size. This means that we need to determine the coefficients γ and the correction term C_i^n so that

$$\gamma_{i,1} p_{i-1}^n + \gamma_{i,2} p_i^n + \gamma_{i,3} p_{i+1}^n + C_i^n \approx \left\{ \frac{\partial (\sigma(v)p)}{\partial v} + \frac{\partial^2 p}{\partial v^2} \right\}_{v = v_i}. \qquad (2.8)$$

At a regular grid point v_i, $i \ne j, j+1$, the coefficients γ in the explicit difference scheme (2.7) are obtained by the standard approximation as follows

$$\gamma_{i,1} = \frac{1}{h^2} - \frac{\sigma(v_{i-1/2})}{2h}, \quad \gamma_{i,2} = \frac{\sigma(v_{i+1/2}) - \sigma(v_{i-1/2})}{2h} - \frac{2}{h^2}, \quad \gamma_{i,3} = \frac{1}{h^2} + \frac{\sigma(v_{i+1/2})}{2h}, \qquad (2.9)$$

and the correction term $C_i^n = 0$.

At the irregular grid point v_j, we expand p_{j-1}^n, p_j^n and p_{j+1}^n in Taylor series at the discontinuous point $v = 0$ to obtain

$$p_{j-1}^n = p^- + v_{j-1} p_v^- + \frac{1}{2} v_{j-1}^2 p_{vv}^- + O(h^3), \qquad (2.10)$$

$$p_j^n = p^- + v_j p_v^- + \frac{1}{2} v_j^2 p_{vv}^- + O(h^3), \qquad (2.11)$$

$$p_{j+1}^n = p^+ + v_{j+1} p_v^+ + \frac{1}{2} v_{j+1}^2 p_{vv}^+ + O(h^3). \qquad (2.12)$$

For Equation (2.12), using (2.1), (2.2) and (2.4), we have

$$p_{j+1}^n = (1 - 2v_{j+1} + v_{j+1}^2) p^- + (v_{j+1} - v_{j+1}^2) p_v^- + \frac{1}{2} v_{j+1}^2 p_{vv}^- + O(h^3). \qquad (2.13)$$

Furthermore, substituting (2.10), (2.11) and (2.13) into (2.8) we have

$$\gamma_{j,1} \left(p^- + v_{j-1} p_v^- + \frac{1}{2} v_{j-1}^2 p_{vv}^- \right) + \gamma_{j,2} \left(p^- + v_j p_v^- + \frac{1}{2} v_j^2 p_{vv}^- \right)$$

$$+ \gamma_{j,3} \left((1 - 2v_{j+1} + v_{j+1}^2) p^- + (v_{j+1} - v_{j+1}^2) p_v^- + \frac{1}{2} v_{j+1}^2 p_{vv}^- \right) + C_j^n + O(h) \qquad (2.14)$$

$$= \left\{ \frac{\partial (\sigma(v)p)}{\partial v} + \frac{\partial^2 p}{\partial v^2} \right\}_{v = 0^-} + O(h).$$

Then by arranging terms we obtain

$$p^- \left(\gamma_{j,1} + \gamma_{j,2} + (1 - 2v_{j+1} + v_{j+1}^2)\gamma_{j,3} \right) + p_v^- \left(v_{j-1}\gamma_{j,1} + v_j\gamma_{j,2} + (v_{j+1} - v_{j+1}^2)\gamma_{j,3} \right)$$

$$+ p_{vv}^- \left(\frac{1}{2} v_{j-1}^2 \gamma_{j,1} + \frac{1}{2} v_j^2 \gamma_{j,2} + \frac{1}{2} v_{j+1}^2 \gamma_{j,3} \right) + C_j^n + O(h) \tag{2.15}$$

$$= -p_v^- + p_{vv}^- + O(h).$$

Comparing both sides of (2.15), one obtains three equations for $\gamma_{j,1}$, $\gamma_{j,2}$ and $\gamma_{j,3}$ as follows

$$\gamma_{j,1} + \gamma_{j,2} + (1 - 2v_{j+1} + v_{j+1}^2)\gamma_{j,3} = 0, \tag{2.16}$$

$$v_{j-1}\gamma_{j,1} + v_j\gamma_{j,2} + (v_{j+1} - v_{j+1}^2)\gamma_{j,3} = -1, \tag{2.17}$$

$$v_{j-1}^2 \gamma_{j,1} + v_j^2 \gamma_{j,2} + v_{j+1}^2 \gamma_{j,3} = 2, \tag{2.18}$$

and the correction term $C_j^n = 0$.

Therefore, one can solve (2.16) - (2.18) to attain the coefficients of Equation (2.7) for $i = j$.

In a similar way, we can compute the coefficients at the irregular grid point v_{j+1} from the equations

$$(1 + 2v_j + v_j^2)\gamma_{j+1,1} + \gamma_{j+1,2} + \gamma_{j+1,3} = 0, \tag{2.19}$$

$$(v_j + v_j^2)\gamma_{j+1,1} + v_{j+1}\gamma_{j+1,2} + v_{j+2}\gamma_{j+1,3} = 1, \tag{2.20}$$

$$v_j^2 \gamma_{j+1,1} + v_{j+1}^2 \gamma_{j+1,2} + v_{j+2}^2 \gamma_{j+1,3} = 2, \tag{2.21}$$

and the correction term $C_{j+1}^n = 0$.

3. Numerical results

For the Fokker-Planck Equation (1.2), using spectral decomposition method, one can get the transition probability distribution in closed analytic form [1] [14]:

$$p(v,t \,|\, v_0, 0) = \frac{1}{D} \hat{p}\left(\frac{1}{D} v, \frac{1}{D} t \,\middle|\, \frac{1}{D} v_0, 0 \right), \tag{3.1}$$

where

$$\hat{p}(x,\tau \,|\, x', 0) = \frac{e^{-\tau/4}}{2\sqrt{\pi\tau}} e^{-(|x|-|x'|)/2} e^{-(x-x')^2/(4\tau)} + \frac{e^{-|x|}}{4} \left[1 + \mathrm{erf}\left(\frac{\tau - (|x| + |x'|)}{2\sqrt{\tau}} \right) \right] \tag{3.2}$$

is the transition probability distribution in non-dimensional units and

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{3.3}$$

is the error function. In addition, when $t \to \infty$ the Fokker-Planck Equation (1.2) admits a steady stationary state

$$p(v) = \frac{1}{2D} e^{-|v|/D}. \tag{3.4}$$

Let $D = 1$, $v_0 = 2$ and the computing interval be $[-10, 10]$. We choose the

space-step $h = 0.01$ and the time-step $\tau = 5 \times 10^{-5}$. For simplicity, we take the analytic distribution (3.1) at time $t = 0.01$ as the initial condition for computing. **Figure 1** shows the comparison of numerical and analytical results of the probability distribution $p(v,t \,|\, v_0, 0)$ at different times. It can be seen that the numerical solutions (points) coincide with the exact solutions (solid lines), indicating the effectiveness of the Scheme (2.7).

To see the accuracy of the scheme numerically, we consider the L_2 and L_∞ errors between the numerical solutions and the exact solutions defined by

$$\left\| Error \right\|_{L_2} = \sqrt{h \sum_{i=1}^{M} (p_i - \bar{p}_i)^2}, \tag{3.5}$$

$$\left\| Error \right\|_{L_\infty} = \max_{1 \le i \le M} |\, p_i - \bar{p}_i \,|, \tag{3.6}$$

where p_i is the numerical solution and $\bar{p}_i$ is the exact solution. Then we calculate the order of accuracy. A small time-step $\tau = 1 \times 10^{-5}$ and $v_0 = 2$ are chosen and the problem is recalculated from time $t = 0.01$ to $t = 1$. As illustrated in **Table 1**, the scheme is approximated second order in the velocity direction.
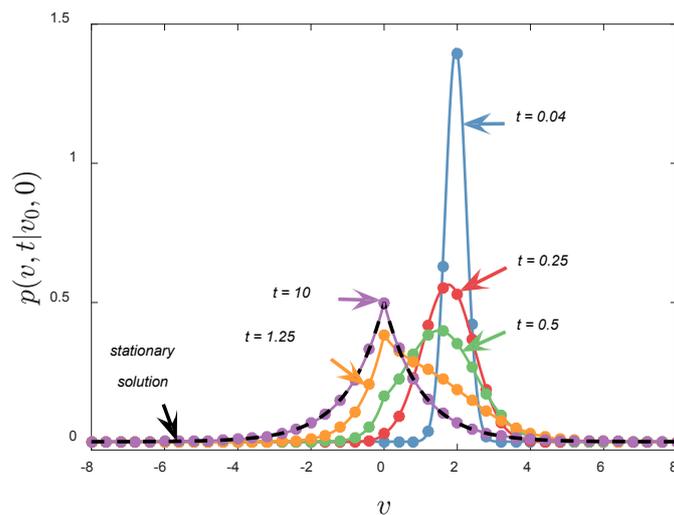
Figure 1. Transition probability distribution $p(v,t \,|\, v_0, 0)$ of Fokker-Planck Equation (1.2) with solid lines corresponding to the exact solutions, points to the numerical solutions, and dashed line to the stationary solution.

Table 1. Accuracy test in the velocity direction for $t = 1$ and $\tau = 1 \times 10^{-5}$.

h	L_2 error	Order	L_∞ error	Order
0.04	6.9701×10^{-4}		8.9056×10^{-4}	
0.02	1.7441×10^{-4}	2.00	2.2432×10^{-4}	1.99
0.01	4.2682×10^{-5}	2.03	5.5016×10^{-5}	2.03
0.005	9.7086×10^{-6}	2.14	1.2371×10^{-5}	2.15

4. Conclusions

We have used the IIM to solve a Fokker-Planck equation with discontinuous drift in this paper. The numerical results show that the developed scheme is effective and has second order of accuracy. Moreover, the scheme can be readily extended to other dry friction models and the numerical results obtained are important references to see whether the dry friction effect exists in engineering applications.

Acknowledgements

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