

# The SMW Formula for Bounded Homogeneous Generalized Inverses with Applications

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## Abstract

In this paper, we present an extension of the so-called classical Sherman-Morrison-Woodbury (for short SMW) formula for bounded homogeneous generalized inverse in Banach spaces. Some particular cases and applications will be also considered. Our results generalize the results of many authors for finite dimensional matrices and Hilbert space operators in the literature.

## Keywords

Homogeneous Generalized Inverse, Perturbation, Sherman-Morrison-Woodbury Formula

## 1. Introduction

It is well known the Sherman-Morrison-Woodbury (for short SMW) formula gives an explicit form for the inverse of matrices of the form  $A + YGZ^*$ :

$$(A + YGZ^*)^{-1} = A^{-1} - A^{-1}Y(G^{-1} + Z^*A^{-1}Y)^{-1}Z^*A^{-1}, \quad (1)$$

where  $A$  and  $G$  be  $n \times n$  and  $r \times r$  nonsingular matrices with  $r \leq n$ . Also, let  $Y$  and  $Z$  be  $n \times r$  matrices such that  $G^{-1} + Z^*A^{-1}Y$  is invertible. The SMW formula (1) is valid only if the matrices  $A$  and  $G^{-1} + Z^*A^{-1}Y$  are invertible. Over the years, Generalizations (see [1] [2] for example) have been considered in the case of singular or rectangular matrices using the concept of Moore-Penrose generalized inverses. Certain results on extending the SMW formula to operators on Hilbert spaces are also considered by many authors (see [3] [4] [5]).

Let  $X$  and  $Y$  be Banach spaces, and  $B(X, Y)$  be the Banach space consisting of all bounded linear operators from  $X$  to  $Y$ . For  $A \in B(X, Y)$ , let  $\mathcal{N}(A)$  (resp.  $\mathcal{R}(A)$ ) denote the kernel (resp. range) of  $A$ . It is well known that for  $A \in B(X, Y)$ , if  $\mathcal{N}(A)$  and  $\mathcal{R}(A)$  are topologically complemented in the

spaces  $X$  and  $Y$ , respectively, then there exists a linear projector generalized inverse  $A^+ \in B(Y, X)$  defined by  $A^+Ax = x, x \in \mathcal{N}(A)^c$  and  $A^+y = 0, y \in \mathcal{R}(A)^c$ , where  $\mathcal{N}(A)^c$  and  $\mathcal{R}(A)^c$  are topologically complemented subspaces of  $\mathcal{N}(A)$  and  $\mathcal{R}(A)$ , respectively. In this case,  $A^+A$  is the projection from  $X$  onto  $\mathcal{N}(A)^c$  along  $\mathcal{N}(A)$  and  $AA^+$  is the projection from  $Y$  onto  $\mathcal{R}(A)^c$  along  $\mathcal{R}(A)$ . But, in general, we know that not every closed subspace in a Banach space is complemented, thus, the linear generalized inverse  $A^+$  of  $A$  may not exist. In this case, we may seek other types of generalised inverses for  $T$ . Motivated by the ideas of linear generalized inverses and metric generalized inverses (cf. [6]), by using the so-called homogeneous (resp. quasi-linear) quasi-linear projectors in Banach space, in [7], the authors defined homogeneous (resp. quasi-linear) projector generalized inverse. Then, in [8] [9], the authors give a further study on this type of generalized inverse in Banach space. More important, from the results in [9], we know that, in some reflexive Banach spaces  $X$  and  $Y$ , for an operator  $T \in B(X, Y)$ , there may exist a bounded homogeneous (quasi-linear) projector generalized inverse of  $T$ , which is generally neither linear nor metric generalized inverse of  $T$ . So, from this point of view, it is important and necessary to study homogeneous (resp. quasi-linear) generalized inverses in Banach spaces. From then on, many research papers about the Moore-Penrose metric generalized inverses have appeared in the literature.

The objectives of this paper are concerned with certain extensions of the so called Sherman-Morrison-Woodbury formula to operators between some Banach spaces. We consider the SMW formula in which the inverse is replaced by bounded homogeneous generalized inverse. More precisely, let  $X, Y$  be Banach spaces, and we denote the set of all bounded linear operators from  $X$  into  $Y$  by  $B(X, Y)$  and by  $B(X)$  when  $X = Y$ . Let  $T \in B(X)$ ,  $G \in B(Y)$ , and  $U \in B(Y, X)$ ,  $V \in B(X, Y)$  such that  $A^h$  and  $G^h$  exist, In the main part of this paper, we will develop some conditions under which the Sherman-Morrison-Woodbury formula can be represented as

$$(T + UGV)^h = T^h - T^hU(G^h + VT^hU)^hVT^h,$$

where  $T^h$  is a bounded homogeneous generalized inverse of  $T$ . As a consequence, some particular cases and applications will be also considered. Our results generalize the results of many authors for liner operator generalized inverses.

## 2. Preliminaries

In this section, we recall some concepts and basic results will be used in this paper. We first present some facts about homogeneous operators. Let  $X, Y$  be Banach spaces. Denote by  $H(X, Y)$  the set of all bounded homogeneous operators from  $X$  to  $Y$ . Equipped with the usual linear operations for  $H(X, Y)$ , and for  $T \in H(X, Y)$ , the norm is defined by  $\|T\| = \sup\{\|Tx\| \mid \|x\| = 1, x \in X\}$ . then similar to the space of bounded linear operators, we can easily prove that  $(H(X, Y), \|\cdot\|)$  is a Banach space. For a bounded homogeneous operator

$T \in H(X, Y)$ , we always assume that  $\mathcal{D}(T) = X$ .

**Definition 2.1** ([8]). Let  $M \subset X$  be a subset and let  $T: X \rightarrow Y$  be a mapping. Then we call  $T$  is quasi-additive on  $M$  if  $T$  satisfies

$$T(x+z) = T(x) + T(z), \quad \forall x \in X, \forall z \in M.$$

For a homogeneous operator  $T \in H(X, X)$ , if  $T$  is quasi-additive on  $\mathcal{R}(T)$ , then we will simply say  $T$  is a quasi-linear operator.

**Definition 2.2** ([8]). Let  $P \in H(X, X)$ . If  $P^2 = P$ , we call  $P$  is a homogeneous projector. In addition, if  $P$  is also quasi-additive on  $\mathcal{R}(P)$ , i.e., for any  $x \in X$  and any  $z \in \mathcal{R}(P)$ ,

$$P(x+z) = P(x) + P(z) = P(x) + z,$$

then we call  $P$  is a quasi-linear projector.

The following concept of bounded homogeneous generalized inverse is also a generalization of bounded linear generalized inverse.

**Definition 2.3** ([8]). Let  $T \in B(X, Y)$ . If there is  $T^h \in H(Y, X)$  such that

$$TT^hT = T, \quad T^hTT^h = T^h,$$

then we call  $T^h$  is a bounded homogeneous generalized inverse of  $T$ .

Definition 2.3 was first given in paper [8] for linear transformations and bounded linear operators. The existence condition of a homogeneous generalized inverse is also given in [8].

### 3. Main Results

In this section, we mainly study the SMW formula for bounded homogeneous generalized inverses of a bounded linear operator in Banach spaces. In order to prove our main theorems, we first need to present some lemmas. The following result is well-known for bounded linear operators, we can generalize it to bounded homogeneous operators as follows.

**Lemma 3.1** ([10]). Let  $T, S \in H(X)$  such that  $T$  is quasi-additive on  $\mathcal{R}(S)$  and  $S$  is quasi-additive on  $\mathcal{R}(T)$ , then  $I+TS$  is invertible if and only if  $I+ST$  is invertible. Specially, when  $T \in B(X)$  and  $S \in H(X)$ , if  $S$  is quasi-additive on  $\mathcal{R}(T)$ , then  $I+TS$  is invertible if and only if  $I+ST$  is invertible.

The following result is well-known for bounded linear operators, we generalize it to the bounded homogeneous operators and metric projections in the following form.

**Lemma 3.2.** Let  $A \in H(X)$ . Let  $L \subset X$  be a subspace and  $P_L$  be the quasi-linear projection from  $X$  onto  $L$ .

- 1)  $P_L A = A$  if and only if  $\mathcal{R}(A) \subset L$ ;
- 2) If  $A$  is quasi-additive on  $L$ , then  $A P_L = A$  if and only if  $\mathcal{N}(P_L) \subset \mathcal{N}(A)$ .

*Proof.* Here, we only prove (1), and (2) can be proved in the same way. On the one hand, if  $P_L A = A$ , then  $\mathcal{R}(A) = \mathcal{R}(P_L A) \subset \mathcal{R}(P_L) = L$ . On the other hand,

for any  $x \in X$ , since,  $\mathcal{R}(A) \subset L$ , we can get that  $(A - P_L A)x = (I - P_L)Ax = 0$ , thus,  $P_L A = A$ . This completes the proof.  $\square$

**Lemma 3.3.** Let  $A \in H(X, Y)$  such that  $A^h$  exists. Then  $\mathcal{R}(A^h A) = \mathcal{R}(A^h)$  and  $\mathcal{N}(AA^h) = \mathcal{N}(A^h)$ .

*Proof.* Since  $A^h = A^h A A^h$ , we have

$$\mathcal{R}(A^h A) \subset \mathcal{R}(A^h) = \mathcal{R}(A^h A A^h) \subset \mathcal{R}(A^h A).$$

So,  $\mathcal{R}(A^h A) = \mathcal{R}(A^h)$ . Similarly, we also have

$$\mathcal{N}(AA^h) = \mathcal{N}(A^h A A^h) = \mathcal{N}(A^h) \subset \mathcal{N}(AA^h).$$

Therefore,  $\mathcal{N}(AA^h) = \mathcal{N}(A^h)$ .  $\square$

**Theorem 3.4.** Let  $T \in B(X)$ ,  $G \in B(Y)$ , and  $U \in B(Y, X)$ ,  $V \in B(X, Y)$  such that  $T^h$  and  $G^h$  exist, also, let  $S = T + UGV \in B(X)$  and  $C = G^h + VT^h U \in H(X)$  such that  $S^h$  and  $C^h$  exist. Suppose that  $S^h$  is quasi-additive on  $\mathcal{R}(T)$  and  $\mathcal{R}(U)$ , if

$$\begin{aligned} \mathcal{R}(T^h) &\subset \mathcal{R}(S^h), \quad \mathcal{N}(T^h) \subset \mathcal{N}(S^h), \\ \mathcal{N}(G^h) &\subset \mathcal{N}(U), \quad \mathcal{N}(C^h) \subset \mathcal{N}(G). \end{aligned} \tag{2}$$

Then  $\bar{T}^h = T^h - T^h U (G^h + VT^h U)^h VT^h$ .

*Proof.* From (2) and Lemma 3.3,

$$\begin{aligned} \mathcal{R}(T^h) &\subset \mathcal{R}(S^h) = \mathcal{R}(S^h S), \\ \mathcal{R}(I_X - TT^h) &= \mathcal{N}(TT^h) = \mathcal{N}(T^h) \subset \mathcal{N}(S^h). \end{aligned} \tag{3}$$

Then, using Lemma 3.2, we obtain

$$T^h = S^h S T^h, \quad S^h (I - TT^h) = 0.$$

Note that  $S^h$  is quasi-additive on  $\mathcal{R}(T)$ , thus, we have  $S^h = S^h T T^h$ , and then

$$\begin{aligned} T^h U &= S^h S T^h U = S^h S T^h U - S^h T T^h U + S^h T T^h U \\ &= S^h (S - T) T^h U + S^h U. \end{aligned}$$

Similarly, by Lemma 3.2, and also note that  $U \in B(Y, X)$ , then, from  $\mathcal{N}(G^h) \subset \mathcal{N}(U)$ , we get  $U = U G G^h$ . Now, Since  $S^h$  is also quasi-additive on  $\mathcal{R}(U)$  and  $S - T = UGV$ , thus

$$T^h U = S^h (S - T) T^h U + S^h U = S^h U G V T^h U + S^h G G U = S^h U G C. \tag{4}$$

Now using Lemma 3.2 again, also note that  $\mathcal{N}(C^h) \subset \mathcal{N}(G)$  and (4), we get

$$S^h U G = S^h U G C C^h = T^h U C^h. \tag{5}$$

Now, using (5), by simple computation, we can obtain

$$\begin{aligned} (T + UGV)^h &= S^h T T^h = S^h (S - UGV) T^h = T^h - T^h U C^h V T^h \\ &= T^h - T^h U (G^h + VT^h U)^h VT^h. \end{aligned}$$

This completes the proof.  $\square$

In above Theorem 3.4, if  $C$  and  $G$  are all invertible, we have the following

result.

**Corollary 3.5.** *Let  $T \in B(X)$ ,  $G \in B(Y)$ , and  $U, V \in B(Y, X)$  such that  $T^h$  and  $G^{-1}$  exist, also, let  $S = T + UGV \in B(X)$  and  $C = G^{-1} + VT^hU \in H(X)$  such that  $S^h$  and  $C^{-1}$  exist. Suppose that  $S^h$  is quasi-additive on  $\mathcal{R}(T)$  and  $\mathcal{R}(U)$ , if*

$$\mathcal{R}(T^h) \subset \mathcal{R}(S^h), \quad \mathcal{R}(T^h) \subset \mathcal{R}(S^h). \tag{6}$$

Then,  $S^h = T^h - T^hU(G^h + VT^hU)^h VT^h$ .

Furthermore, if  $A$  is invertible and  $G = I_Y$  in Corollary 3.5, then we also have the following result.

**Corollary 3.6** ([1], Theorem 2.1). *Let  $A \in B(X)$ ,  $U \in B(Y, X)$ , and  $V \in B(X, Y)$  such that  $A$  is invertible. Then  $A + UV$  is invertible if and only if  $I_Y + VA^{-1}U$  is invertible. Furthermore, when  $A + UV$  is invertible, then*

$$(A + UV)^{-1} = A^{-1} - A^{-1}U(I_Y + VA^{-1}U)^{-1}VA^{-1}. \tag{7}$$

*Proof.* Since  $A^{-1}$  exists, then, from Lemma 3.1, we see  $I_X + A^{-1}UV$  is invertible if and only if  $I_Y + VA^{-1}U$  is invertible. Now, using the equality  $A + UV = A(I_X + A^{-1}UV)$ , we see  $A + UV$  is invertible if and only if  $I_X + A^{-1}UV$  is invertible. The formula (7) can be obtained by some simple computations.  $\square$

**Theorem 3.7.** *Let  $T \in B(X)$ ,  $G \in B(Y)$ , and  $U \in B(Y, X)$ ,  $V \in B(X, Y)$  such that  $T^h$  and  $G^h$  exist, also, let  $S = T + UGV \in B(X)$  and  $C = G^h + VT^hU \in H(Y)$  such that  $S^h$  and  $C^h$  exist. Suppose that  $T^h$  and  $S^h$  are quasi-additive on  $\mathcal{R}(T)$  and  $\mathcal{R}(U)$ . If any of the following conditions holds:*

- i)  $\mathcal{N}(T^hT) \subset \mathcal{N}(V)$ ,  $\mathcal{N}(C^hC) \subset \mathcal{N}(U)$ ,  $\mathcal{R}(V) \subset \mathcal{R}(G^h)$ ,
- ii)  $\mathcal{N}(G^h) \subset \mathcal{N}(U)$ ,  $\mathcal{R}(V) \subset \mathcal{R}(C^hC)$ ,  $\mathcal{R}(U) \subset \mathcal{R}(TT^h)$ ,

then  $S^h = T^h - T^hU(G^h + VT^hU)^h VT^h$ .

*Proof.* For convenience, set  $\Gamma = T^h - T^hU(G^h + VT^hU)^h VT^h$ , we will show that  $S^h = \Gamma$ . Here, we only give the proof under the assumption (i). Another can be proved similarly. Note that, if  $\mathcal{N}(T^hT) \subset \mathcal{N}(V)$ ,  $\mathcal{N}(C^hC) \subset \mathcal{N}(U)$ ,  $\mathcal{R}(V) \subset \mathcal{R}(G^h)$  then by Lemma 3.2, we have

$$V(I - T^hT) = 0, \quad U(I - C^hC) = 0, \quad (I - G^hG)V = 0.$$

Consequently, we obtain

$$\begin{aligned} \Gamma S &= (T^h - T^hU(G^h + VT^hU)^h VT^h)(T + UGV) \\ &= T^hT + T^hUGV - T^hU(G^h + VT^hU)^h VT^hT \\ &\quad - T^hU(G^h + VT^hU)^h VT^hUGV \\ &= T^h + T^hU(G^h + VT^hU)^h VT^h \\ &= T^hT + T^hU(I - C^hC)GV - T^hUC^h(I - G^hG)V \\ &= T^hT. \end{aligned}$$

Similarly, we can also check that  $S\Gamma = TT^h$ . Thus, we have

$$\Gamma S \Gamma = T^h T \Gamma = \Gamma, \quad S \Gamma S = T T^h S = S.$$

From Definition 2.3, we have  $\Gamma = S^h$ . This completes the proof.  $\square$

If we let  $G = I_Y$  and assume that  $I_Y + VT^hU$  is invertible in Theorem 3.7, then we can get the following result.

**Corollary 3.8.** *Let  $T \in B(X)$ ,  $U \in B(Y, X)$  and  $V \in B(X, Y)$  such that  $T^h$  exists. Suppose that  $T^h$  is quasi-additive on  $\mathcal{R}(T)$ . If  $I_Y + VT^hU \in H(Y)$  is invertible and  $\mathcal{N}(T) \subset \mathcal{N}(V)$ ,  $\mathcal{R}(U) \subset \mathcal{R}(T)$  then  $(T + UV)^h = T^h - T^h U (I_Y + VT^hU)^{-1} VT^h$  and*

$$\left\| (T + UV)^h - T^h \right\| \leq \frac{\|T^h U\| \|VT^h\|}{1 - \|VT^h U\|}.$$

## 4. Conclusions

In this paper, we develop conditions under which the so-called Sherman-Morrison-Woodbury formula can be represented by the bounded homogeneous generalized inverse. More precisely, we will develop some conditions under which the Sherman-Morrison-Woodbury formula holds for the bounded homogeneous generalized inverse in Banach space. Note that this is the first related results about nonlinear generalized inverse. As a result, our results generalize the results of many authors for finite dimensional matrices and Hilbert space operators in the literature.

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