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Oscillation and Asymptotic Behaviour of Solutions of Nonlinear Two-Dimensional Neutral Delay Difference Systems

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Abstract

This paper deals with the some oscillation criteria for the two-dimensional neutral delay difference system of the form $\Delta(x_n + p_n x_{n-k}) = b_n y_n, \Delta(y_n) = -a_n x_{n-l+1}, n \in \mathbb{N}(n_0) = 1, 2, 3, \cdots$ Examples illustrating the results are inserted.

Keywords

Asymptotic, Two-Dimensional Neutral Delay Difference Systems

1. Introduction

Consider a nonlinear neutral type two-dimensional delay difference system of the form

$$\begin{aligned} &\Delta(x_n + p_n x_{n-k}) = b_n y_n \\ &\Delta(y_n) = -a_n x_{n-l+1}, \quad n \in \mathbb{N}(n_0) = 1, 2, 3, \cdots \end{aligned} \tag{1.1}$$

Subject to the following conditions:

(c_1), $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences such that $\sum_{n=1}^{\infty} b_{n_0} = \infty$.

(c_2), $\{p_n\}$ is a positive real sequence.

(c_3), f,g : $\mathbb{R} \to \mathbb{R}$ are continous non-decreasing with $uf(u) > 0$, $ug(u) > 0$, for $u \neq 0$ and $|f(u)| \geq k|u|$, where k is a constant.

(c_4), k and l are nonnegative integers.

Let $\theta = \max\{k, 1\}$. By a solution of the system (1.1), we mean a real sequence $\{x_n, y_n\}$ which is defined for all $n \geq n_0 - \theta$ and satisfies (1.1) for all $n \in \mathbb{N}(n_0)$.

Let W be the set of all solutions $X=\{x_n, y_n\}$ of the system (1.1) which exists for $n\in\mathbb{N}(n_0)$ and satisfies

$$\sup\{|x_n|+|y_n|; n\geq N\}>0 \quad \text{for any integer } N\geq N_0.$$

A real sequence defined on $\mathbb{N}(n_0)$ is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

A solution $X\in W$ is said to be oscillatory if both components are oscillatory and it will be called nonoscillatory otherwise.

Some oscillation results for difference system (1.1) when $p_n=0$ for $n\in N(N_0)$ and $n-l+1=n$ have been presented in [1], In particular when $b_n>0$ for all $n\in N(n_0)$. The difference system (1.1) reduces to the second order nonlinear neutral difference equation

$$\Delta\left(\frac{1}{b_n}\Delta(x_n+p_nx_{n-k})\right)=-a_nx_{n-l+1}. \tag{1.2}$$

If $b_n=1$, in Equation (1.2), we have a second order linear equation

$$\Delta^2(x_n+p_nx_{n-k})=-a_nx_{n-l+1}. \tag{1.3}$$

For oscillation criteria regarding Equations (1.1)-(1.3), we refer to [2]-[12] and the references cited therein. In Section 2, we present some basic lemmas. In Section 3, we establish oscillation criteria for oscillation of all solutions of the system (1.1). Examples are given in Section 4 to illustrate our theorems.

2. Some Basic Lemmas

Denote $A_n=\sum_{s=n_0}^{n-1}a_s, n\in\mathbb{N}(n_0)$, For any x_n, we define z_n by

$$z_n=x_n+p_nx_{n-k} \tag{2.1}$$

We begin with the following lemma.

2.1. Let $(c_1)-(c_4)$ hold and let $\{(x_n,y_n)\}\in W$ be a solution of system (1.1) with $\{x_n\}$ either eventually positive or eventually negative for $n\in\mathbb{N}(n_0)$. Then $\{(x_n,y_n)\}$ is nonoscillatory and $\{z_n\}$ and $\{y_n\}$ are monotone for $n\in\mathbb{N}(N)$ for $N\in\mathbb{N}(n_0)$.

Proof. Let $\{(x_n,y_n)\}\in W$ and let $\{x_n\}$ be nonoscillatory on $\mathbb{N}(n_0)$. Then from the second equation of system (1.1), we have $\Delta y_n\leq 0$ for all $n\geq N_1\in\mathbb{N}(n_0)$ and Δy_n, and y_n are not identically zero for infinitely many values of *n*. Thus $\{y_n\}$ is monotone for $n\geq N$. Hence $\{y_n\}$ is either eventually positive or eventually negative for $n\geq N_1$. Then, $\{(x_n,y_n)\}$ is nonoscillatory. Further from the first equation of the system (1.1). We have $\Delta z_n>0$ or $\Delta z_n<0$ eventually. Hence $\{z_n\}$ is monotone and nonoscillatory for all $n\geq N\geq N_1$. The proof is similar when $\{x_n\}$ is eventually negative.

Lemma 2.2. In addition to conditions $(c_1)-(c_2)$ assume that $0<p_n\leq 1$ for all $n\in\mathbb{N}(n_0)$. Let $\{x_n\}$ be a nonoscillatory solution of the inequality

$$x_n(x_n+p_nx_{n-k})\geq 0 \tag{2.2}$$

for sufficiently large *n*. If for $n-k$ for all $n\in\mathbb{N}(n_0)$. Then, $\{x_n\}$ is bounded.

Proof. Without loss of generality we may assume that $\{x_n\}$ be an eventually positive solution of the inequality (2.1), the proof for the case $\{x_n\}$ eventually negative is similar. From (2.1) we have

$$(x_n + p_n x_{n-k}) \geq 0, \quad \text{for} \quad n \geq \mathbb{N}(n_0).$$

and $0 < p_n \leq 1$, we have from (2.2), $x_{n-k} \leq p_n x_{n-k} \leq x_n$ for all $n \geq N$. Hence $\{x_n\}$ is bounded.

Next, we state a lemma whose proof can be found in [1].

Lemma 2.3. Assume that $\{a_n\}$ is a non negative real sequence and not identically zero for infinitely many values of n and l is a positive integer. If

$$\liminf_{n\to\infty} \sum_{s=n-l+1}^{n-1} a_s > \left(\frac{l}{l+1}\right)^{l+1}$$

Then the difference inequality

$$\Delta y_n + a_n x_{n-l+1} \leq 0 \quad n \in \mathbb{N}(n_0)$$

cannot have an eventually positive solution and

$$\Delta y_n + a_n x_{n-l+1} \geq 0 \quad n \in \mathbb{N}(n_0)$$

cannot have an eventually negative solution.

3. Oscillation Theorems for the System (1.1)

Theorem 3.1. Assume that $\{p_n\}$ is bounded and there exists an integer j such that $l > j + k + 2$. If

$$\limsup_{n\to\infty} A_n \sum_{s=n-l+1}^{\infty} a_s > \frac{1}{k\beta} \tag{3.1}$$

and

$$\liminf_{n\to\infty} \sum_{s=n-(l-j-k)}^{n-1} k\beta b_s \left(\sum_{t=s}^{s+j} \frac{a_t}{p_{t-l+k+1}} \right) > \left(\frac{l-j-k}{l-j-k+2} \right)^{l-j-k+2} \tag{3.2}$$

Then every solution $\{(x_n, y_n)\} \in W$ is a nonoscillatory solution of system (1.1), with $\{x_n\}$ bounded. Without loss of generality we may assume that $\{x_n\}$ is eventually positive and bounded for all $n \geq n_1 \in N(n_0)$. From the second equation of (1.1), we obtain $\Delta y_n \leq 0$ for sufficiently large $n \geq n_2 \in N(n_1)$. In view of Lemma 2.1, we have two cases for sufficiently large $n_3 \in N(n_2)$:

1) $y_n < 0$ for $n \geq n_3$;

2) $y_n > 0$ for $n \geq n_3$.

Case (1). Because $\{y_n\}$ is negative and nonincreasing there is constant $L > 0$. Such that

$$y_n \leq -L \quad \text{for all } n \geq n_3 \tag{3.3}$$

Since $\{x_n\}$ and $\{p_n\}$ are bounded. $\{z_n\}$ defined by (2.1) is bounded. Summing the first equation of (1.1) from n_3 to $n-1$ and then using (3.3), we obtain

$$z_n - z_{n_0} \leq -L \sum_{s=n_3}^{n-1} a_s, \quad n \geq n_3. \tag{3.4}$$

From (3.3), we see that $\lim_{n\to\infty} z_n = -\infty$ which contradicts the fact that $\{z_n\}$

is bounded. Case (1) cannot occur.

Case (2). Let $z_n > 0$ for $n \ge n_4$ where $n_4 \in N(n_3)$ is sufficiently large. Because $\{z_n\}$ is nondecreasing there is a positive constant *M*, such that

$$z_n \ge M, \quad \text{for all } n \ge n_4. \tag{3.5}$$

From (2.1), we have $z_n > x_n$, and by hypothesis, we obtain

$$a_n z_{n-l+1} \ge a_n \frac{x_{n-l+1}}{k}, \quad n \ge n_5 \in N(n_4) \tag{3.6}$$

summing the second equation of (1.1) from *n* to *i*, using (3.5) and then letting $i \to \infty$, we obtain

$$y_n \ge k \sum_{s=n}^{\infty} a_s z_{s-l+1}, \quad n \ge n_5. \tag{3.7}$$

From condition (3.1), we have

$$\frac{1}{k\beta} < \limsup_{n\to\infty} A_s a_s \tag{3.8}$$

we claim that the condition (3.1) implies

$$\sum_{n=N}^{\infty} A_n a_n = \infty, \quad N \in N(n_0). \tag{3.9}$$

Otherwise, if $\sum_{n=N}^{\infty} A_n a_n < \infty$, we can choose an integer $N_1 \ge N$. So large that $\sum_{n=N_1}^{\infty} A_n a_n < \frac{1}{k\beta}$ which contradicts (3.6).

Using a summation by parts formula, we have

$$\sum_{s=N}^{n-1} A_{s+1} \Delta g(y_s) = A_n y_n - A_N y_N - z_n - z_N. \tag{3.10}$$

From (3.3), (3.4) and (3.6) and the second equation of (1.1), we have

$$\begin{aligned}\sum_{s=N}^{n-1} A_{s+1} \Delta g(y_s) &\le \beta \sum_{s=N}^{n-1} A_{s+1} \Delta y_s \\ &\le -Mk\beta \sum_{s=N}^{n-1} A_{s+1} y_s \\ &\le -Mk\beta \sum_{s=N}^{n-1} A_s y_s\end{aligned}$$

$$Mk\beta \sum_{s=N}^{n-1} A_s y_s = -A_n y_n + A_N y_N + z_n - z_N, n \ge N.$$

combining (3.6) with (3.8), we obtain

$$\lim_{n\to\infty} (z_n - A_n y_n) = \infty.$$

and

$$z_n \ge A_n g(y_n) \ge \beta A_n y_n, \quad n \ge n_6 \in \mathbb{N}(n_5).$$

The last inequality together with (3.4) and the monotonocity of $\{z_n\}$ implies

$$\begin{aligned}z_n &\ge k\beta A_n \sum_{s=n}^{\infty} a_s z_{s-l+1} \ge k\beta A_n \sum_{s=n-l+1}^{\infty} a_s z_{s-l+1} \\ &\ge k\beta A_n z_n \sum_{s=n+l-1}^{\infty} a_s\end{aligned}$$

and $1 \ge k\beta A_n z_n \sum_{s=n+l-1}^{\infty} a_s$, $n \in \mathbb{N}(n_6)$ which contradicts (1.1). This case cannot occur. The proof is complete.

Theorem 3.2. Assume that $0 < p_n \le 1$, then there exists an integer *j* such that $l > j + k$ and the conditions (3.1) and (3.2) are satisfied. Then all solutions of (1.1) are oscillatory.

Proof . Let $\{(x_n, y_n)\} \in W$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $\{x_n\}$ is positive for n $n \in \mathbb{N}(n_1)$. As in the proof of above theorem we have two cases.

Case (1). Analogus to the proof of case (1) of above theorem, we can show that $\lim_{n\to\infty} z_n = -\infty$. By Lemma 2.2, $\{x_n\}$ is bounded and hence $\{z_n\}$ is bounded which is a contradiction. Hence case (1) cannot occur.

Case (2). The proof of case (2) is similar to that of the above theorem and hence the details are omitted. The proof is now complete.

Theorem 3.3. Assume that $0 < p_n \le 1$ and

$$\limsup_{n\to\infty} \sum_{s=n-k-l+1}^{n-1} \frac{k\beta(A_n - A_{s+1})}{p_{s-l-k+1}} a_s > 1. \tag{3.14}$$

$$\sum_{n=N}^{\infty} b_n \left(\sum_{s=n}^{\infty} a_s \right) = \infty, \quad N \in N(n_0) \tag{3.15}$$

$$\limsup_{n\to\infty} \left(k\beta A_n \sum_{s=n}^{\infty} a_s \right) > 1. \tag{3.16}$$

Then all solutions of (1.1) are oscillatory.

Proof. Let $\{(x_n, y_n)\} \in W$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $\{x_n\}$ is positive for $n \in \mathbb{N}(n_1)$. As in the proof of above theorem we have two cases.

Case 1. From (2.1), we have

$$z_n > p_n x_{n-k}, \quad n \ge n_3 \in N(n_0)$$

and

$$f(x_{n-l+1}) \ge k x_{n-l+1} > k \frac{z_{n-l-k+1}}{p_{n-l-k+1}}, \quad n \ge n_4 \tag{3.17}$$

where $n_4 \in N(n_3)$ is sufficiently large. Then the following equality

$$z_n = z_i + (A_n - A_i) y_i + \sum_{s=i}^{n-1} (A_n - A_{s+1}) \Delta y_s$$

$$z_n < \sum_{s=i}^{n-1} (A_n - A_{s+1}) \Delta y_s, \quad n > i \ge n_5.$$

Combining the last inequality with the second equations of (1.1) and (3.17), we have

$$z_n < \beta \sum_{s=i}^{n-1} (A_n - A_{s+1})(-a_s f(x_{s-l}))$$

$$< k\beta \frac{\sum_{s=i}^{n-1} (A_n - A_{s+1}) a_s z_{s-l-k+1}}{p_{s-l-k+1}}, n > i \ge n_5.$$

Let $i = n - l + k + 1$ and using the monotonocity of $\{z_n\}$, from the last inequality, we obtain

$$z_n < z_n \sum_{s=n-l-k+1}^{n-1} k\beta \frac{(A_n - A_{s+1}) a_s}{p_{s+l+k-1}}$$

and

$$1 > z_n \sum_{s=n-l-k+1}^{n-1} k\beta \frac{(A_n - A_{s+1}) a_s}{p_{s+l+k-1}}$$

which contradicts the condition (3.14).

Case 2. The proof for this case is similar to that of Theorem (3.1). Here we use condition (3.16) instead of condition (2.1). The proof is complete.

4. Examples

Example 4.1. Consider the difference system

$$\begin{aligned} &\Delta\left(x_n + \frac{1}{2}x_{n-3}\right) = \frac{1}{n} y_n \\ &\Delta y_n = -n x_{n-2},\ n \ge 1. \end{aligned} \tag{4.1}$$

The conditions (3.1) and (3.2) are

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{s=n+2}^{\infty} s = \infty.$$

$$\liminf_{n\to\infty} \sum_{s=n-4}^{n-3} \frac{1}{s}\left(\sum_{t=s-1}^{s} 2t\right) = 4.$$

All conditions of Theorem 3.2 are satisfied and so all solutions of the system (4.1) are oscillatory.

Example 4.2. Consider the difference systems

$$\begin{aligned} &\Delta\left(x_n + \frac{1}{4}x_{n-2}\right) = (n+1) y_n \\ &\Delta y_n = \frac{-c}{n+1} x_{n-1}, n \ge 1, \end{aligned} \tag{4.2}$$

where c is a positive constant. The conditions (3.1) and (3.2) are

$$\limsup_{n\to\infty} (n+1) \sum_{s=n+1}^{\infty} \frac{c}{s+1} = \infty$$

and

$$\liminf_{n\to\infty} \sum_{s=n-3}^{n-2} (s+1)\left(\sum_{t=s}^{s+1} \frac{-4c}{t+1}\right) = 12c.$$

For $c > \frac{1}{12}$, all conditions of Theorem 3.2 are satisfied and so all solutions of the system (4.2) are oscillatory.

References

[1] Agarwal, R.P. (2000) Difference Equations and Inequalities. 2nd Edition, Marcel Dekkar, New York.

[2] Elizebeth, S., Graef, J.R., Sundaram, P. and Thandapani, E. (2005) Classifying Non-oscillatory Solutions and Oscillations of Neutral Difference Equations. *Journal of Difference Equations and Applications*, **11**, 605-618.

https://doi.org/10.1080/10236190412331334491

[3] Sternal, A. and Szamanda, B. (1996) Asymptotic and Oscillatory Behaviour of Certain Difference Equations. *Le Matematiche*, **51**, 77-86.

[4] Sternal, A., Szamanda, B. and Szafranski, Z. (1998) Oscillatory and Asymptotic Behaviour of Some Difference Equations. *Publications. DE L'INSTITUT MATHEMATQUE*, **63**, 66-74.

[5] Thandapani, E. (1992) Asymptotic and Oscillatory Behaviour of Solutions of a Second Order Neutral Difference Equations. *Riv. Mat. Univ. Parma*, **1**, 105-113.

[6] Thandapani, E. and Mahalingam, K. (2003) Necessary and Sufficient Condition for Oscillation of Second Order Neutral Difference Equations. *Tamkang Journal of Mathematics*, **34**, 137-145.

[7] Thandapani, E. and Mohankumar, P. (2007) Oscillation of Difference Systems of the Neutral type. *Computers and Mathematics with Applications*, **54**, 556-566. https://doi.org/10.1016/j.camwa.2006.12.029

[8] Thandapani, E., Sundaram, P., Graef, J.R. and Spikes, P.W. (1995) Asymptotic Properties of Solutions of Nonlinear Second Order Neutral Delay Difference Equations. *Dynam. Systems Appl.*, 4, 125-136.

[9] Thandapani, E. and Sundaram, P. (2000) Oscillation and Nonoscillation Theorems for Second Order Quasilinear Functional Difference Equations. *Indian J. Pure Appl. Math*, **31**, 37-47.

[10] Zhang, B.G. and Saker, S.H. (2003) Kamenev-Type Oscillation Criteria for Nonlinear Neutral Delay Difference Equations. *Indian J. Pure Appl. Math.*, **34**, 1571-1584.

[11] Zhang, G. (2002) Oscillation for Nonlinear Neutral Difference Equations. *Applied Math. E-Notes*, **2**, 22-24.

[12] Zhang, Z., Chen, J. and Zhang, C. (2001) Oscillation of Solutions of Second Order Nonlinear Difference Equations with Nonlinear Neutral Term. *Computers and Mathematics with Applications*, **41**, 1571-1584. https://doi.org/10.1016/s0898-1221(01)00113-4