

# Playing Rock, Paper, Scissors in Non-Transitive Statistical Thermodynamics

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## Abstract

Does non-transitivity in information theory have an analog in thermodynamics? A non-transitive game, “Swap”, is used as a toy thermodynamic model to explore concepts such as temperature, heat flow, equilibrium and entropy. These concepts, found to be inadequate for non-transitive thermodynamic, need to be generalized. Two kinds of temperatures, statistical and kinetic, are distinguished. Statistical temperature is a parameter in statistical distributions. Kinetic temperature is proportional to the expected kinetic energy based on its distribution. Identical for Maxwell-Boltzmann statistics, these temperatures differ in non-Maxwellian statistics when a force is present. Fourier’s law of conduction and entropy should be expressed using statistical temperature, not kinetic temperature. Kinetic temperature is always scalar but statistical temperature and statistical entropy in non-transitive systems have circulation, thereby allowing continuous and circular heat flow. Entropy is relative to underlying statistics, in analogy to the Kullback-Leibler divergence in information theory. The H-theorem, limited by assumptions of homogeneity and indistinguishability, only covers statistically homogeneous systems. The theorem does not cover non-transitive, statistically heterogeneous systems combining different distributions such as Maxwell-Boltzmann, biased half-Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein. The second law can be preserved if generalized by expressing it in terms of statistical temperature and statistical entropy.

## Keywords

Statistical Thermodynamics, Non-Transitivity, Second Law, Detailed Balance, Kinetic Temperature, Statistical Temperature, Non-Transitive Game, Non-Transitive Thermodynamics, Entropy, Arrow of Time

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*“All reality is a game. Physics at its most fundamental, the very fabric of our universe, results directly from the interaction of certain fairly simple rules, and chance.”*

Iain M. Banks, *The Player of Games* [1]

## 1. Introduction

Thermodynamics and information theory provide different perspectives for the same physical phenomena such as heat, temperature and entropy. Yet non-transitive games as described in information theory, have so far, no known correspondence in thermodynamics. The purpose of this paper is to show that such counterparts do exist and to explore the consequences.

Non-transitive games, for example, rock, paper, scissors are well known. In such a game, items being compared, for example A, B, C, must be capable of being ordered in a ring.

$$A > B > C > A \quad (1)$$

Non-transitivity can be produced in several ways. If the items are mathematical, the modulo operator can be used. If the items are physical, then a magnetic field or a spinning wheel can be employed to define a clockwise or counter-clockwise direction.

Objects with different statistics can also display non-transitivity. Several games [2] [3] [4] use this effect. For example, three non-transitive dice [4] A, B, C can be designed by assigning different values to their faces such that:

$$\begin{aligned} \text{Dice A has faces : } & 2, 2, 4, 4, 9, 9 \\ \text{Dice B has faces : } & 1, 1, 6, 6, 8, 8 \\ \text{Dice C has faces : } & 3, 3, 5, 5, 7, 7 \end{aligned} \quad (2)$$

The outcome of a game played by three players holding dices A, B and C is summarized by the **Tables 1-3** in which the letter entry is the winning player.

**Table 1.** Player A with a dice producing random numbers 2, 4 and 9 plays against player C with a dice producing random numbers 3, 5, 7. Player C wins 5 out of 9 games.

	A	2	4	9
C				
3		C	A	A
5		C	C	A
7		C	C	A

**Table 2.** Player B with a dice producing random numbers 1, 6, 8 plays against Player A with a dice producing random numbers 2, 4 and 9. Player A wins 5 out of 9 games.

	B	1	6	8
A				
2		A	B	B
4		A	B	B
9		A	A	A

**Table 3.** Player C with a dice producing random numbers 3, 5, 7 plays against Player B with a dice producing random numbers 1, 6 and 8. Player C wins 5 out of 9 games.

	C	3	5	7
B				
1		C	C	C
6		B	B	C
8		B	B	B

The **Tables 1-3** indicate that C beats A 5/9 of the time. Similarly, A beats B 5/9th of the time and B beats C 5/9th of the time.

This paper will discuss non-transitive thermodynamics as follows:

- 1) A transitive game “Swap,” involving the exchange of tokens between two players will be presented. Thermodynamic concepts associated with the game, such as temperature, heat flow, equilibrium and entropy will be developed.
- 2) The game’s statistics will be skewed to favor one player over the other, thereby requiring the generalization of the thermodynamics concepts.
- 3) A non-transitive game between three or more players will then be developed, showing that, tokens, which are the analog of heat, can flow in circles in clear violation of the zeroth law.
- 4) The recipe used to convert the game from transitive to non-transitive shall be applied to a physical system. It will be shown that the skewing of statistics required for non-transitivity can be achieved by heterogeneous system that combine different statistics such as Maxwell-Boltzmann, biased half Maxwellian, Fermi-Dirac and Bose Einstein.

## 2. Transitive Game

This game may be called “Swap.” Two players, A and B, each holds a bag containing a mixture of black and white tokens, a white token having value  $k$ , a black token having zero value. Player A begins with a total of  $N_A$  tokens of which  $n_A$  are white, the rest being black, and player B begins with a total of  $N_B$  tokens of which  $n_B$  are white, the rest being black.

The players randomly select a token from their bags, swap them and put them back in their bags without necessarily looking at them. Player A “wins” if he picks up a black token, B picks up a white token and they swap. After the swap, A comes out ahead because he has acquired a white token. Player B wins if the reverse occurs. Otherwise the game is a tie. The total number of tokens transferred from one player to another is  $Q$ , which is of course the analog of heat.

The game is a model of diffusion and obviously extremely boring-only a die-hard thermodynamicist could really enjoy it. However, it provides a basis for a more complex non-transitive game to be discussed in the next section.

The probability for player A to pick a white token from his bag is

$$\frac{n_A}{N_A} \quad (3)$$

and the probability for B to pick a white token is

$$\frac{n_B}{N_B} \quad (4)$$

Because the players *swap* tokens, the conditional probability  $p_{A/B}$  for A to win against B is the product of the probabilities of A picking a *black* token and of B picking a *white* token:

$$p_{A/B} = \frac{N_A - n_A}{N_A} \cdot \frac{n_B}{N_B} \quad (5)$$

Similarly, the conditional probability  $p_{B/A}$  for B to win against A is

$$p_{B/A} = \frac{N_B - n_B}{N_B} \cdot \frac{n_A}{N_A} \quad (6)$$

This game provides a vehicle for discussing some thermodynamic concepts such as:

- 1) Fourier's law of heat conduction.
- 2) Thermal equilibrium.
- 3) Entropy.

## 2.1. Fourier Law of Heat Conduction

Since each white token has value  $k$ , the expected token value transfer  $\Delta Q_{A \rightarrow B}$  from A to B after each round is the difference between an expected loss  $kp_{B/A}$  and an expected win  $kp_{A/B}$  by A. After one round, the net win for A is:

$$\Delta Q_{A \rightarrow B} = k(p_{B/A} - p_{A/B}) \quad (7)$$

Combining Equations (5) and (6) with (7) yields:

$$\Delta Q_{A \rightarrow B} = k \left( \frac{N_B - n_B}{N_B} \cdot \frac{n_A}{N_A} - \frac{N_A - n_A}{N_A} \cdot \frac{n_B}{N_B} \right) \quad (8)$$

Simplifying:

$$\Delta Q_{A \rightarrow B} = -k \left( \frac{n_B}{N_B} - \frac{n_A}{N_A} \right) \quad (9)$$

To further the analogy with thermodynamics one could define temperature  $T$  as the "density" of white token in the bags of the players:

$$T_A = \frac{n_A}{N_A} \quad \text{and} \quad T_B = \frac{n_B}{N_B} \quad (10)$$

Therefore, the expected currency transfer from A to B per round can be expressed as

$$\Delta Q_{A \rightarrow B} = -k(T_B - T_A) \quad (11)$$

which is, of course, a discrete version of Fourier's equation for heat conduction where  $k$  is the conductivity.

## 2.2. Equilibrium

Equations (8) and (11) indicate that if player A is at a higher temperature  $T_A >$

$T_B$  than B, then A is likely to lose white tokens. Tokens flow from A to B until both players are at the same temperature. They are then in equilibrium which corresponds to:

$$T_A = T_B \quad (12)$$

### 2.3. Entropy

The tokens  $\Delta Q_{A \rightarrow B}$  moving from A to B cause a change in entropy in A:

$$\Delta S_{A/B} = -\frac{\Delta Q_{A \rightarrow B}}{T_A} \quad (13)$$

(The number of token  $N_A$  is assumed to be large so that  $T_A$  does not change appreciably after a single token swap.) Combining Equations (13) with (11) yields the change in entropy for A:

$$\Delta S_{A/B} = k \frac{T_B - T_A}{T_A}. \quad (14)$$

For both players, the combined change in entropy is:

$$\Delta S_{A/B+B/A} = \Delta S_{A/B} + \Delta S_{B/A} = k \frac{T_B - T_A}{T_A} + k \frac{T_A - T_B}{T_B}. \quad (15)$$

which simplifies to:

$$\Delta S_{A/B+B/A} = k \frac{(T_A - T_B)^2}{T_A T_B} \quad (16)$$

Hence

$$\Delta S_{A/B+B/A} \geq 0 \quad (17)$$

the above derivation being a simple H-theorem for the game being analyzed.

This game provides thermodynamic analogs of heat flow, equilibrium and entropy. Tokens flows from a rich player to a poor player. Equilibrium is reached when both players at the same temperature. Entropy never decreases.

## 3. Two-Player Game with Skewed Statistics

Before undertaking the analysis of the non-transitive game, a simpler two-player game shall be discussed, in which statistics are skewed to give a player an advantage over the other. This section shows that concepts such as temperature, equilibrium and entropy break down when games (or systems) involve skewed or different statistics.

Skewing statistics in one player's favor can be implemented in several ways. It could be in the form of a physical effect such as giving the white tokens and the black tokens different weights to enable their separation as the players shake their bags. The players would then pick from the lightest tokens at the top. Another, approach is to assign numbers to the tokens as was done with the non-transitive dice discussed in the introduction. Yet a third approach would be to display rock, paper, scissors on tokens. A more direct method discussed in [Appendix A](#) is to embed the bias directly in the rules of the game.

In any case, the actual method of biasing the statistics, whether through the rules of the game or by a physical effect is unimportant. The following discussion shall assume that the statistics are a-priori skewed.

Consider the “swap” game discussed above in which the statistics are skewed by parameter  $\alpha$ . The conditional probability for A of picking a white token when playing against B is:

$$\frac{\alpha_{A/B}n_A}{N_A} \quad (18)$$

and the conditional probability for B of picking a white token when playing against A is:

$$\frac{\alpha_{B/A}n_B}{N_B} \quad (19)$$

where  $\alpha_{A/B}$  and  $\alpha_{B/A}$  are statistics-skewing parameters defined as

$$\alpha_{A/B} = 1 - \delta \quad (20)$$

and

$$\alpha_{B/A} = 1 + \delta \quad (21)$$

(When  $\delta = 0$ , Equations (18) and (19) revert to (3) and (4), and the game returns to being transitive.)

Player A wins by swapping a black token with a white token from B. Therefore, the conditional probability  $p_{A/B}$  for A to win against B is the product of the probability of A picking a black token and B picking a white token:

$$p_{A/B} = \frac{N_A - \alpha_{A/B}n_A}{N_A} \cdot \frac{\alpha_{B/A}n_B}{N_B} \quad (22)$$

Similarly, the conditional probability  $p_{B/A}$  for B to win against A is

$$p_{B/A} = \frac{N_B - \alpha_{B/A}n_B}{N_B} \cdot \frac{\alpha_{A/B}n_A}{N_A} \quad (23)$$

### 3.1. Breakdown of Conventional Thermodynamics

The concepts of temperature, heat flow and entropy successfully used in the transitive game example are inadequate when applied to system with skewed or multiple statistics as shall be shown.

The expected transfer in currency  $\Delta Q_{A \rightarrow B}$  from A to B is the difference between the expected value loss  $kp_{B/A}$  by A and the expected value win  $kp_{A/B}$  by A:

$$\Delta Q_{A \rightarrow B} = k(p_{B/A} - p_{A/B}) \quad (24)$$

Combining Equations (22) and (23) with (24) yields

$$\Delta Q_{A \rightarrow B} = k \left( \frac{N_B - \alpha_{B/A}n_B}{N_B} \cdot \frac{\alpha_{A/B}n_A}{N_A} - \frac{N_A - \alpha_{A/B}n_A}{N_A} \cdot \frac{\alpha_{B/A}n_B}{N_B} \right) \quad (25)$$

Simplifying yields

$$\Delta Q_{A \rightarrow B} = -k \left( \alpha_{B/A} \frac{n_B}{N_B} - \alpha_{A/B} \frac{n_A}{N_A} \right) \quad (26)$$

Defining temperature as per Equation (10), yields

$$\Delta Q_{A \rightarrow B} = -k(\alpha_{B/A}T_B - \alpha_{A/B}T_A) \quad (27)$$

leading to an impasse. Equation (27) describes token flow but does not have the same form as Fourier's law of conduction as expressed in Equation (11) because the temperatures  $T_A$  and  $T_B$  are scaled by the statistics-biasing parameters  $\alpha_{A/B}$  and  $\alpha_{B/A}$ .

Furthermore the equilibrium state  $\Delta Q_{A \rightarrow B} = 0$ , requires  $\alpha_{B/A}T_B = \alpha_{A/B}T_A$  from Equation (27), not  $T_A = T_B$ , and therefore violates Clausius' formulation of the second law as currently understood.

In addition, the change in entropy caused by a transfer of tokens  $\Delta Q_{A \rightarrow B}$  (as calculated with the method used in Section 1 on transitive games) does not make sense as it leads to *decreasing entropy*. Combining Equations (27) and (13), one gets for player A playing against B:

$$\Delta S_{A/B} = k \frac{\alpha_{B/A}T_B - \alpha_{A/B}T_A}{T_A} \quad (28)$$

The total change in entropy for both players is:

$$\Delta S_{A/B+B/A} = \Delta S_{A/B} + \Delta S_{B/A} = k \frac{\alpha_{B/A}T_B - \alpha_{A/B}T_A}{T_A} + k \frac{\alpha_{A/B}T_A - \alpha_{B/A}T_B}{T_B} \quad (29)$$

Combining the above with Equations (20) and (21), one gets

$$\Delta S_{A/B+B/A} = k \left( \frac{(1-\delta)T_A^2 + (1+\delta)T_B^2 - 2T_A T_B}{T_A T_B} \right) \quad (30)$$

Simplifying yields:

$$\Delta S_{A/B+B/A} = k \left( \frac{(T_A - T_B)^2 - \delta(T_A^2 - T_B^2)}{T_A T_B} \right) \quad (31)$$

This expression is negative (note that  $T_A > T_B$  because tokens flows from A to B as per Equation (27)) for the statistical skewing parameter  $\delta$  given by

$$\Delta S_{A/B+B/A} < 0 \quad \text{if } \delta > \frac{(T_A - T_B)^2}{T_A^2 - T_B^2} \quad (32)$$

which renders this formulation of entropy unusable.

The impasse is caused by the different or skewed statistics employed by the players against each other. This problem can be resolved by including statistical terms in the formulation of temperature and entropy, in effect making these quantities relative to statistical distributions. This idea is not entirely new being akin to concepts in information theory such as Shannon's entropy, mutual information, conditional information and the Kullback-Leibler divergence [5].

The proposed temperature shall be called "statistical temperature" and is represented by the symbol  $\theta$ . The symbol  $T$  shall be called "kinetic temperature" (for reasons explained in section 4 "Non-Transitive Physical Systems."). In this section,  $T$  defined by Equation (10) is proportional to the density of white tokens

held by a player. Accordingly, the statistical temperature  $\theta_{A/B}$  of player A playing against (relative to) player B is *defined* as:

$$\theta_{A/B} = \alpha_{A/B} T_A = \alpha_{A/B} \frac{n_A}{N_A} \quad (33)$$

$\theta$  is a relative quantity that depends on the statistical skewing factor  $\alpha$ . It provides a measure of the temperature of one player who uses a given statistical distribution in a game against another player who uses a different statistical distribution. Similarly, the statistical temperature  $\theta_{B/A}$  of player B relative to player A is

$$\theta_{B/A} = \alpha_{B/A} T_B \quad (34)$$

Correspondingly, a formulation of entropy shall be defined, called statistical entropy (related to Shannon's entropy as shall be discussed later) expressed in terms of statistical temperature. For the example above, the change in statistical entropy caused by a transfer of tokens  $\Delta Q_{A \rightarrow B}$  from player A to player B is given by

$$\Delta S_{A/B} = -\frac{\Delta Q_{A \rightarrow B}}{\theta_{A/B}}. \quad (35)$$

Statistical entropy is related to the Kullback-Liebler divergence, a relative quantity that depends on the statistical distributions used by the players as explained in **Appendix B**.

### 3.2. Extended Fourier's Law of Heat Conduction

The token flow from A to B can properly be expressed in the format of the Fourier heat conduction equation by combining (27) with (33) and (34):

$$\Delta Q_{A \rightarrow B} = -k(\theta_{B/A} - \theta_{A/B}) \quad (36)$$

This result agrees with Tolman ([6], pages 551, 552) who shows that heat flow depends on *statistical temperature*, not *kinetic temperature*. These terms will be further clarified in Section 5 on non-transitive physical systems.

### 3.3. Equilibrium

The equilibrium state  $\Delta Q_{A \rightarrow B} = 0$  occurs as per Equation (36) when

$$\theta_{B/A} = \theta_{A/B} \quad (37)$$

which implies from Equations (33) and (34) that

$$\alpha_{B/A} T_B = \alpha_{A/B} T_A \quad (38)$$

Since the statistics are skewed,  $\alpha_{A/B} < \alpha_{B/A}$ , surprisingly one finds that at equilibrium:

$$T_B < T_A \quad (39)$$

even though  $\theta_{B/A} = \theta_{A/B}$ .

### 3.4. Entropy

Statistical temperature can be used to calculate the change entropy  $S_{A/B}$  for a



transfer of tokens  $\Delta Q_{A \rightarrow B}$  from A to B. Combining Equations (33), (34), (35) and (36), yields for player A:

$$\Delta S_{A/B} = -\frac{\Delta Q_{A \rightarrow B}}{\theta_{A/B}} = k \frac{\alpha_{B/A} T_B - \alpha_{A/B} T_A}{\alpha_{A/B} T_A} \tag{40}$$

For both players (A against B, and B against A) the total change in entropy is:

$$\begin{aligned} \Delta S_{A/B+B/A} &= \Delta S_{A/B} + \Delta S_{B/A} \\ &= k \frac{\alpha_{B/A} T_B - \alpha_{A/B} T_A}{\alpha_{A/B} T_A} + k \frac{\alpha_{A/B} T_A - \alpha_{B/A} T_B}{\alpha_{B/A} T_B} \end{aligned} \tag{41}$$

Simplifying produces

$$\Delta S_{A/B+B/A} = k \frac{(\alpha_{A/B} T_A - \alpha_{B/A} T_B)^2}{\alpha_{A/B} \alpha_{B/A} T_A T_B} \tag{42}$$

Since all terms are positive, the above result implies

$$\Delta S_{A/B+B/A} \geq 0 \tag{43}$$

One concludes that statistical temperature and statistical entropy are appropriate tools for describing the thermodynamics of a two-player game involving different or skewed statistics

### 4. Non-Transitive Game

This thought experiment illustrates non-transitivity between three or more players. In this example, four players are placed at the four corners of a square as shown in **Figure 1**.

Player A, located at the southwest corner, plays against B at the northwest corner. B plays against C at the northeast corner, C against D at the southeast corner and D against A.

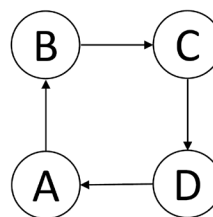
The geometry of a game involving four rather than three players facilitates the analogy with the physical systems described in Section 4, in which fields are used to skew statistics.

The physical implementation method of skewing the statistics is unimportant. Statistics shall be assumed to be modified by parameters  $\alpha$ 's as described in Section 2 "Two-Player Game with Skewed Statistics."

Accordingly, the conditional probability for A to win against B (A picking a black token, B picking a white token and swapping) is as per Equation (22):

$$P_{A/B} = \frac{N_A - \alpha_{A/B} n_A}{N_A} \cdot \frac{n_B}{N_B} \tag{44}$$

Similarly, the other interactions are:



**Figure 1.** Four players arranged in square and playing a non-transitive game.

$$\begin{aligned}
p_{C/B} &= \frac{N_C - \alpha_{C/B}n_C}{N_C} \cdot \frac{n_B}{N_B} \\
p_{C/D} &= \frac{N_C - \alpha_{C/D}n_C}{N_C} \cdot \frac{n_D}{N_D} \\
p_{D/C} &= \frac{N_D - \alpha_{D/C}n_D}{N_D} \cdot \frac{n_C}{N_C} \\
p_{D/A} &= \frac{N_D - \alpha_{D/A}n_D}{N_D} \cdot \frac{n_A}{N_A} \\
p_{A/D} &= \frac{N_A - \alpha_{A/D}n_A}{N_A} \cdot \frac{n_D}{N_D}
\end{aligned} \tag{45}$$

The game is played in two phases. The initial phase utilizes transitive rules in which all interactions are unbiased and fair. In other words, the statistics skewing parameters  $\alpha_s$  are all set to 1 making all interactions transitive. Eventually equilibrium is reached at which point the token flow is zero, the players are at the same kinetic temperature as per Equation (12),

$$T_A = T_B = T_C = T_D \tag{46}$$

and all statistical temperatures are equal as per Equation (37) with  $\alpha = 1$ :

$$\theta_{A/B} = \theta_{B/A} = \theta_{B/C} = \theta_{C/B} = \theta_{C/D} = \theta_{D/C} = \theta_{D/A} = \theta_{A/D} \tag{47}$$

The second phase begins after equilibrium is reached. The rules of the game are suddenly changed, making the game non-transitive. For the sake of simplicity, only the interaction  $A \rightarrow B$  becomes biased, the other interactions remaining unbiased. In other words, the new rules are of the form  $A \rightarrow B \leftrightarrow C \leftrightarrow D \leftrightarrow A$ . Therefore, all statistic skewing parameters  $\alpha$ 's remain equal to 1 except  $\alpha_{B/A} < \alpha_{A/B}$  corresponding to the interaction between A and B. As shown below, biasing a single interaction causes the whole game to become non-transitive.

#### 4.1. Token Flow

Immediately after the rule change and before tokens have time to flow, the kinetic temperature of the players is still  $T_A = T_B = T_C = T_D$  because no token has been exchanged.

The statistical temperatures for the *transitive interactions*  $B \leftrightarrow C \leftrightarrow D \leftrightarrow A$  are not affected by the change of rules. Therefore  $\theta_{B/C} = \theta_{C/B} = \theta_{C/D} = \theta_{D/C} = \theta_{D/A} = \theta_{A/D}$  because the corresponding  $\alpha$ 's remain = 1.

However, the onset of the non-transitive  $A \rightarrow B$  interaction requires  $\alpha_{B/A} < \alpha_{A/B}$  and therefore the statistical temperatures  $\theta_{A/B} = \alpha_{A/B}T_A$  and  $\theta_{B/A} = \alpha_{B/A}T_B$  for the  $A \rightarrow B$  interaction are changed. Since no token has yet flowed,  $T_A = T_B$ . Therefore:

$$\frac{\theta_{A/B}}{\alpha_{A/B}} = \frac{\theta_{B/A}}{\alpha_{B/A}} \tag{48}$$

and because  $\alpha_{B/A} < \alpha_{A/B}$

$$\theta_{A/B} > \theta_{B/A} \tag{49}$$

This statistical temperature difference causes white tokens to flow *down the statistical temperature difference*, from A to B according to the extended Fourier

law of conduction expressed in Equation (36). As the white tokens leave A and accumulate in B the kinetic temperature of A and B changes such that

$$T_A < T_B \quad (50)$$

implying that *white tokens flow up the kinetic temperature difference* from A to B.

The accumulation of white tokens in B causes the statistical temperature of B to increase, thereby affecting the transitive B-C interaction, and prompting tokens to flow from B to C according to Equation (11). In addition, the depletion of white tokens in A affects the A-D interaction causing a token flow from D to A.

## 4.2. Dynamic Equilibrium

Eventually the white tokens also flow from C to D and around the A, B, C, D players' loop, in a continuous fashion until a stable dynamic equilibrium is reached. This is expected since the game is non-transitive.

The flow of tokens in each interaction is proportional to the statistical temperature difference for the interaction as per Equation (36). Therefore:

$$\Delta\theta_{BA} = \theta_{B/A} - \theta_{A/B} > 0 \quad (51)$$

$$\Delta\theta_{CB} = \theta_{C/B} - \theta_{B/C} > 0 \quad (52)$$

$$\Delta\theta_{DC} = \theta_{D/C} - \theta_{C/D} > 0 \quad (53)$$

$$\Delta\theta_{DA} = \theta_{A/D} - \theta_{D/A} > 0 \quad (54)$$

Taking a loop summation of the statistical temperature differences

$$\Delta\theta_{BA} + \Delta\theta_{CB} + \Delta\theta_{DC} + \Delta\theta_{AD} > 0 \quad (55)$$

results in a non-zero value, indicating that statistical temperature has circulation, a property related to curl but more appropriate for discrete systems. Accordingly, statistical temperature is not a scalar quantity.

The kinetic temperature difference between A and B as expressed in Equation (10) is

$$\Delta T_{BA} = T_B - T_A = \frac{n_B}{N_B} - \frac{n_A}{N_A} \quad (56)$$

In contrast to statistical temperature, a loop summation of kinetic temperature differences

$$\begin{aligned} & \Delta T_{BA} + \Delta T_{CB} + \Delta T_{DC} + \Delta T_{AD} \\ &= \left( \frac{n_B}{N_B} - \frac{n_A}{N_A} \right) + \left( \frac{n_C}{N_C} - \frac{n_B}{N_B} \right) + \left( \frac{n_D}{N_D} - \frac{n_C}{N_C} \right) + \left( \frac{n_A}{N_A} - \frac{n_D}{N_D} \right) = 0 \end{aligned} \quad (57)$$

is zero confirming that kinetic temperature is scalar because tokens are conserved.

## 4.3. Entropy

The change in statistical entropy is described for the statistically biased interaction by Equation (42), and for the unbiased interactions by Equation (16):

$$\Delta S_{A/B+B/A} = k \frac{(\alpha_{A/B}T_A - \alpha_{B/A}T_B)^2}{\alpha_{A/B}\alpha_{B/A}T_A T_B} \quad (58)$$

$$\Delta S_{B/C+C/B} = k \frac{(T_B - T_C)^2}{T_B T_C} \quad (59)$$

$$\Delta S_{C/D+D/C} = k \frac{(T_C - T_D)^2}{T_C T_D} \quad (60)$$

$$\Delta S_{D/A+A/D} = k \frac{(T_D - T_A)^2}{T_D T_A} \quad (61)$$

Therefore:

$$\Delta S_{Total} = \Delta S_{A/B+B/A} + \Delta S_{B/C+C/B} + \Delta S_{C/D+D/C} + \Delta S_{D/A+A/D} > 0 \quad (62)$$

Entropy and temperature must be expressed relatively to the different statistics used by each player. These statistics make the game non-transitive, resulting in the continuous flow of tokens around the loop—as expected for a non-transitive game. Statistical temperature and statistical entropy have circulation, topics to be elaborated in the next section.

## 5. Non-Transitive Physical Systems

Can one achieve non-transitivity in thermodynamics as in the Swap game? In the game, the statistics were skewed to favor one player against another, resulting in  $\theta_{B/A} = \theta_{A/B}$  but  $T_B < T_A$  as indicated in Equations (37) and (39) respectively.

In thermodynamics, the corresponding first step would be to produce a spontaneous steady state temperature gradient between two points  $A$  and  $B$ . However, before discussing how to achieve such a gradient, one needs to be clear about what is meant by temperature and how temperature relates to statistics.

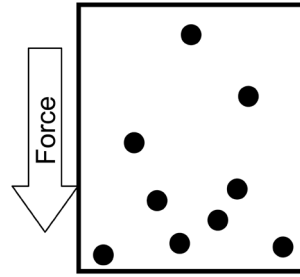
The term “thermodynamic temperature” shall be avoided as it carries too much baggage and has too many meanings to be helpful for the purpose of this paper. As in the game, “kinetic temperature” and “statistical temperature” shall be used. In the game,  $T$  is called “kinetic temperature” and refers to the density of white tokens held by a player. In analogy,  $T$  shall describe the energy packets density *i.e.*, the *kinetic energy* per degree of freedom, per particle. Pursuing the analogy,  $\theta_T$  represents the parameter that defines the kinetic energy distribution and is called “statistical temperature”. Therefore,  $T$  and  $\theta_T$  are not identical. The distinction between these two temperatures is not new [7] [8].

Consider a gas column in thermal equilibrium and subjected to a force as shown in **Figure 2**.

The kinetic temperature  $T(E_p, \theta_T)$  of the gas is proportional to the first moment of the kinetic energy distribution and is given by

$$T(E_p, \theta_T) = \frac{2}{3k_B} \int_0^\infty E_k f(E_k, E_p, \theta_T)_{\text{normalized}} dE_k \quad (63)$$

where  $E_k$  is the kinetic energy and  $E_p$  is the potential energy of the gas at a given elevation. The gas column forms a single thermodynamic ensemble in which all



**Figure 2.** Gas subjected to a force. A low density enclosed gas is subjected to an external force such as gravity.

particles share the same statistical distribution. Obviously then,  $\theta_T$  is invariant for the ensemble and therefore invariant with elevation in compliance with the second law. Is  $T$  also invariant with elevation? This question is addressed in the next two sections covering maxwellian gases and non-maxwellian gases respectively.

### 5.1. Maxwellian Gas

For a maxwellian gas, the distribution,  $f_{MB}(E_k, E_p, \theta_T)$  is given by:

$$f_{MB}(E_k, E_p, \theta_T) = 2 \left( \frac{1}{k_B \theta_T} \right)^{3/2} \sqrt{\frac{E_k}{\pi}} \exp\left(\frac{-E_k}{k_B \theta_T}\right) \exp\left(\frac{-E_p}{k_B \theta_T}\right) \quad (64)$$

where  $E_p$  is the potential energy of the gas molecules at a given elevation. Introducing the  $E_p$  term denormalizes the distribution. Renormalizing Equation (64) for a given elevation eliminates  $E_p$  (because it is constant), [9] [10] [11] [12] resulting in:

$$f_{MB}(E_k, \theta_T)_{\text{normalized}} = 2 \left( \frac{1}{k_B \theta_T} \right)^{3/2} \sqrt{\frac{E_k}{\pi}} \exp\left(\frac{-E_k}{k_B \theta_T}\right) \quad (65)$$

The *normalized* Maxwell-Boltzmann distribution is independent of elevation as shown in **Figure 3**, therefore, the gas column is isothermal in compliance with the second law.

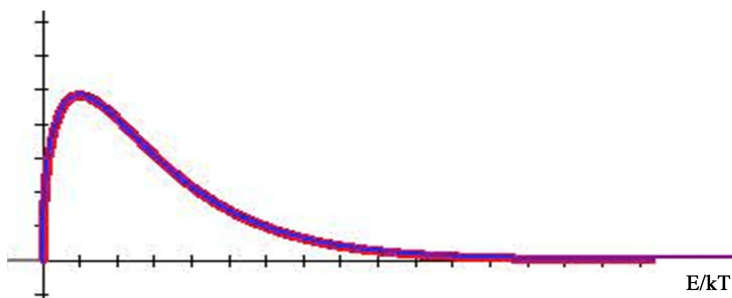
The temperature of the gas column is obtained by combining Equations (63) and (65):

$$T(\theta_T) = \frac{4}{3k_B} \sqrt{\frac{1}{\pi}} \int_0^\infty \left( \frac{E_k}{k_B \theta_T} \right)^{3/2} \exp\left(\frac{-E_k}{k_B \theta_T}\right) dE_k \quad (66)$$

Integrating using the Gamma function  $\Gamma(5/2)$  produces:

$$T = \theta_T = \text{constant with elevation} \quad (67)$$

indicating that statistical temperature and kinetic temperature are identical.  $\theta_T$  is invariant with elevation because it is shared by all particles in the statistical ensemble forming the column. As per Equation (67)  $T = \theta_T$ , therefore  $T$  is also invariant with elevation. This result complies with the second law as commonly understood.



**Figure 3.** The thick red curve is the distribution at the bottom of a gas column and the thin blue curve, is the distribution at the top of the column. The two curves are congruent showing that the distribution of a Maxwellian gas is invariant with elevation. This figure has been generated by a calculator program [13].

## 5.2. Non-Maxwellian Gas

In a non-Maxwellian gas column  $T$  and  $\theta_T$  can be different. Consider the gas column of **Figure 2** in which particles are fermions. Their Fermi-Dirac (FD) distribution is given by [9] [10]

$$f_{\text{Fermions}}(E_k, E_p, \theta_T)_{\text{normalized}} = A(E_p, \theta_T) \sqrt{E_k} \frac{1}{1 + \exp\left(\frac{E_k + E_c + E_p - E_F}{k_B \theta_T}\right)} \quad (68)$$

which includes  $E_p$ . Introducing the potential energy term denormalizes the distribution. Renormalization, however, does not eliminate  $E_p$ . Therefore, the distribution is a function of elevation as shown in **Figure 4**.

The temperature of the fermion column is obtained by combining Equations (63) and (68) yielding:

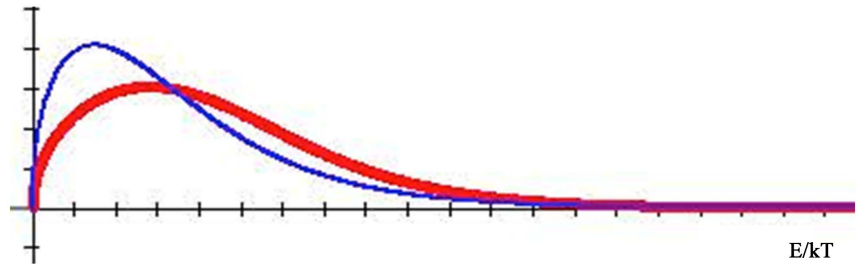
$$T(E_p, \theta_T) = \frac{2A(E_p, \theta_T)}{3k_B} \int_0^\infty (E_k)^{3/2} \frac{1}{1 + \exp\left(\frac{E_k + E_c + E_p - E_F}{k_B \theta_T}\right)} dE_k \quad (69)$$

No closed form solution is known for this integral. However, **Figure 4** clearly shows a shift of the first moment of the distribution to the left indicating a decrease in temperature with elevation. In other words, kinetic temperature decreases with elevation.

$$\frac{dT}{dz} < 0 \quad (70)$$

One must emphasize that statistical temperature  $\theta_T$  remains constant with elevation, conforming with the second law (if the law is defined in terms of statistical temperature). This chapter is summarized in **Figure 5** below.

Can the temperature difference produced by a non-Maxwellian gas be used to produce power in violation of the second law, for example by connecting a heat engine to the top and bottom of a Fermion-filled gas column of **Figure 2**? Such a violation would be truly extraordinary, upending more than two centuries of established thermodynamics science. Two kinds of systems shall be discussed:



**Figure 4.** Fermion energy distribution at the bottom of an energy well is shown as the thick red curve. The distribution at a higher elevation shown as the thin blue curve is shifted to the left and has a smaller moment than the one at a lower elevation, implying a lowering of temperature with elevation. This figure has been generated by a calculator program [13].

	MB <span style="color:red">denormalizes distribution</span>	FD <span style="color:red">denormalizes distribution</span>
Distribution	$f_{MB} = 2 \left( \frac{1}{k_B \theta_T} \right)^{3/2} \sqrt{\frac{E_K}{\pi}} \frac{1}{\exp\left(\frac{E_K + E_p}{k_B \theta_T}\right)}$	$f_{Fermions} = A \sqrt{E_K} \frac{1}{1 + \exp\left(\frac{E_K + E_c - E_f + E_p}{k_B \theta_T}\right)}$
$E_p = 0$	normalized distribution	normalized distribution
$E_p > 0$	renormalized distribution $f_{MB}^{Normalized}(E_p > 0) = f_{MB}^{Normalized}(E_p = 0)$ $E_p$ eliminated by renormalization	renormalized distribution $f_{FD}^{Normalized}(E_p > 0) \neq f_{FD}^{Normalized}(E_p = 0)$ $E_p$ NOT eliminated by renormalization
	$T = \theta_T ; \frac{d\theta_T}{dz} = 0 ; \frac{dT}{dz} = 0 ;$ isothermal	$T \neq \theta_T ; \frac{d\theta_T}{dz} = 0 ; \frac{dT}{dz} < 0$

**Figure 5.** Maxwell-Boltzmann (MB) gases remain isothermal with elevation. Non-MB such as Fermi-Dirac (FD) gases change kinetic temperature with elevation.

- 1) Homogeneous non-maxwellian systems.
- 2) Heterogeneous non-maxwellian systems.

**5.2.1. Homogeneous Non-Maxwellian Systems**

The second law, as stated by the H-Theorem [14] [15] [16] [17] [18], makes the critically important assumption that particles must be homogeneously distributed and indistinguishable. If the heat engine is connected to the top and bottom of the gas column by heat carrying particles homogenous and indistinguishable from the particles in the column, a temperature difference will form in the connector, which would exactly cancel the temperature difference produced by the column. Even though there is a temperature difference, this difference cannot be used by the engine. The entire system including the thermal connectors falls within the confines of the H-Theorem, thereby preserving the second law.

This situation is reminiscent of the impossibility of extracting energy from the built-in potential across a semiconductor junction. The leads connected to the junction produce a voltage that exactly cancels the built-in potential because all charge carriers in the circuit are identically subjected to the potential field which is scalar.

### 5.2.2. Heterogeneous Non-Maxwellian Systems

Heterogeneous systems comprise particles with different statistics thereby falling outside the coverage of the H-Theorem. **Table 4** provides a summary of their characteristics.

### 5.3. Non-Transitive Heterogeneous Systems

To be non-transitive, a system requires the following:

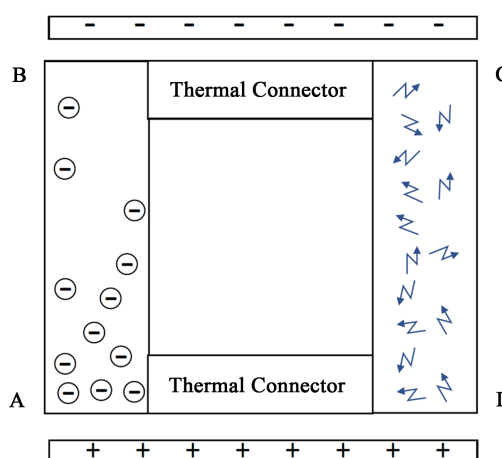
1. The system must be statistically heterogeneous, that is it must include different species of particles, some species having non-maxwellian statistics.
2. A force field must also be present, that affects the different species differently.
3. The species must be thermally connected, forming a closed thermal loop.

Several aspects of non-transitive systems need to be discussed in detail:

1. Statistical temperature;
2. Statistical entropy;
3. Proposed generalization of the second law.

#### 5.3.1. Statistical Temperature for Non-Transitive Systems

*Statistical temperature* in physical systems can have circulation. Consider the statistically heterogeneous system depicted in **Figure 6**.



**Figure 6.** Fermion and phonon columns develop different temperature gradients. A fermion column and a phonon column are thermally connected and develop their own different statistical and kinetic temperature gradients.

**Table 4.** Classification of homogeneous and heterogeneous systems showing that heterogeneous systems that include non-maxwellian components fall outside the coverage of the H-Theorem.

	MB	Non-MB
<b>Homogeneous</b>	No spontaneous temperature gradient.	Spontaneous temperature gradients possible when a field is present. Covered by H-Theorem
<b>Heterogeneous</b>	No spontaneous temperature gradient.	Spontaneous temperature gradients possible when a field is present. Outside coverage of H-Theorem.



This system is the thermodynamic analog of **Figure 1**. A column of low density non-degenerate fermion gas, for example electrical carriers in a thermoelectric material, is subjected to an electric field. The density of the gas is low enough that space charges are insufficient to cancel the field through the material. The gas develops a distribution as depicted in **Figure 4**, and Equation (70), acquiring a higher kinetic temperature at the bottom A of the column than at the top B.

The bottom A and the top B of the column are thermally connected by a thermally conductive material forming a path, B connected to C, C connected to D and D connected to A.

Let the system be in quasi-open loop by assigning to the C-D branch a very low (infinitesimal but not zero) thermal conductivity compared to the other branches. This branch comprises phonons which are not affected by the field and therefore have no tendency to develop a spontaneous temperature difference. However, a small (infinitesimal) amount of heat does flow through C-D, causing a drop in temperature  $\Delta T_{CD}$ .

The A-B path is parallel to the field, and comprises fermions. Therefore, this path develops a kinetic temperature difference even when the heat flow is zero or near zero. Hence  $\Delta T_{AB} > 0$ .

The B-C path and D-A paths are perpendicular to the field and therefore, irrespective of the heat carriers, have no tendency to develop a spontaneous kinetic temperature difference. Because of these paths' high thermal conductivity compared to C-D, one can approximate  $\Delta T_{BC} = 0$  and  $\Delta T_{DA} = 0$ .

Since *kinetic temperature* is scalar (because energy is scalar) the circulation of kinetic temperature around the A-B-C-D loop is zero.

$$\Delta T_{AB} + \Delta T_{BC} + \Delta T_{CD} + \Delta T_{DA} = 0 \quad (71)$$

Since  $\Delta T_{BC} = 0$  and  $\Delta T_{DA} = 0$ ,

$$\Delta T_{CD} = -\Delta T_{AB} < 0 \quad (72)$$

As described by Equation (36) for the game, and proven by Tolman ([6], pages 551, 552) for physical systems, heat flows down the *statistical* temperature gradient. Because of the quasi-open loop and very small heat flow, one can make the approximations,  $\Delta \theta_{AB} = 0$ ,  $\Delta \theta_{BC} = 0$ , and  $\Delta \theta_{DA} = 0$ . The CD branch which has a very low conductivity is occupied by phonons unaffected by the field, therefore, kinetic and statistical temperature differences are identical:

$$\Delta \theta_{CD} = \Delta T_{CD} \quad (73)$$

Therefore,  $\Delta \theta_{CD} < 0$  and a loop summation of statistical temperature around A-B-C-D is not zero,

$$\Delta \theta_{AB} + \Delta \theta_{BC} + \Delta \theta_{CD} + \Delta \theta_{DA} < 0 \quad (74)$$

indicating, *as expected for a non-transitive phenomenon*, that statistical temperature has circulation. Relaxing the quasi-open loop assumption redistributes the temperature drops around the loop but does not qualitatively change this conclusion.

Circulation makes the calculation of differences in statistical temperature

*path-dependent* unlike kinetic temperature which is scalar. The statistical temperature difference calculated directly along D-C is  $\Delta\theta_{CD} < 0$ . However, if it is calculated around B-C-D-A one finds  $\Delta\theta_{DA} + \Delta\theta_{AB} + \Delta\theta_{BC} = 0$ . This is just another manifestation of circulation.

### 5.3.2. Statistical Entropy for Non-Transitive Systems

Circulation in statistical temperature causes heat to flow in a circle around the A-B-C-D loop. This flow is accompanied by an increase in statistical entropy. This increase can be calculated as follows.

Consider the thermal loop in **Figure 6**. The branch C-D experiences the only significant statistical temperature drop. Therefore, C-D is the only branch in which entropy increases. Furthermore, statistical temperature and kinetic temperatures are identical in C-D, because the heat carriers are phonons. Therefore, the increase in statistical entropy caused by  $\Delta Q$  going around the loop can be conventionally calculated in terms of kinetic temperature:

$$\Delta S = \frac{\Delta Q}{T_D} - \frac{\Delta Q}{T_C} = \Delta Q \left( \frac{T_C - T_D}{T_C T_D} \right) = \Delta Q \left( \frac{\Delta T_{CD}}{T_C T_D} \right) \quad (75)$$

Since  $\Delta\theta_{CD} = \Delta T_{CD}$ , the change of entropy can also be expressed in terms of the drop in statistical temperature.

$$\Delta S = \Delta Q \left( \frac{\Delta\theta_{CD}}{T_C T_D} \right) \quad (76)$$

Expressing the above equation differentially for an infinitesimally thin stratum of gas in the column, a transfer of heat  $\Delta Q$  causes an increase in entropy:

$$dS = \Delta Q \frac{d\theta_{\text{strata}}}{T^2} \quad (77)$$

where  $d\theta_{\text{strata}}$  is the statistical temperature difference across the strata, and  $T$  is the kinetic temperature of the gas.

The heat flow around the A-B-C-D loop is continuous and irreversible, driven by statistical temperature circulation, and by the statistical entropy difference  $\Delta S$ . A state of dynamic equilibrium is reached leading to a paradox. If the system is in dynamic equilibrium, then it can reach two identical states and have the same entropy at two different times  $t_1$  and  $t_2$ , yet, Equation (76) implies that an increase in statistical entropy  $\Delta S$  drives the heat flow  $\Delta Q$  around the loop. How can  $S$  always increase, yet remain the same? The paradox is resolved by assigning circulation to statistical entropy.

Circulation in entropy is highly counterintuitive. As shown for the non-transitive dice in the introduction of this paper, heterogeneous systems are possible in which cyclic transitions from state to state are driven by non-transitive statistics. Statistical entropy has circulation, the arrow of time is preserved. The second law should therefore be generalized.

### 5.3.3. Proposed Generalization of Second Law for Non-Transitive Systems

Heterogeneous systems which include heat carriers with different statistics fall

outside the coverage of the H-Theorem which requires homogeneity and indistinguishability. These systems violate the second law [19] as currently understood. Accordingly, the following proposed amendments to the law specify that temperature and entropy must be statistical.

- 1) Statistical entropy never decreases (paraphrasing Boltzmann's formulation of the law as per the H-Theorem).
- 2) Heat flows down the statistical temperature gradient (paraphrasing Clausius' formulation) but possibly up the kinetic temperature gradient.

Other formulations of the law, notably by Kelvin and Planck are violated: heat can be converted to work and perpetual motion machines are possible.

## 6. Examples of Non-Transitive Systems

Examples of thermodynamic systems capable of supporting non-transitivity include:

### 6.1. Thermoelectric Junctions

In the presence of the built-in potential across a junction, electrical carriers (fermions) develop a temperature gradient. In conventional semiconductors, this gradient is quickly shorted by heat phonons but can be observed in semiconductors designed to minimize the coupling between electrical carriers and phonons, such as high performance ( $ZT$ ) thermoelectric materials. Experimental evidence supporting the above argument includes data gathered [9] [20] during very precise and careful measurement of the Seebeck effect in thermoelectric materials. Despite a meticulous experimental procedure, researchers failed to eliminate offsets in the temperature-voltage relation. The Seebeck curve failed to pass through the origin resulting in a temperature difference output for a zero-voltage input, and a voltage output for a zero-temperature difference input.

### 6.2. The Reciprocal Hall Effect

In the presence of a magnetic field parallel to a surface, and an electric field, perpendicular to the surface, electrical carriers at the surface follow interrupted cyclotron orbits thereby generating a voltage perpendicular to both fields and parallel to the surface [21]. This effect combines:

1. the statistics (e.g., Maxwellian or Fermi-Dirac) of the electrical carriers at the surface. These statistics are half distributions because they are truncated by the surface and they are also biased by the magnetic field with respect to the normal to the surface.
2. the statistics (Maxwellian or Fermi-Dirac) of the electrical carriers in the connecting lead which are unbiased and not distorted by the surface.

### 6.3. The Faraday Isolator

This device [22] consists of a polarizer, a rotator and an analyzer, and operates as a light valve even with thermal radiation. When inserted between two black bodies, it produces a temperature difference between the black bodies. The Fa-

raday isolator combines the statistics of the photons traversing the isolator and subjected to a magnetic field, with the statistics of heat carrying particles (e.g., phonons) in the connections outside the isolator.

#### 6.4. Epicatalysis

This phenomenon refers to a gas-surface effect in which a catalyst changes the chemical equilibrium of a reaction, thereby violating detailed balance. High temperature epicatalysis on metals has been experimentally demonstrated by Sheehan [23] [24] [25] [26] [27].

#### 6.5 The Q Machine

The Q machine is a low temperature plasma device that also exhibits well documented epicatalytic behavior [28] [29].

#### 6.6. Unpowered LED Emission

Light emission from unbiased LED's has been experimentally demonstrated by Orem [30].

### 7. Conclusions

Information theory, more particularly the non-transitive "Swap" game was used to explore non-transitive thermodynamic concepts such as temperature, heat flow, equilibrium and entropy. These concepts were found to be inadequate and had to be generalized.

Two kinds of temperatures, statistical and kinetic, need to be distinguished. Statistical temperature is a parameter of statistical distributions. Kinetic temperature is proportional to the expectation value of the kinetic energy based on its distribution. These two temperatures are identical when energy has a Maxwell-Boltzmann distribution, however, they are different in non-Maxwellian distributions such as half-maxwellian, Fermi-Dirac's or Bose-Einstein and when a force is present. Kinetic temperature is always scalar. Statistical temperature can have circulation, thereby enabling non-transitivity.

Heat flow was found to depend more accurately on the statistical temperature gradient, rather than on the kinetic temperature gradient.

Fourier's law of conduction and entropy should be stated in terms of statistical temperature, not kinetic temperature. Equilibrium corresponds to a zero-statistical temperature gradient. Kinetic temperature is always scalar. Static equilibrium corresponds to unchanging maximum entropy but dynamic equilibrium requires continuously increasing entropy. Entropy was found to be relative to the underlying statistics, in analogy to the Kullback-Leibler divergence in information theory.

Maxwellian systems in thermal equilibrium are always isothermal. In the presence of a force, non-maxwellian systems can produce temperature gradients in the direction of the force. In homogeneous non-maxwellian systems such gradients cannot be used to produce work, in accordance with the H-Theorem.

However, heterogeneous non-maxwellian systems comprised of particles with different statistics, fall outside of the H-Theorem. In such systems temperature gradients can be used to produce usable work. When expressed in terms relative to the underlying statistics, entropy always increases even in heterogeneous non-maxwellian systems.

Several experimentally demonstrated systems are described, capable of generating spontaneous temperature gradients, and supporting thermodynamic non-transitivity.

The formulations of the second law by Clausius and Boltzmann can be preserved if stated in terms of statistical temperature and statistical entropy.

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## Appendix A

Non-transitivity can be built into the rules of a game. Consider players A and B, each selecting *two* tokens for each round. After displaying their tokens to each other, the players swap them in accordance with the table below.

		A			
		Black-Black	Black-White	White-Black	White-White
B	Black-Black	No Swap	Swap (B wins 1)	Swap (B wins 1)	Swap (B wins 2)
	Black-White	No Swap	No Swap	No Swap	Swap (B wins 1)
	White-Black	No Swap	No Swap	No Swap	Swap (B wins 1)
	White-White	Swap (A wins 2)	No Swap	No Swap	No Swap

As can be seen, the asymmetry in the table gives B an advantage.

## Appendix B

For two probability distributions  $p$  and  $q$ , the Kullback-Liebler divergence  $D$  is:

$$D = p \log \frac{p}{q}. \quad (78)$$

where one can consider the  $p/q$  ratio to be the statistical skewing parameter of Equation (18), *i.e.*,  $p/q = \alpha$ . Hence from Equations (33) and (35) one can infer that

$$D = p \log \alpha \quad (79)$$

and statistical entropy is given by

$$\Delta S = \frac{1}{\alpha} \cdot \frac{Q}{T} \quad (80)$$



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