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Decay Mode Solutions to (2 + 1)-Dimensional Burgers Equation, Cylindrical Burgers Equation and Spherical Burgers Equation

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Abstract

Three (2 + 1)-dimensional equations—Burgers equation, cylindrical Burgers equation and spherical Burgers equation, have been reduced to the classical Burgers equation by different transformation of variables respectively. The decay mode solutions of the Burgers equation have been obtained by using the extended $\left(\frac{G'}{G}\right)$-expansion method, substituting the solutions obtained into the corresponding transformation of variables, the decay mode solutions of the three (2 + 1)-dimensional equations have been obtained successfully.

Keywords

Decay mode Solution, (2 + 1)-Burgers Equation, (2 + 1)-Cylindrical Burgers Equation, (2 + 1)-Spherical Burgers Equation, Transformation of Variables, Extended $\left(\frac{G'}{G}\right)$-Expansion Method

1. Introduction

Many famous nonlinear evolution equations such as Korteweg-de Vries (KdV), Modified KdV (mKdV), Kadomstev-Perviashvili (KP), Coupled KP and Zakharov-Kuznetsov (ZK) have been obtained by using the standard reductive perturbation method in nonlinear propagation of dust-acoustic wave, especially, the dust-acoustic solitary wave (DASW) in space and laboratory plasma [1]-[7]. Recent theoretical studies for ion-acoustic/dust-acoustic waves show that the properties of solitary waves in bounded nonplanar cylindrical/spherical geometry differ from that in unbounded planar geometry. The effects of dissipation on the propagation of soliton waves are scarcely discussed, especially, for nonplanar waves [8]. A dissipative cylindrical/spherical KdV is obtained by using the standard reductive perturbation method in Ref. [8]. It is known that the transverse perturbations always exist in the higher-dimensional system. Anisotropy is introduced into the system and the wave structure and stability are modified by the transverse perturbation. A spherical KP (SKP) equation is obtained by using the standard reductive perturbation method [9]. The cylindrical KP Equation (CKP) is also be introduced by Johnson [10] [11] to describe surface wave in a shallow incompressible fluid.

Following the extension sense of the KP equation Ref. [12] developed the (2 + 1)-Burgers equation in the form

$$\left(u_t + uu_x + vu_{xx}\right)_x + \lambda u_{yy} = 0, \tag{1}$$

where v is a constant that defines the kinematic viscosity, λ is a constant. If surface tension is weak compared to gravitational forces, then $\lambda > 0$ is used. However if surface tension is strong, then $\lambda < 0$ is used. The kink solutions and periodic solutions were obtained by using the tanh-coth method, *N*-soliton solutions were established by applying the powerful Hirota's bilinear method in Ref. [12].

In this work, following the extension sense of (2 + 1)-Burgers equation [12] and cylindrical KP equation [10] [11] [13] [14] the (2 + 1)-cylindrical Burgers equation can be developed in the form

$$\left(u_t + uu_x + vu_{xx} + \frac{1}{2t}u\right)_x + \frac{\lambda}{t^2}u_{yy} = 0, \tag{2}$$

where v is a constant that defines the kinematic viscosity, λ is a constant.

Following the (2 + 1)-Burgers equation and spherical KP equation [9], the (2 + 1)-spherical Burgers equation can be developed in the form

$$\left(u_t + uu_x + vu_{xx} + \frac{1}{t}u\right)_x + \frac{\lambda}{t^2}\left(u_{yy} + \frac{1}{y}u_y\right) = 0, \tag{3}$$

where v is a constant that defines the kinematic viscosity, λ is a constant.

The nonlinear evolution equation can describe various motions. So it is important to study their exact solutions. There exist many kinds of solutions to some integrable equations such as soliton, complexiton, negaton, rational and periodic solutions [13]. Besides these solutions, there exists the decay mode soliton which was proposed by Nakamura [13] [15]. Physically, the only difference between soliton and decay mode is that soliton is absolutely stable while decay mode vanishes eventually as time passes. The decay mode solutions for the cylindrical KP equation were obtained by the Backlund transformation and Hirota method in Ref. [13]. The solutions of the CKP in Ref. [13] were expressed in terms of the Airy functions. The 1-decay mode and 2-decay mode solutions of the CKP equation have been obtained in terms of the nonlinear transformation derived by using the simplified homogeneous balance method (SHB) [14] [16] [17] in Ref. [14].

In the present paper, the aim is to study the decay mode solutions of Equations (1), (2) and (3). The paper is organized as follows: In Section 2, making transformation of variables, by which reduction of (2 + 1)-dimensional Burgers equation, cylindrical Burgers equation and spherical Burgers equation to the classical Burgers equation; In Section 3, the decay mode solutions of the classical Burgers equation are obtained by using the extended $\left(\frac{G'}{G}\right)$-expansion method (the original $\left(\frac{G'}{G}\right)$-expansion method can be found in Ref. [18]); In Section 4, using the results obtained in Section 3, the decay mode solutions of Equations (1), (2) and (3) can be obtained by using different transformations of variables, respectively; In Section 5, some conclusions are made.

2. Reduction of (2 + 1)-Dimensional Burgers Equation, Cylindrical Burgers Equation and Spherical Burgers Equation

In Equation (1), assume that

$$u = w(\xi, t),\ \ \xi = x - q(y, t), \tag{4}$$

where $q = q(y,t)$ is to be determined later. Substituting Equation (4) into Equation (1), yields an equation as follows

$$\frac{\partial}{\partial \xi}\left(w_t + w w_\xi + v w_{\xi\xi}\right) - \lambda q_{yy} w_\xi + \left(\lambda q_y^2 - q_t\right) w_{\xi\xi} = 0. \tag{5}$$

Setting the coefficients of w_ξ and $w_{\xi\xi}$ to zero, yields

$$-\lambda q_{yy} = 0,\ \ \lambda q_y^2 - q_t = 0. \tag{6}$$

the system (6) admits a solution:

$$q(y,t) = \mu y + \lambda \mu^2 t, \tag{7}$$

where μ is a nonzero arbitrary constant. Using Equation (7) the expression (4) becomes

$$u = w(\xi, t),\ \ \xi = x - \left(\mu y + \lambda \mu^2 t\right), \tag{8}$$

and after integrating Equation (5) with respect to ξ once and taking the constant of integration to zero, Equation (5) becomes the classical Burgers equation for $w = w(\xi, t)$

$$w_t + w w_\xi + v w_{\xi\xi} = 0. \tag{9}$$

From the discussion above, the conclusion can be made that the (2 + 1)-dimensional Burgers Equation (1) for $u = u(x, y, t)$ is reduced to the Burgers Equation (9) for $w = w(\xi, t)$ by using the transformation of variables (8), if $w(\xi, t)$ is a solution of Burgers Equation (9), substituting it into Equation (8), then the exact solution of the (2 + 1)-Burgers equation can be obtained.

Similarly, the (2 + 1)-dimensional cylindrical Burgers Equation (2) for $u = u(x, y, t)$ is reduced to Equation (9) for $w = w(\xi, t)$ by using the transformation of variables

$$u = w(\xi, t),\ \ \xi = x - \left(\frac{1}{4\lambda} y^2 t + \mu y t + \lambda \mu^2 t\right), \tag{10}$$

where μ is a nonzero arbitrary constant, if $w(\xi,t)$ is a solution of Equation (9), and substituting it into Equation (10), the exact solution of Equation (2) can be obtained.

The conclusion can be made that the (2 + 1)-dimensional spherical Burgers Equation (3) for $u=u(x,y,t)$ is reduced to Equation (9) for $w=w(\xi,t)$ by using the transformation of variables

$$u=w(\xi,t),\ \ \xi=x-\left(\frac{1}{4\lambda}y^2t+\mu\right), \tag{11}$$

where μ is a nonzero arbitrary constant, if $w(\xi,t)$ is a solution of Equation (9), and substituting it into Equation (11), the exact solution of Equation (3) can be obtained.

3. Decay Mode Solution of Burgers Equation

Considering the homogeneous balance between ww_ξ and $w_{\xi\xi}$ in Equation (9) ($2m+1=m+2\to m=1$), according to the extended $\left(\frac{G'}{G}\right)$-expansion method, a suppose can be made that the solution of Equation (9) is of the form

$$w=v_1(\xi,t)\frac{G'(\varphi)}{G(\varphi)}+v_0(\xi,t), \tag{12}$$

where $v_1=v_1(\xi,t),\varphi=\varphi(\xi,t),v_0=v_0(\xi,t)$ are to be determined later (Noticed that in the original $\left(\frac{G'}{G}\right)$-expansion, v_1,v_0 are constants, $\varphi=\xi=x-vt$ is traveling waves), and $G=G(\varphi)$ satisfies the second order LODE

$$G''+\delta G=0, \tag{13}$$

where δ is a constant to be determined later.

When $\delta<0$, ODE (13) has general solution

$$G(\varphi)=A\cosh\left(\sqrt{-\delta}\varphi\right)+B\sinh\left(\sqrt{-\delta}\varphi\right),\ \ A,B\ \text{ are constants}$$

then

$$\frac{G'(\varphi)}{G(\varphi)}=\sqrt{-\delta}\,\frac{A\sinh\left(\sqrt{-\delta}\varphi\right)+B\cosh\left(\sqrt{-\delta}\varphi\right)}{A\cosh\left(\sqrt{-\delta}\varphi\right)+B\sinh\left(\sqrt{-\delta}\varphi\right)}. \tag{14}$$

Choose $B=0$, then

$$\frac{G'(\varphi)}{G(\varphi)}=\sqrt{-\delta}\tanh\left(\sqrt{-\delta}\varphi\right). \tag{15}$$

Now our main goal is to determine $v_1=v_1(\xi,t),\varphi=\varphi(\xi,t),v_0=v_0(\xi,t)$ and constant δ, such that expression (12) satisfies Equation (9).

Substituting Equation (12) into the left side of Equation (9), collecting the coefficients of each power of $\left(\frac{G'}{G}\right)^i$, ($i=0,1,2,3$), setting each coefficient to zero, the PDEs can be obtained for $v_1=v_1(\xi,t),\varphi=\varphi(\xi,t),v_0=v_0(\xi,t)$ as follows

$$\left(\frac{G'}{G}\right)^3 : -v_1\varphi_\xi\left(v_1 - 2v\varphi_\xi\right) = 0, \tag{16}$$

$$\left(\frac{G'}{G}\right)^2 : -2v\varphi_\xi v_{1\xi} - v_1\left(\varphi_t + v_0\varphi_\xi - v_{1\xi} + v\varphi_{\xi\xi}\right) = 0, \tag{17}$$

$$\left(\frac{G'}{G}\right) : v_{1t} - \delta v_1^2\varphi_\xi + v_1\left(2v\delta\varphi_\xi^2 + v_{0\xi}\right) + v_0 v_{1\xi} + v v_{1\xi\xi} = 0, \tag{18}$$

$$\left(\frac{G'}{G}\right)^0 : v_{0t} + v_0 v_{0\xi} - 2v\delta\varphi_\xi v_{1\xi} - \delta v_1\left(\varphi_t + v_0\varphi_\xi + v\varphi_{\xi\xi}\right) + v v_{0\xi\xi} = 0. \tag{19}$$

Simplifying Equations (16)-(19), then

$$v_1 = 2v\varphi_\xi, \tag{20}$$

$$-2v\varphi_\xi\left(\varphi_t + v_0\varphi_\xi + v\varphi_{\xi\xi}\right) = 0, \tag{21}$$

$$2v\frac{\partial}{\partial\xi}\left(\varphi_t + v_0\varphi_\xi + v\varphi_{\xi\xi}\right) = 0, \tag{22}$$

$$\left(v_{0t} + v_0 v_{0\xi} + v v_{0\xi\xi}\right) - 2\delta v\varphi_\xi\left(\varphi_t + v_0\varphi_\xi - 3v\varphi_{\xi\xi}\right) = 0. \tag{23}$$

Noticed that in Equations (20)-(23), if

$$\varphi = v_0 = \frac{\xi}{t}, \quad \text{then} \quad v_1 = 2v\varphi_\xi = \frac{2v}{t}, \tag{24}$$

Equations (20)-(23) are satisfied completely for arbitrary constant $\delta < 0$.

Substituting Equation (15) and Equation (24) into Equation (12), the decay mode solution for Equation (9) can be expressed as follows

$$w(\xi,t) = \frac{2v\sqrt{-\delta}}{t}\tanh\left(\frac{\sqrt{-\delta}\xi}{t}\right) + \frac{\xi}{t}, \delta < 0. \tag{25}$$

As far as we know, the solution (25) has never seen in early literatures.

4. Decay Mode Solutions of (2 + 1)-Burgers Equation, Cylindrical Burgers Equation and Spherical Burgers Equation

Substituting Equation (25) into Equation (8), the decay mode solution for Equation (1) can be obtained as follows

$$u(x,y,t) = \frac{2v\sqrt{-\delta}}{t}\tanh\left(\frac{\sqrt{-\delta}\xi}{t}\right) + \frac{\xi}{t}, \tag{26}$$

where $\delta < 0$ and $\xi = x - \left(\mu y + \lambda\mu^2 t\right)$.

Substituting Equation (25) into Equation (10), the decay mode solution for Equation (2) can be obtained as follows

$$u(x,y,t) = \frac{2v\sqrt{-\delta}}{t}\tanh\left(\frac{\sqrt{-\delta}\xi}{t}\right) + \frac{\xi}{t}, \tag{27}$$

where $\delta < 0$ and $\xi = x - \left(\frac{1}{4\lambda}y^2 t + \mu y t + \lambda\mu^2 t\right)$.

Substituting Equation (25) into Equation (11), the decay mode solution for Equation (3) can be obtained as follows

$$u(x,y,t)=\frac{2v\sqrt{-\delta}}{t}\tanh\left(\frac{\sqrt{-\delta}\xi}{t}\right)+\frac{\xi}{t}, \tag{28}$$

where $\delta<0$ and $\xi=x-\left(\frac{1}{4\lambda}y^2t+\mu\right)$.

5. Conclusion

In this paper, by making corresponding transformation of variables, the (2 + 1)-dimensional Burgers equation, (2 + 1)-dimensional cylindrical Burgers equation and (2 + 1)-dimensional spherical Burgers equation are all reduced to the classical Burgers equation, which can be solved by using extended $\left(\frac{G'}{G}\right)$-expansion method to obtain a novel type of decay mode solution. Substituting the novel solution of the Burgers equation into the corresponding transformation of variables, the decay mode solutions of the (2 + 1)-dimensional Burgers equation, (2 + 1)-dimensional cylindrical Burgers equation and (2 + 1)-dimensional spherical Burgers equation have been obtained for the first time, respectively. The analysis may be extended to other works to make further progress.

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