Decay Rate for a Viscoelastic Equation with Strong Damping and Acoustic Boundary Conditions

Zhiyong Ma

College of Science, Shanghai Second Polytechnic University, Shanghai, China
Email: mazhiyong1980@hotmail.com

Abstract

This paper is concerned with a nonlinear viscoelastic equation with strong damping: $\left| u_t \right|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(x,s) \mathrm{d}s - \Delta u_t = 0,$. The objective of the present paper is to provide some results on the long-time behavior to this equation with acoustic boundary conditions. By using the assumptions on the relaxation function due to Tatar [1], we show an arbitrary rate of decay with not necessary of an exponential or polynomial one and without the assumption $\int_0^{\infty} g(s) \mathrm{d}s < \frac{1}{2}$ condition. The result extends and improves some results given in Cavalcanti [2].

Keywords

Viscoelastic Equation, Decay Rate, Acoustic Boundary Condition

1. Introduction

In this paper, we investigate the following viscoelastic system with acoustic boundary conditons

$$\left| u_t \right|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(x,s) \mathrm{d}s - \Delta u_t = 0, \quad (x,t) \in (0,+\infty), \quad (1.1)$$

$$\frac{\partial u_t}{\partial \nu}(x,t) = 0 \qquad (x,t) \in \Gamma \times [0,+\infty), \quad (1.2)$$

$$u(x,t) = 0, \qquad (x,t) \in \Gamma_1 \times [0,+\infty), \quad (1.3)$$

$$\frac{\partial u_{tt}}{\partial \nu}(x,t) + \frac{\partial u}{\partial \nu}(x,t) - \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(x,s) \mathrm{d}s = y_t \quad (x,t) \in \Gamma_0 \times [0,+\infty), \quad (1.4)$$

$$u_t(x,t) + p(x) y_t + q(x) y(x,t) = 0 \qquad (x,t) \in \Gamma_0 \times [0,+\infty), \quad (1.5)$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x \in \Omega, \tag{1.6}$$

where $\Omega \subseteq \mathbb{R}^n (n=1,2)$ is a bounded domain with smooth boundary $\Gamma = \Gamma_0 \bigcup \Gamma_1$, ν is the unit outward normal to Γ, the function g represents the kernel of a memory, p and q are specific functions, and ρ is a real number such that

$$1 < \rho \leq \frac{2}{n-2} \quad \text{if} \quad n \geq 3; \quad \rho > 1 \quad \text{if} \quad n = 1,2. \tag{1.7}$$

Our problem is of the form

$$f(u_t)u_{tt} - \Delta u - \Delta u_{tt} = 0, \tag{1.8}$$

which has several modeling features. In the case, $f(u_t)$ is a constant; Equation (8) has been used to model extensional vibrations of thin rods (see Love [3], Chapter 20). In the case, $f(u_t)$ is not a constant; Equation (8) can model materials whose density depends on the velocity u_t, for instance, a thin rod which possesses a rigid surface and with an interior which can deform slightly. We refer the reader to Fabrizio and Morro [4] for several other related models.

Recently, Liu [5] considered the following viscoelastic problem with acoustic boundary conditions

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = 0, \quad (x,t) \in (0,+\infty), \tag{1.9}$$

$$u(x,t) = 0, \qquad (x,t) \in \Gamma_1 \times [0,+\infty), \tag{1.10}$$

$$\frac{\partial u}{\partial \nu}(x,t) - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(x,s)\mathrm{d}s = y_t \quad (x,t) \in \Gamma_0 \times [0,+\infty), \tag{1.11}$$

$$u_t(x,t) + p(x)y_t + q(x)y(x,t) = 0 \quad (x,t) \in \Gamma_0 \times [0,+\infty), \tag{1.12}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.13}$$

the authors obtain an arbitrary decay rate of the energy. In the pioneering paper [6], Beale and Rosencrans considered the acoustic boundary condition (1.12) and the coupled impenetrability boundary condition (1.11) with a general form, which had the presence of y_{tt} in (1.2), in a study of the model for acoustic wave motion of a fluid interacting with a so-called locally reacting surface. Recently, many authors treated wave equations with acoustic boundary conditions, see [7] [8] [9] [10] and references therein. For instance, Rivera and Qin [10] proved the polynomial decay for the wave motion with general acoustic boundary conditions by using the Lyapunov functional technique. Frota and Larkin [8] established global solvability and the exponential decay for problems (1.9)-(1.13) with $g \equiv 0$. They overcame the difficulties which were arisen due to the absence of y_{tt} in (1.12) by using the degenerated second order equation. Recently, Park and Park [9] investigated problems (1.9)-(1.13) and proved general rates of decay which depended on the behavior of g, under the additional assumption of that $\int_0^{+\infty} g(s)\mathrm{d}s$.

Many authors have focused on the viscoelastic problem. In the pioneer work of Dafermos [11] [12], existence and asymptotic stability for a one-dimensional vis-

coelastic problem were proved but no rate of decay has been specified. Since then problems related to viscoelasticity have attracted a great deal of attention [13] [14] [15]. It seems all started with kernels of the form $g(t) = e^{-\beta t}$, $\beta > 0$, then with kernels satisfying $-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t)$, for all $t \geq 0$, for some constants ξ_1 and ξ_2 and some other conditions on the second derivative, Cavalcanti *et al.* [2] studied the following equation with Dirichlet boundary conditions

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + g * \Delta u - \gamma \Delta u_t = 0 \tag{1.14}$$

where $g * \Delta u = \int_0^t g(t-s) \Delta u(s) \mathrm{d}s$. They established a global existence result for $\gamma \geq 0$ and an exponential decay of energy for $\gamma > 0$, and studied the interaction within the $|u_t|^\rho u_{tt}$ and the memory term $g * \Delta u$. Messaoudi and Tatar [16] established, for small initial data, the global existence and uniform stability of solutions to the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + g * \Delta u = b|u|^{p-2} u \tag{1.15}$$

with Dirichlet boundary condition, where $\gamma \geq 0, \rho, b > 0, p > 2$ are constants. In the case $b = 0$ in (15), Messaoudi and Tatar [17] proved the exponential decay of global solutions to (15) without smallness of initial data, considering only the dissipation effect given by the memory.

In [18] [19], the condition has been replaced by $g'(t) \leq -\xi(t) g(t)$, where $\xi(t)$ is a positive function. Similarly, Han and Wang [20] proved the energy decay for the viscoelastic equation with nonlinear damping

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + g * \Delta u + |u_t|^m u_t = 0, \tag{1.16}$$

with Dirichlet boundary condition, where $\rho > 0, m > 0$ are constants. Then Park and Park [21] established the general decay for the viscoelastic problem with nonlinear weak damping

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + g * \Delta u + h(u_t) = 0, \tag{1.17}$$

with the Dirichlet boundary condition, where $\rho > 0$ is a constant. We also mention that Fabrizio and Polidoro [22] obtained the exponential decay result under the conditions that $g'(t) \leq 0$ and $e^{\alpha t} g(t) \in L^1(0, +\infty)$ for some $\alpha > 0$. Recently, Tatar [23] improved these results by removing the last condition and established a polynomial asymptotic stability. In fact, he considered the kernels having small flat zones and these zones are not too big (see also [24] for the case of coupled system). More recently, under the assumptions that $g'(t) \leq 0$ and $g(t) \gamma(t) \in L^1(0, +\infty)$ for some nonnegative function $\gamma(t)$, Tatar [1] generalized these works to an arbitrary decay for wave equation with a viscoelastic damping term. Moreover, we would like to mention some results in [25]-[30].

The rest of our paper is organized as follows. In Section 2, we give some preparations for our consideration and our main result. The statements and the proofs of our main results will be given in Section 3.

For convenience, we denote the norm and scalar product in $L^2(\Omega)$ by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. C denotes a general positive constant, which may be different in different estimates.

2. Preliminaries and Main Result

For the memory kernel g we assume that:

(H_1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing differentiable function satisfying that

$$g(0) > 0, \quad l = 1 - \int_0^{+\infty} g(s)\,\mathrm{d}s > 0. \tag{2.1}$$

(H_2) suppose that there exists a nondecreasing function $\gamma(t) > 0$ such that $\dfrac{\gamma'(t)}{\gamma(t)} = \eta(t)$ is a decreasing function and $\int_0^{+\infty} g(s)\gamma(s)\,\mathrm{d}s < +\infty$.

For the functions p and q, we assume that $p, q \in C(\Gamma_0)$ and $p(x) > 0$ and $q(x) > 0$ for all $x \in \Gamma_0$. This assumption implies that there exist positive constants $p_i, q_i \, (i = 0,1)$ such that

$$p_0 \le p(x) \le p_1, \quad q_0 \le q(x) \le q_1, \quad x \in \Gamma_0. \tag{2.2}$$

We use the notation

$$V = \left\{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1 \right\},$$
$$(u, v) = \int_\Omega u(x)v(x)\,\mathrm{d}x, \text{ and } (u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\,\mathrm{d}\Gamma.$$

Let λ and $\tilde{\lambda}$ be the smallest positive constants such that

$$\|u\|^2 \le \lambda \|\nabla u\|^2, \quad \|u\|_{\Gamma_0}^2 \le \tilde{\lambda} \|\nabla u\|^2. \tag{2.3}$$

Firstly, we have the following existence and uniqueness results, it can be established by adopting the arguments of [2] [31].

Theorem 2.1 Let $(u_0, u_1) \in \left(V \bigcap H^2(\Omega)\right) \times V$. Assume that H_1, H_2 and (2.2) hold. There exists a unique pair of functions (u, y_t), which is a solution to the problem (1.1) in the class

$$u \in L^\infty\left(0, T, V \cap H^2(\Omega)\right), \quad u_t \in L^\infty(0, T, V), \tag{2.4}$$

$$u_{tt} \in L^\infty\left(0, T, L^2(\Omega)\right), \quad y, y^t \in L^2\left(\mathbb{R}^+; L^2(\Gamma_0)\right). \tag{2.5}$$

We introduce the modified energy functional

$$\begin{aligned}
E(t) &= \frac{1}{\rho + 2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\
&\quad + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\int_{\Gamma_0} q(x)|y(x,t)|^2\,\mathrm{d}\Gamma,
\end{aligned} \tag{2.6}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2\,\mathrm{d}s.$$

Clearly

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla u_t(t)\|^2 - \frac{1}{2}g(t)\|\nabla u\|^2 + \frac{1}{2}(g' \circ \nabla u) - \int_{\Gamma_0} p y_t^2. \tag{2.7}$$

To state our main result, we introduce the following notations as in [32]. For every measurable set $\mathcal{A} \subset \mathbb{R}^+$, we define the probability measure $\hat{g}$ by

$$\hat{g}(\mathcal{A}) = \frac{1}{1-l}\int_{\mathcal{A}} g(s)\,\mathrm{d}s. \tag{2.8}$$

The flatness set and the flatness rate of g are defined by

$$\mathcal{F}_g = \left\{ s \in \mathbb{R}^+ : g(s) > 0 \text{ and } g'(s) = 0 \right\} \tag{2.9}$$

and

$$\mathcal{R}_g = \hat{g}(\mathcal{F}_g) = \frac{1}{1-l} \int_{\mathcal{F}_g} g(s) \mathrm{d}s \tag{2.10}$$

respectively. We denote

$$G_\gamma(t) = \gamma(t)^{-1} \int_t^{+\infty} g(s) \gamma(s) \mathrm{d}s. \tag{2.11}$$

Now, we are in a position to state our main result.

Theorem 2.2 ([23]) *Let* $(u_0, u_1) \in \left(V \bigcap H^2(\Omega)\right) \times V$, *Assume that* (2.1)-(2.2) *hold and* $\mathcal{R}_g < \frac{1}{2}$. *If* $G_\gamma(0) < \frac{(1-l)(2-l)}{2}$, *then there exist positive constants* C *and* v *such that*

$$E(t) \le C\gamma(t)^{-v}, \quad t \ge 0. \tag{2.12}$$

3. Arbitrary Rate of Decay

Now we define

$$\Phi(t) = \frac{1}{\rho+1} \int_\Omega |u_t|^\rho u_t u \,\mathrm{d}x + \int_\Omega \nabla u_t \cdot \nabla u \,\mathrm{d}x + \frac{1}{2} \int_{\Gamma_0} p y^2 \mathrm{d}\Gamma + \int_{\Gamma_0} u y \,\mathrm{d}\Gamma. \tag{3.1}$$

Using (1.1) and (3.1), we have

$$\Phi'(t) = \frac{1}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} - \|\nabla u\|^2 + \|\nabla u_t\|^2 + \int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) \mathrm{d}s \mathrm{d}x \\ + \int_\Omega \Delta u_t u \,\mathrm{d}\Gamma + 2 \int_{\Gamma_0} u y_t \mathrm{d}\Gamma - \int_{\Gamma_0} q(x) y^2 \mathrm{d}\Gamma. \tag{3.2}$$

We use here the following identity due to [1], to give a better estimate for the term $\int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) \mathrm{d}s \mathrm{d}x$:

$$\int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) \mathrm{d}s \mathrm{d}x \\ = \frac{1}{2} \left(\int_0^t g(s) \mathrm{d}s \right) \|\nabla u\|^2 + \frac{1}{2} \int_0^t g(t-s) \|\nabla u(s)\|^2 \,\mathrm{d}s - \frac{1}{2} (g \circ \nabla u)(t). \tag{3.3}$$

From (2.1), (3.2) and (3.3), integration by parts and Young's inequality, we derive for any $\delta_0 > 0$,

$$\Phi'(t) \le \frac{1}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} + (1+\delta_0) \|\nabla u_t(t)\|^2 - \left(\frac{1+l}{2} - \delta_0 \tilde{\lambda} \right) \|\nabla u\|^2 \\ + \frac{1}{2} \int_0^t g(t-s) \|\nabla u(s)\|^2 \,\mathrm{d}s - \frac{1}{2} (g \circ u)(t) \tag{3.4} \\ + \frac{1}{\delta_0} \|y_t\|_{\Gamma_0}^2 - \int_{\Gamma_0} q(x) y^2 \mathrm{d}\Gamma.$$

As in [5], we have:

Lemma 3.1 *For* $u \in H_0^1(\Omega)$, *we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s)) \mathrm{d}s \right)^2 \mathrm{d}x \le \lambda(1-l)(g \circ \nabla u)(t). \tag{3.5}$$

Now we define the functional

$$\Psi(t) = \int_\Omega \left(\Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t \right) \int_0^t g(t-s)(u(t)-u(s)) \mathrm{d}s \mathrm{d}x. \tag{3.6}$$

It follows from (1.1) and (3.6) that

$$
\begin{aligned}
\Psi'(t) &= \int_\Omega \Delta u_t \int_0^t g'(t-s)(u(t)-u(s)) \mathrm{d}s \mathrm{d}x - \left(\int_0^t g(s) \mathrm{d}s \right) \|\nabla u_t\|^2 \\
&\quad + \left(1 - \int_0^t g(s) \mathrm{d}s \right) \int_\Omega \nabla u(t) \cdot \left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s)) \mathrm{d}s \right) \mathrm{d}x \\
&\quad + \int_\Omega \left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s)) \mathrm{d}s \right)^2 \mathrm{d}x - \frac{\int_0^t g(s) \mathrm{d}s}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} \\
&\quad + \int_\Omega \nabla u_t \int_0^t g(t-s)(\nabla u(t)-\nabla u(s)) \mathrm{d}s \mathrm{d}x \\
&\quad - \int_\Omega \frac{1}{\rho+1}|u_t|^\rho u_t \int_0^t g'(t-s)(u(t)-u(s)) \mathrm{d}s \mathrm{d}x \\
&\quad - \int_{\Gamma_0} y_t \left(\int_0^t g(t-s)(u(t)-u(s)) \mathrm{d}s \right) \mathrm{d}\Gamma \\
&= I_1 - I_2 + \left(1 - \int_0^t g(s) \mathrm{d}s \right) I_3 + I_4 - I_5 + I_6 - I_7 - I_8.
\end{aligned}
\tag{3.7}
$$

For any $\delta > 0$, we have

$$I_1 \le \delta \|\nabla u_t(t)\|^2 - \frac{g(0)}{4\delta} \lambda (g'(s) \circ \nabla u)(t). \tag{3.8}$$

For all measurable sets $\mathcal{A}$ and $\mathcal{F}$ such that $\mathcal{A} = \mathbb{R}^+ \setminus \mathcal{F}$, I_3, I_4 and I_6 can be estimated as in [1]:

$$
\begin{aligned}
I_3 &\le \delta_1 \|\nabla u\|^2 + \frac{1-l}{4\delta_1} \int_\Omega \int_{\mathcal{A}_t} g(t-s)|\nabla u(t)-\nabla u(s)|^2 \mathrm{d}s \mathrm{d}x \\
&\quad + \frac{3}{2}(1-l)\hat{g}(\mathcal{F}) \|\nabla u\|^2 + \frac{1}{2} \int_{\mathcal{F}_t} g(t-s) \|\nabla u(s)\|^2 \mathrm{d}s, \quad \delta_1 > 0,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
I_4 &\le \left(1 + \frac{1}{\delta_2} \right)(1-l) \int_\Omega \int_{\mathcal{A}_t} g(t-s)|\nabla u(t)-\nabla u(s)|^2 \mathrm{d}s \mathrm{d}x \\
&\quad + (1+\delta_2)(1-l)\hat{g}(\mathcal{F}) \int_\Omega \int_{\mathcal{F}_t} g(t-s)|\nabla u(t)-\nabla u(s)|^2 \mathrm{d}s \mathrm{d}x, \quad \delta_2 > 0,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
I_6 &\le \delta_1 \|\nabla u_t\|^2 + \frac{1}{4\delta_1} \int_\Omega \int_{\mathcal{A}_t} g(t-s)|\nabla u(t)-\nabla u(s)|^2 \mathrm{d}s \mathrm{d}x \\
&\quad + \frac{3}{2}\hat{g}(\mathcal{F}) \|\nabla u_t\|^2 + \frac{1}{2} \int_{\mathcal{F}_t} g(t-s) \|\nabla u(s)\|^2 \mathrm{d}s, \quad \delta_1 > 0,
\end{aligned}
\tag{3.11}
$$

where $\hat{g}$ is defined in (2.8). For any $\delta > 0$,

$$I_7 \le \delta \|\nabla u_t(t)\|^2 - \frac{g(0)}{4\delta} \lambda (g'(s) \circ \nabla u)(t). \tag{3.12}$$

For I_8, for $\delta_3, \delta_4 > 0$, we use a different estimate as

$$I_8 = \int_{\Gamma_0} y_t \left(\int_{\mathcal{A}_t} g(t-s)(u(t)-u(s)) ds \right) d\Gamma$$

$$+ \int_{\Gamma_0} y_t \left(\int_{\mathcal{F}_t} g(t-s)(u(t)-u(s)) ds \right) d\Gamma$$

$$\leq \frac{1}{2} \|y_t\|_{\Gamma_0}^2 + \frac{\tilde{\lambda}(1-l)}{2} \int_\Omega \int_{\mathcal{A}_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, dsdx$$

$$+ \frac{1}{4\delta_3} \hat{g}(\mathcal{F}) \|y_t\|_{\Gamma_0}^2 \tag{3.13}$$

$$+ \delta_3 \tilde{\lambda} \hat{g}(\mathcal{F}) \|\nabla u\|^2 + \frac{1}{4\delta_4} \|y_t\|_{\Gamma_0}^2$$

$$+ \delta_4 \tilde{\lambda}(1-l) \int_{\mathcal{F}_t} g(t-s) \|\nabla u(s)\|^2 \, ds.$$

Taking into account these estimates in (3.6), let t_* be a number such that $\int_0^{t_*} g(s) ds = g_*$, we obtain that

$$\Psi'(t) \leq \left(-\frac{g_*}{2} + \delta_1 \right) \|\nabla u_t\|^2 - \frac{g_*}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2}$$

$$+ \left\{ (1-g_*) \left(\delta_1 + \frac{3}{2}(1-l)\hat{g}(\mathcal{F}) \right) + \delta_3 \tilde{\lambda} \hat{g}(\mathcal{F}) + \delta \right\}$$

$$\times \|\nabla u\|^2 (1-l) \left(\frac{1-g_*}{4\delta_1} + \frac{1+\delta_2}{\delta_2} + \frac{\tilde{\lambda}}{2} \right) \int_\Omega \int_{\mathcal{A}_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, dsdx$$

$$- \frac{3}{4\delta} g(0) \lambda (g' \circ \nabla u)(t) \tag{3.14}$$

$$+ (1+\delta_2)(1-l)\hat{g}(\mathcal{F}) \int_\Omega \int_{\mathcal{F}_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, dsdx$$

$$+ \left(\frac{1-g_*}{2} + \delta_4 \tilde{\lambda}(1-l) \right) \int_{\mathcal{F}_t} g(t-s) \|\nabla u(s)\|^2$$

$$+ \left(\frac{1}{2} + \frac{\hat{g}(\mathcal{F})}{4\delta_3} + \frac{1}{4\delta_4} \right) \|y_t\|_{\Gamma_0}^2.$$

Let

$$I(t) = \int_\Omega \int_0^t G_\gamma(t-s) |\nabla u(s)|^2 \, dsdx, \tag{3.15}$$

and $G_\gamma(t)$ is given in (2.11), we define the following functional

$$F(t) = ME(t) + \varepsilon\Phi(t) + \Psi(t) + \epsilon I(t), \tag{3.16}$$

then we know from [1] that

$$I'(t) \leq G_\gamma(0) \|\nabla u\|^2 - \eta(t) \int_0^t G_\gamma(t-s) \|\nabla u(s)\|^2 \, ds$$

$$- \int_0^t g(t-s) \|\nabla u(s)\|^2 \, ds. \tag{3.17}$$

At the same time, we have the following lemmas.

Lemma 3.2 *For* M *large enough, there exist two positive constants* ρ_1 *and* ρ_2 *such that*

$$\rho_1 (E(t) + I(t)) \leq F(t) \leq \rho_2 (E(t) + I(t)). \tag{3.18}$$

Proof. See, e.g. Liu [5].

Proof of Theorem 2.2 By using (2.7), (3.4), (3.13)-(3.16), a series of com-

putations yields, for $t \geq t_*$,

$$
\begin{aligned}
F'(t) \leq & \left(\frac{M}{2} - \frac{3}{4\delta} g(0)\lambda \right)(g' \circ \nabla u)(t) - \left(\frac{g_*}{\rho+1} - \frac{\varepsilon}{1+\rho} \right) \|u_t(t)\|_{\rho+2}^{\rho+2} \\
& - \left[\frac{M}{2} + \frac{g_*}{2} - \delta_1 - \varepsilon(1+\delta_0) \right] \|\nabla u_t(t)\|^2 \\
& + (1+\delta_2)(1-l)\hat{g}(\mathcal{F}) \int_\Omega \int_{\mathcal{F}_t} g(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds dx \\
& + \left\{ (1-g_*)\left(\delta_1 + \frac{3}{2}(1-l)\hat{g}(\mathcal{F}) \right) + \delta_3 \tilde{\lambda}\hat{g}(\mathcal{F}) + \delta + eG_\gamma(0) - \varepsilon\left(\frac{1+l}{2} - \delta_0\tilde{\lambda} \right) \right\} \|\nabla u\|^2 \\
& + (1-l)\left(\frac{1-g_*}{4\delta_1} + \frac{1+\delta_2}{\delta_2} + \frac{\tilde{\lambda}}{2} \right) \int_\Omega \int_{\mathcal{A}_t} g(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds dx \\
& - \left(\epsilon - \frac{\varepsilon}{2} \right) \int_0^t g(t-s)\|\nabla u(s)\|^2 \, ds - \frac{\varepsilon}{2}(g \circ \nabla u)(t)\left(\frac{1-g_*}{2} + \delta_4 \tilde{\lambda}(1-l) \right) \int_{\mathcal{F}_t} g(t-s)\|\nabla u(s)\|^2 \, ds dx \\
& - \epsilon\eta(t) - \varepsilon \int_{\Gamma_0} q(x) y^2 d\Gamma - \left[Mp_0 - \frac{\varepsilon}{\delta_0} - \left(\frac{1}{2} + \frac{\hat{g}(\mathcal{F})}{4\delta_3} + \frac{1}{4\delta_4} \right) \right] \|y_t\|_{\Gamma_0}^2 .
\end{aligned}
\tag{3.19}
$$

For $n \in \mathbb{N}$, as in [32] we introduce the sets

$$
\mathcal{A}_n = \left\{ s \in \mathbb{R}^+ : ng'(s) + g(s) \leq 0 \right\}.
\tag{3.20}
$$

It is easy to see that

$$
\bigcup_n \mathcal{A}_n = \mathbb{R}^+ \setminus \left\{ \mathcal{F}_g \cup \mathcal{N}_g \right\},
\tag{3.21}
$$

where $\mathcal{F}_g$ is given in (2.9) and $\mathcal{N}_g$ is the null set where g' is not defined. Additionally, we denote $\mathcal{F}_n = \mathbb{R}^+ \setminus \mathcal{A}_n$, then

$$
\lim_{n \to \infty} \hat{g}(\mathcal{F}_n) = \hat{g}(\mathcal{F}_g),
\tag{3.22}
$$

since $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for all n and $\bigcap_n \mathcal{F}_n = \mathcal{F}_g \cup \mathcal{N}_g$. Then, we take $\mathcal{A} = \mathcal{A}_n$ and $\mathcal{F} = \mathcal{F}_n$ in (3.18), it follows that

$$
\begin{aligned}
F'(t) \leq & \left(\frac{M}{2} - \frac{3}{4\delta} g(0)\lambda \right)(g' \circ \nabla u)(t) - \left(\frac{g_*}{\rho+1} - \frac{\varepsilon}{\rho+1} \right) \|u_t(t)\|_{\rho+2}^{\rho+2} \\
& - \left[\frac{M}{2} + \frac{g_*}{2} - \delta_1 - \varepsilon(1+\delta_0) \right] \|\nabla u_t(t)\|^2 \\
& - \left(\epsilon - \frac{1+\varepsilon-g_*}{2} - \delta_4 \tilde{\lambda}(1-l) \right) \int_0^t g(t-s)\|\nabla u(s)\|^2 \, ds \\
& + \left\{ (1-g_*)\left(\delta_1 + \frac{3}{2}(1-l)\hat{g}(\mathcal{F}_n) \right) + \delta_3 \tilde{\lambda}\hat{g}(\mathcal{F}_n) + \delta + eG_\gamma(0) \right. \\
& \qquad \left. - [\sigma + (1-\sigma)]\varepsilon \frac{1+l}{2} + \varepsilon\delta_0\tilde{\lambda} \right\} \|\nabla u\|^2 \\
& + (1-l)\left(\frac{1-g_*}{4\delta_1} + \frac{1+\delta_2}{\delta_2} + \frac{\tilde{\lambda}}{2} \right) \int_\Omega \int_{\mathcal{A}_t} g(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds dx \\
& - \left(\frac{\varepsilon}{2} - (1+\delta_2)(1-l)\hat{g}(\mathcal{F}_n) \right)(g \circ \nabla u)(t) \\
& - \epsilon\eta(t)I(t) - \varepsilon \int_{\Gamma_0} q(x) y^2 d\Gamma - \left[Mp_0 - \frac{\varepsilon}{\delta_0} - \left(\frac{1}{2} + \frac{\hat{g}(\mathcal{F}_n)}{4\delta_3} + \frac{1}{4\delta_4} \right) \right] \|y_t\|_{\Gamma_0}^2 ,
\end{aligned}
\tag{3.23}
$$

for some $0 < \delta < 1$. Since $\mathcal{R}_g = \hat{g}(\mathcal{F}_g) < \dfrac{1}{2}$, we can choose ε, δ_2 small enough and n, t_* large enough such that

$$\frac{\varepsilon}{2} - (1 + \delta_2)(1 - l)\hat{g}(\mathcal{F}_n) \geq 0 \tag{3.24}$$

and

$$\frac{3}{2}(1 - l)(1 - g_*)\hat{g}(\mathcal{F}_n) - \sigma\varepsilon\frac{1 + l}{2} < 0 \tag{3.25}$$

with $\sigma = \dfrac{3(1 - l)(1 - g_*)}{2g_*(1 + l)}$. Note that for t_* large enough. Furthermore, we require that

$$\frac{1 + \varepsilon - g_*}{2} + \delta_4\tilde{\lambda}(1 - l) \leq \epsilon \leq \frac{1}{G_\gamma(0)}\left((1 - \sigma)\varepsilon\frac{1 + l}{2}\right. \tag{3.26}$$
$$\left. -(1 - g_*)\delta_1 - \delta_3\tilde{\lambda}\hat{g}(\mathcal{F}_n) - \varepsilon\delta_0\tilde{\lambda} + \delta\right).$$

Combining (3.24) and (3.25), we obtain

$$(1 - g_*)\left(\delta_1 + \frac{3}{2}(1 - l)\hat{g}(\mathcal{F}_n)\right) + \delta_3\tilde{\lambda}\hat{g}(\mathcal{F}_n) + \delta + eG_\gamma(0) - \varepsilon\frac{1 + l}{2} + \varepsilon\delta_0\tilde{\lambda} < 0 \tag{3.27}$$

Choose our constants properly so that:

$$\frac{M}{2} - \frac{3}{4\delta}g(0)\lambda \geq \frac{M}{4}, \tag{3.28}$$

$$Mp_0 - \frac{\varepsilon}{\delta_0} - \left(\frac{1}{2} + \frac{\hat{g}(\mathcal{F}_n)}{4\delta_3} + \frac{1}{4\delta_4}\right) \geq 0, \tag{3.29}$$

$$(1 - l)\left(\frac{1 - g_*}{4\delta_1} + \frac{1 + \delta_2}{\delta_2} + \frac{\tilde{\lambda}}{2}\right) - \frac{M}{4n} < 0 \tag{3.30}$$

together with (3.22) yield

$$F'(t) \leq -C_1 E(t) - \epsilon\eta(t)I(t), \quad t \geq t_*. \tag{3.31}$$

As $\eta(t)$ is decreasing, we have $\eta(t) \leq \eta(0)$ for all $t \geq t_*$. Then (3.30) becomes

$$F'(t) \leq -\frac{C_1}{\eta(0)}\eta(t)E(t) - \epsilon\eta(t)I(t), \quad t \geq t_*.$$

Since $F(t)$ is equipped with $E(t) + I(t)$, we get

$$F'(t) \leq -C_2\eta(t)F(t), \tag{3.32}$$

integrating (3.31) over $[t_*, t]$ yields

$$F(t) \leq e^{-C_2\int_{t_*}^t \eta(s)\mathrm{d}s}F(t_*), \quad t \geq t_*.$$

Then using the left hand side inequality in (3.17), we get

$$\rho_1(E(t) + I(t)) \leq e^{-C_2\int_{t_*}^t \eta(s)\mathrm{d}s}F(t_*), \quad t \geq t_*.$$

By virtue of the continuity and boundedness of $E(t)$ in the interval $[0, t_*]$, we conclude that

$$E(t) \leq C\gamma^{-\nu}(t), \quad t \geq 0 \tag{3.33}$$

for some positive constants C and ν.

Acknowledgements

References

[1] Tatar, N. (2011) Arbitrary Decays in Linear Viscoelasticity. *Journal of Mathematical Physics*, **52**, Article ID: 013502.

[2] Cavalcanti, M.M., Domingos Cavalcanti, V.N. and Ferreira, J. (2001) Existence and Uniform Decay for a Non-Linear Viscoelastic Equation with Strong Damping. *Mathematical Methods in the Applied Sciences*, **24**, 1043-1053.
https://doi.org/10.1002/mma.250

[3] Love, A.E.H. (1944) A Treatise on the Mathematical Theory of Elasticity. Dover, New York.

[4] Fabrizio, M. and Morro, A. (1992) Mathematical Problems in Linear Viscoelasticity. SIAM Studies in Applied Mathematics, No. 12, Philadelphia.

[5] Liu, W. (2014) Arbitrary Rate of Decay for a Viscoelastic Equation with Acoustic Boundary Conditions. *Applied Mathematics Letters*, **38**, 155-161.
https://doi.org/10.1090/S0002-9904-1974-13714-6

[6] Beale, J.T. and Rosencrans, S.I. (1974) Acoustic Boundary Conditions. *Bulletin of the AMS*, **80**, 1276-1278.

[7] Frota, C.L. and Goldstein, J.A. (2000) Some Nonlinear Wave Equations with Acoustic Boundary Conditions. *Journal of Differential Equations*, **164**, 92-109.
https://doi.org/10.1006/jdeq.1999.3743

[8] Frota, C.L. and Larkin, N.A. (2006) Uniform Stabilization for a Hyperbolic Equation with Acoustic Boundary Conditions in Simple Connected Domains. In: Cazenave, T., Costa, D., Lopes, O., Manásevich, R., Rabinowitz, P., Ruf, B. and Tomei, C., Eds., *Contributions to Nonlinear Analysis*, Vol. 66, Birkhauser, Basel, 297-312.
https://doi.org/10.1007/3-7643-7401-2_20

[9] Park, J.Y. and Park, S.H. (2011) Decay Rate Estimates for Wave Equations of Memory Type with Acoustic Boundary Conditions. *Nonlinear Analysis*, **74**, 993-998.

[10] Munoz Rivera, J.E. and Qin, Y. (2003) Polynomial Decay for the Energy with an Acoustic Boundary Condition. *Applied Mathematics Letters*, **16**, 249-256.

[11] Dafermos, C.M. (1970) Asymptotic Stability in Viscoelasticity. *Archive for Rational Mechanics and Analysis*, **37**, 297-308. https://doi.org/10.1007/BF00251609

[12] Dafermos, C.M. (1970) An Abstract Volterra Equation with Applications to Linear Viscoelasticity. *Journal of Differential Equations*, **7**, 554-569.

[13] Berrimi, S. and Messaoudi, S.A. (2006) Existence and Decay of Solutions of a Viscoelastic Equation with a Nonlinear Source. *Nonlinear Analysis*, **64**, 2314-2331.

[14] Cavalcanti, M.M., Domingos Cavalcanti, V.N. and Martinez, P. (2008) General Decay Rate Estimates for Viscoelastic Dissipative Systems. *Nonlinear Analysis*, **68**, 177-193.

[15] Liu, W.J. and Sun, Y. (2014) General Decay of Solutions for a Weak Viscoelastic Equation with Acoustic Boundary Conditions. *Zeitschrift für Angewandte Mathematik und Physik*, **65**, 125-134. https://doi.org/10.1007/s00033-013-0328-y

[16] Messaoudi, S.A. and Tatar, N.-E. (2007) Global Existence and Uniform Stability of Solutions for a Quasilinear Viscoelastic Problem. *Mathematical Methods in the Applied Sciences*, **30**, 665-680. https://doi.org/10.1002/mma.804

[17] Messaoudi, S.A. and Tatar, N.-E. (2009) Exponential Decay for a Quasilinear Viscoelastic Equation. *Mathematische Nachrichten*, **282**, 1443-1450. https://doi.org/10.1002/mana.200610800

[18] Messaoudi, S.A. (2008) General Decay of Solutions of a Viscoelastic Equation. *Journal of Mathematical Analysis and Applications*, **341**, 1457-1467.

[19] Wu, S.T. (2012) General Decay of Energy for a Viscoelastic Equation with Damping and Source Terms. *Taiwanese Journal of Mathematics*, **16**, 113-128.

[20] Han, X. and Wang, M. (2009) General Decay of Energy for a Viscoelastic Equation with Nonlinear Damping. *Mathematical Methods in the Applied Sciences*, **32**, 346-358. https://doi.org/10.1002/mma.1041

[21] Park, J.Y. and Park, S.H. (2009) General Decay for Quasiliear Viscoelastic Equations with Nonlinear Weak Damping. *Journal of Mathematical Physics*, **50**, 1-10. https://doi.org/10.1063/1.3187780

[22] Fabrizio, M. and Polidoro, S. (2002) Asymptotic Decay for Some Differential Systems with Fading Memory. *Applicable Analysis*, **81**, 1245-1264. https://doi.org/10.1080/0003681021000035588

[23] Tatar, N. (2008) Polynomial Stability without Polynomial Decay of the Relaxation Function. *Mathematical Methods in the Applied Sciences*, **31**, 1874-1886. https://doi.org/10.1002/mma.1018

[24] Yu, S.Q. (2009) Polynomial Stability of Solutions for a System of Non-Linear Viscoelastic Equations. *Applicable Analysis*, **88**, 1039-1051. https://doi.org/10.1080/00036810903114825

[25] Araújo, R.O., Ma, T. and Qin, Y. (2013) Long-Time Behavior of a Quasilinear Viscoelastic Equation with Past History. *Journal of Differential Equations*, **254**, 4066-4087.

[26] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Prates Filho, J.A. and Soriano, J.A. (2001) Existence and Uniform Decay Rates for Viscoelastic Problems with Non-Linear Boundary Damping. *Differential and Integral Equations*, **14**, 85-116.

[27] Chueshov, I. and Lasiecka, I. (2008) Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping. Vol. 195, Memoirs of the American Mathematical Society, Providence.

[28] Munoz Rivera, J.E., Lapa, E.C. and Barreto, R. (1996) Decay Rates for Viscoelastic Plates with Memory. *Journal of Elasticity*, **44**, 61-87. https://doi.org/10.1007/BF00042192

[29] Wu, S.T. (2011) General Decay of Solutions for a Viscoelastic Equation with Nonlinear Damping and Source Terms. *Acta Mathematica Scientia*, **31**, 1436-1448.

[30] Han, X. and Wang, M. (2009) Global Existence and Uniform Decay for a Nonlinar Viscoelastic Equation with Damping. *Nonlinear Analysis*, **70**, 3090-3098.

[31] Qin, Y., Feng, B. and Zhang, M. (2014) Uniform Attractors for a Non-Autonomous Viscoelastic Equation with a Past History. *Nonlinear Analysis*, **101**, 1-15.

[32] Pata, V. (2006) Exponential Stability in Linear Viscoelasticity. *Quarterly of Applied Mathematics*, **6**, 499-513. https://doi.org/10.1090/S0033-569X-06-01010-4