Journal of Applied Mathematics and Physics, 2017, 5, 844-852
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Exact Solution of the Space-Time Fractional Modified Kdv-Zakharov-Kuznetsov Equation

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How to cite this paper: Jin, Q.Y., Xia, T.C. and Wang, J.B. (2017) The Exact Solution of the Space-Time Fractional Modified Kdv-Zakharov-Kuznetsov Equation. *Journal of Applied Mathematics and Physics*, **5**, 844-852.
https://doi.org/10.4236/jamp.2017.54074

Received: March 3, 2017
Accepted: April 25, 2017
Published: April 28, 2017

Abstract

In this paper, we get many new analytical solutions of the space-time nonlinear fractional modified KDV-Zakharov Kuznetsov (mKDV-ZK) equation by means of a new approach namely method of undetermined coefficients based on a fractional complex transform. These solutions have physics meanings in natural sciences. This method can be used to other nonlinear fractional differential equations.

Keywords

Analytical Solutions, the Space-Time Fractional Modified KDV-ZK Equation,Nonlinear Fractional Differential Equation, Modified Riemann-Liouville Derivative

1. Introduction

Nonlinear fractional differential equations (NFDEs) are universal models of the classical differential equations of integer order. In recent years, the fractional order derivative and integral is becoming a hot spot of international research; it can more accurately describe the nonlinear phenomena in physics. Such as chemical kinematics, chemical physics and geochemistry, communication, physics, biology, engineering, mathematics, diffusion processes in porous media, in vibrations in a nonlinear string, power-law non-locality, and power-law long-term memory can use NFDEs as models to express these problem [1] [2] [3] [4] [5]. In the last few years, it has become an important issue and matter of interest for researchers about the study of analytical and numerical solutions of fractional differential equations (FDEs). There are a lot of effective methods which can be used to study soliton, such as the fractional functional sub-equation method [6], the fractional modified trial equation method [7], the first integral method

[8], the fractional functional variable method [9], the extended tanh-function method [10], the (G'/G)-expansion method [11] [12] and so on.

The present article aims to find out the modified KDV-Zakharov Kuznetsov [13] [14] [15] equation's exact solutions by using named method of undetermined coefficients. The following is the organization of this paper. Some basic definitions and mathematical preliminaries of the fractional calculus are introduced in the next section. Investigated method of undetermined coefficients applied to solve fractional differential equations based on a fractional complex transform is presented in Section 3. In Section 4, we apply method of undetermined coefficients to the space-time nonlinear fractional modified KDV-ZK equation. Finally, we give some conclusions.

2. Basic Definitions

Fractional calculus is a generalization of classical calculus. There are a lot of approaches developed over years to generalize the concept of fractional order derivative, such as, Riemann-Liouville, Grünwald-Letnikow, Caputo [16], Kolwankar-Gangal, Oldham and Spanier, Miller and Ross, Cresson have presented many methods, and Jumnarie put forward a modified Riemann-Liouville derivative [17] [18].

In the section, the some properties and definitions of the modified Riemann-Liouville derivative that will be applied in the sequel of the work were given.

The following is the modified Riemann-Liouville derivative defined by Jumarie [17] [18]

$$D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^t (t-\xi)^{-\alpha-1} \left[f(\xi) - f(0) \right] \mathrm{d}\xi, \ \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^t (t-\xi)^{-\alpha} \left[f(\xi) - f(0) \right] \mathrm{d}\xi, \ 0 < \alpha < 1, \\ \left(f^{(n)}(t) \right)^{(\alpha-n)}, \ n \le \alpha \le n+1, \ n \ge 1. \end{cases} \tag{1}$$

Remark1. $f: R \to R, t \to f(t)$ denote a continuous but not necessarily differentiable function.

The probability calculus, fractional Laplace problems, and fractional variational calculus successfully applied Jumarie's modified Riemann-Liouville derivative. To summarize a few useful formulae by Jumarie's modified Riemann-Liouville derivative in [17] [18], we give some properties as follows

$$D_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0, \tag{2}$$

$$D_t^{\alpha} \left(c f(t) \right) = c D_t^{\alpha} f(t), \quad c = \text{constant}, \tag{3}$$

$$D_t^{\alpha} f\left[g(t) \right] = f_g'\left[g(t) \right] D_t^{\alpha} g(t), \tag{4}$$

$$D_t^{\alpha} f\left[g(t) \right] = D_g^{\alpha} f\left[g(t) \right] (g')^{\alpha}, \tag{5}$$

$$D_t^\alpha\left[af(t)+bg(t)\right]=aD_t^\alpha f(t)+bD_t^\alpha g(t). \tag{6}$$

Remark 2. J. H. He *et al.* in [19] modified the chain rule given by Equation (5) to the formula

$$D_t^\alpha f\left[g(t)\right]=\sigma_t' f_g'\left[g(t)\right]D_t^\alpha g(t), \tag{7}$$

where σ_t' is called the sigma indexes (see [19]). Therefore, Equation (5) is modified to the forms

$$D_t^\alpha f\left[g(t)\right]=\sigma_t' D_g^\alpha f\left[g(t)\right](g')^\alpha . \tag{8}$$

3. Method of Undetermined Coefficients

In the section, we introduce the generally steps of method of undetermined coefficients

Step 1: We set a nonlinear fractional order partial differential equation as follows

$$P\left(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, \cdots\right)=0,\ 0<\alpha,\beta<1 \tag{9}$$

where u is an unknown function about x,t two independent variables, $D_t^\alpha u, D_x^\alpha u$ modified Riemann-Liouville derivative of u, and P is a polynomial of u and its partial fractional derivatives, in which includes the highest order derivatives and the nonlinear terms.

Step 2: By using the traveling wave transformation

$$\begin{aligned} &u(x,t)=U(\xi),\\ &\xi=\frac{kx^\beta}{\Gamma(\beta+1)}-\frac{ct^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{10}$$

where k and c are non zero arbitrary constants. And by using the chain rule

$$\begin{aligned} D_t^\alpha u&=\sigma_t'\frac{\mathrm{d}U}{\mathrm{d}\xi}D_t^\alpha\xi,\\ D_x^\alpha u&=\sigma_x'\frac{\mathrm{d}U}{\mathrm{d}\xi}D_x^\alpha\xi, \end{aligned} \tag{11}$$

where σ_t' and σ_x' are called the sigma index. The sigma index usually is determined by gamma function [20]. In general, we can take $\sigma_t'=\sigma_x'=l$, where l is a constant.

Substituting (10) along with (2) and (11) into (9), we can rewrite Equation (9) in the following nonlinear ordinary differential equation

$$Q\left(U, U', U'', U''', \cdots\right)=0, \tag{12}$$

where the prime denotes the derivative with respect to ξ. For the convenience of calculation, we should obtain a new equation by integrating Equation (12) term by term one or more times.

Step 3: By the following form [21], assume that solution of the Equation (14) can be represented

$$U(\xi)=A\operatorname{sech}^m\xi, \tag{13}$$

where A is nonzero constant, m is obtained by balancing the highest order term and nonlinear term of Equation (9) or Equation (12).

Step 4: Substituting the constant A and m into Equation (14), we can obtain the solution of the fractional order Equation (9).

4. The (3 + 1) Dimensional Space-Time Fractional mKDV-ZK Equation

In this current sub-section, we apply method of undetermined coefficients to solve the (3 + 1) dimensional space-time fractional mKDV-ZK equation of the form,

$$D_t^{\alpha}u + \mathrm{d}u^2u_x + eu_{xxx} + fu_{xyy} + gu_{xzz} = 0, \quad t > 0,\ 0 < \alpha < 1, \tag{14}$$

where d, e, f and g are nonzero constants, α is a parameter describing the order of the fractional space-time-derivative. When $f = 0$, $g = 0$, d, $e \neq 0$, Equation (14) is called the fractional modified KDV equation

$$D_t^{\alpha}u + \mathrm{d}u^2u_x + eu_{xxx} = 0, \quad t > 0,\ 0 < \alpha < 1, \tag{15}$$

when $\alpha = 1$, Equation (14) is called the modified KDV-ZK equation

$$u_t + \mathrm{d}u^2u_x + eu_{xxx} + fu_{xyy} + gu_{xzz} = 0, \quad t > 0. \tag{16}$$

The modified KDV-ZK equation is applied in many physical areas. Existence of the solutions for this equation has been considered in several papers, see references in [22] [23]. Next, we will obtain the non-topological soliton and dark soliton solutions to Equation (14) by method of undetermined coefficients [24] [25].

Therefore, we use the following transformations,

$$u(x, y, z, t) = U(\xi), \quad \xi = kx + py + qz - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}, \tag{17}$$

Where k, p, q, λ are nonzero constants.

Substituting Equation (17) with Equation (2) and Equation (11) into Equation (14), we have

$$-\lambda U' + kdU^2U' + k^3eU''' + kfp^2U''' + kgq^2U''' = 0, \tag{18}$$

where "U'" $= \frac{\mathrm{d}U}{\mathrm{d}\xi}$. By once integrating and setting the constants of integration to zero, we obtain

$$-\lambda U + \frac{kd}{3}U^3 + k\left(ek^2 + fp^2 + gq^2\right)U'' = 0. \tag{19}$$

4.1. The Non-Topological Soliton Solution

To get the non-topological soliton solution of Equation (19), we can make the assumption,

$$U(\xi) = A\mathrm{sech}^m \xi, \tag{20}$$

where

$$\xi = kx + py + qz - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}, \tag{21}$$

where k, p, q, λ are nonzero constants coefficients. The m is unknown at this point and will be determined later. From the Equation (20)-(21), we obtain

$$\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi} = -Am\operatorname{sech}^{m+1}\xi \sinh\xi, \tag{22}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2U(\xi)}{\mathrm{d}\xi^2} &= -Am(m+1)\operatorname{sech}^{m}\xi(-\operatorname{sech}\xi \tanh\xi)\sinh\xi + \left(-Am\operatorname{sech}^{m+1}\xi\cosh\xi\right)\\ &= Am(m+1)\operatorname{sech}^{m+2}\xi\sinh^2\xi - Am\operatorname{sech}^{m}\xi\\ &= Am(m+1)\operatorname{sech}^{m+2}\xi\left(\frac{1}{\sec h^2\xi}-1\right) - Am\operatorname{sech}^{m}\xi\\ &= Am^2\operatorname{sech}^{m}\xi - Am(m+1)\operatorname{sech}^{m+2}\xi,\end{aligned} \tag{23}$$

and

$$U^3(\xi) = A^3\operatorname{sech}^{3m}\xi. \tag{24}$$

Thus, substituting the ansatz (23)-(27) into Equation (21), yields to

$$\begin{aligned}&-\lambda A\operatorname{sech}^{m}\xi + \frac{kd}{3}A^3\operatorname{sech}^{3m}\xi\\ &+k\left(ek^2+fp^2+gq^2\right)\left(Am^2\operatorname{sech}^{m}\xi - Am(m+1)\operatorname{sech}^{m+2}\xi\right)=0.\end{aligned} \tag{25}$$

Now, from Equation (25), equating the exponents $m+2$ and $3m$ leads to

$$m+2=3m, \tag{26}$$

so that

$$m=1. \tag{27}$$

From Equation (25), setting the coefficients of $\operatorname{sech}^{m+2}\xi$ and $\operatorname{sech}^{3m}\xi$ terms to zero, we obtain

$$\frac{kd}{3}A^3 - Ak\left(ek^2+fp^2+gq^2\right)m(m+1)=0, \tag{28}$$

by using Equation (27) and after some calculations, we have

$$A = \pm\sqrt{\frac{6\left(ek^2+fp^2+gq^2\right)}{d}}. \tag{29}$$

We find, from setting the coefficients of $\operatorname{sech}^{m}\xi$ terms in Equation (25) to zero

$$-\lambda A + Am^2k\left(ek^2+fp^2+gq^2\right)=0, \tag{30}$$

also we get

$$\lambda = k\left(ek^2+fp^2+gq^2\right). \tag{31}$$

From Equation (29), it is important to note that

$$d\left(ek^2+fp^2+gq^2\right)>0. \tag{32}$$

Thus finally, the 1-soliton solution of Equation (14) is given by:

$$u_1\left(x,y,z,t\right)=\sqrt{\frac{6\left(ek^2+fp^2+gq^2\right)}{d}}\operatorname{sech}\left(kx+py+qz-\frac{d\left(ek^2+fp^2+gq^2\right)t^\alpha}{\Gamma\left(1+\alpha\right)}\right), \tag{33}$$

$$u_2\left(x,y,z,t\right)=-\sqrt{\frac{6\left(ek^2+fp^2+gq^2\right)}{d}}\operatorname{sech}\left(kx+py+qz-\frac{d\left(ek^2+fp^2+gq^2\right)t^\alpha}{\Gamma\left(1+\alpha\right)}\right). \tag{34}$$

4.2. The Dark Soliton Solution

In order to start off with the solution hypothesis, we use the solitary wave ansatz of the form

$$U\left(\xi\right)=A\tanh^m\xi, \tag{35}$$

and

$$\xi=kx+py+qz-\frac{\lambda t^\alpha}{\Gamma\left(1+\alpha\right)}, \tag{36}$$

where k, p, q, λ are the free parameters. Also the m is unknown at this point and will be determined later.

From Equations (35)-(36), we obtain

$$\frac{\mathrm{d}U\left(\xi\right)}{\mathrm{d}\xi}=Am\left(\tanh^{m-1}\xi-\tanh^{m+1}\xi\right) \tag{37}$$

and

$$\frac{\mathrm{d}^2U\left(\xi\right)}{\mathrm{d}\xi^2}=Am\left\{\left(m-1\right)\tanh^{m-2}\xi-2m\tanh^m\xi+\left(m+1\right)\tanh^{m+2}\xi\right\}, \tag{38}$$

and

$$U^3\left(\xi\right)=A^3\tanh^{3m}\xi. \tag{39}$$

Substituting Equations (35)-(39) into Equation (19), gives

$$\begin{aligned}&-\lambda A\tanh^m\xi+\frac{kd}{3}A^3\tanh^{3m}\xi\\&+Amk\left(ek^2+fp^2+gq^2\right)\left\{\left(m-1\right)\tanh^{m-2}\xi\right.\\&\left.-2m\tanh^m\xi+\left(m+1\right)\tanh^{m+2}\xi\right\}=0.\end{aligned} \tag{40}$$

Now, from Equation (40), equating the exponents of $\tanh^{3m}\xi$ and $\tanh^{m+2}\xi$ gives,

$$3m=m+2, \tag{41}$$

which yields

$$m=1. \tag{42}$$

Setting the coefficients of $\tanh^{3m}\xi$ and $\tanh^{m+2}\xi$ terms in Equation (40) to zero, we have

$$\frac{kd}{3}A^3 + Akm(m+1)\left(ek^2 + fp^2 + gq^2\right) = 0, \tag{43}$$

then, we get

$$A = \pm\sqrt{-\frac{6\left(ek^2 + fp^2 + gq^2\right)}{d}}. \tag{44}$$

Again, from Equation (40) setting the coefficients of $\tanh^m \xi$ terms to zero,

$$-\lambda A - 2m^2 Ak\left(ek^2 + fp^2 + gq^2\right) = 0, \tag{45}$$

and from Equation (45) we have

$$\lambda = -2k\left(ek^2 + fp^2 + gq^2\right). \tag{46}$$

Equation (46) prompts the constraint

$$d\left(ek^2 + fp^2 + gq^2\right) < 0. \tag{47}$$

Thus finally, the dark soliton solution for the (3 + 1) dimensional space-time fractional mKDV-ZK equation is given by:

$$u_3(x,y,z,t) = \sqrt{-\frac{6\left(ek^2 + fp^2 + gq^2\right)}{d}}\tanh\left(kx + py + qz + \frac{2k\left(ek^2 + fp^2 + gq^2\right)}{\Gamma(1+\alpha)}\right), \tag{48}$$

$$u_4(x,y,z,t) = -\sqrt{-\frac{6\left(ek^2 + fp^2 + gq^2\right)}{d}}\tanh\left(kx + py + qz + \frac{2k\left(ek^2 + fp^2 + gq^2\right)}{\Gamma(1+\alpha)}\right). \tag{49}$$

5. Conclusion

In this article, we have got the new solutions for the (3 + 1) dimensional space-time fractional mKDV-ZK equation by using the method of undetermined coefficients. Up to now, we could not find that these solutions were reported in other papers. In order to solve many systems of nonlinear fractional partial differential equations in mathematical and physical sciences, such as, the space-time fractional mBBM equation, the time fractional mKDV equation, the nonlinear fractional Zoomeron equation and so on, we can use the method of undetermined coefficients recommended herein would be general to a certain extent.

Acknowledgements

This work is in part supported by the Natural Science Foundation of China (Grant Nos. 11271008, 61072147).

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