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Travelling Waves in Space-Fractional Nonlinear Diffusion with Linear Convection

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Abstract

In this paper we investigate anomalous diffusion coupled with linear convection, using fractional calculus to describe the anomalous associated memory effects in diffusive term. We get an explicit travelling wave solution, wavefront, with finite propagation. We comment the properties of the solution, including the stationary case.

Keywords

Fractional Convection-Diffusion, Riesz Derivative, Travelling Wave, Wavefront

1. Introduction

The fractional calculus is almost as old as the calculus of integer order, but only in 1974 was realized the first Conference on Fractional Calculus its Applications. In the decade 90s fractional calculus was recognized and specific journals and textbooks were being published [1] [2] [3]. Nowadays, many definitions have appeared in fractional calculus that shows us some difficulties and limitations of the theory with applications [4] [5] [6] [7]. The researchers have investigated in diffusion processes the anomalous related memory effects [8] [9] [10] [11], for example, some materials the moisture propagates according to the $x/t^{\alpha/2}$ scaling, with $0<\alpha<2$ [12] [13]. Costa *et al.* [14] comment that water transport for large distance in a relatively short time (groundwater infiltration problem) can be described for a fractional space-time nonlinear diffusion equation.

We study the diffusion with linear convection, that is, a nonlinear convection-diffusion problem. Appearing in several physical, biological and chemical applications, B. H. Gilding and R. Kersner [15] highlight the study of pattern formation by bacterial colonies analyzed in [16]. The investigation of that problem with the fractional operators has shown that fractional derivatives in convection-diffusion equation give more information about the anomalous associated with effect memory [17] [18] [19]. We are interested in travelling wave solution given by similarity reductions to fractional equation.

Our purpose is to find an explicit solution, so that we can investigate its properties in the fractional model. In Section 2 we define the Riesz fractional derivative. In Section 3 the fractional model used to describe the convection-diffusion equation using Riesz fractional derivative. In Section 4 we use the similarity reduction method in fractional equation. In Section 5 we calculate the particular travelling wave, called a wavefront. Section 6 makes our concluding remarks.

2. Riesz Fractional Derivative

The Riesz fractional derivative of order α, with $0<\alpha<2$ and $\alpha\neq 1$ is defined by:

$$D_x^\alpha f(x) = -\frac{D_+^\alpha f(x) + D_-^\alpha f(x)}{2\cos(\alpha\pi/2)}, \tag{1}$$

where $D_\pm^\alpha f(x)$ are Weyl fractional derivatives [20] [21].

Theorem 1. *Let be* $h(x)=|x|^{-\alpha-1}$, *with* $1<\alpha<2$. *We describe the Riesz fractional derivative of order* α *for an appropriate Fourier convolution product is given by*:

$$D_x^\alpha f(x) = d_\alpha (f * h)(x), \tag{2}$$

with $d_\alpha = -\dfrac{1}{2\Gamma(-\alpha)\cos\left(\dfrac{\alpha\pi}{2}\right)}$.

This result is an improvement of the theorem developed by E. C. Grigoletto and E. C. de Oliveira [22].

3. Fractional Convection-Diffusion

Analyze a fractional space nonlinear convection-diffusion equation, in which we apply the Riesz fractional derivative in diffusive term. If we consider only diffusion, Equation (3) defines a fractional porous medium equation [23] [24] [25].

$$\frac{\partial}{\partial t}u(x,t) = D_x^\alpha u^n(x,t) + \frac{\partial}{\partial x}u(x,t), \quad n>1, t>0 \text{ and } x\in\mathbb{R}, \tag{3}$$

where D_x^α is Riesz fractional derivative, with $1<\alpha<2$. In the case $\alpha\to 2$, we get the integer classical convection-diffusion equation. The parameter n appears, when we consider the diffusion coefficient $nu^{n-1}(x,t)$, that is, density dependent diffusivity, in that n is determined by properties of the medium and phenomenon investigated.

An important application arises in a model describing the unsaturated flow of a fluid through a homogeneous porous column under the influence of capillary pressure and gravity. As example, we cite water movement in a vertical column of the medium, if water movement in a horizontal column of the medium, the problem reduces to the flow porous medium.

4. Travelling waves

Use the similarity reduction method in Equation (3), to find the travelling wave solutions $g(x \pm ct)$, where $g(x \pm ct) = x^d g\left(1 \pm c\frac{t}{x}\right)$, with d is related to the homogeneity degree.

$$u(x,t) = x^a U(\eta), \quad \eta = 1 \pm c\frac{t}{x}, \tag{4}$$

where $t > 0$, $x \in \mathbb{R} - \{0\}$ and a is unknown.

We apply Equation (4) on the left-side hand of Equation (3):

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}\left[x^a U\left(1 \pm c\frac{t}{x}\right)\right] = (\pm c)x^{a-1}\frac{\partial}{\partial \eta}U(\eta). \tag{5}$$

In relation to the right-side hand of Equation (3). We calculate Riesz fractional derivative, using **Theorem 1**:

$$\begin{aligned} D_x^\alpha u^n(x,t) &= D_x^\alpha\left[x^{an}U^n\left(1 \pm c\frac{t}{x}\right)\right] = d_\alpha\left[x^{an}U^n\left(1 \pm c\frac{t}{x}\right)\right] * |x|^{-\alpha-1} \\ &= d_\alpha \int_{-\infty}^{+\infty} s^{an}U^n\left(1 \pm c\frac{t}{s}\right)|x-s|^{-\alpha-1}\,\mathrm{d}s \\ &= d_\alpha\left[\int_x^{+\infty}(s-x)^{-\alpha-1}s^{an}U^n\left(1 \pm c\frac{t}{s}\right)\mathrm{d}s + \int_{-\infty}^x (x-s)^{-\alpha-1}s^{an}U^n\left(1 \pm c\frac{t}{s}\right)\mathrm{d}s\right]. \end{aligned} \tag{6}$$

The integrals are given follows:

$$\begin{aligned} &\int_{-\infty}^x (x-s)^{-\alpha-1}s^{an}U^n\left(1 \pm c\frac{t}{s}\right)\mathrm{d}s \\ &= x^{an-\alpha}(1-\eta)^{1+an}\int_1^\eta(\eta-\tau)^{-1-\alpha}(1-\tau)^{-1+\alpha-an}U^n(\tau)\,\mathrm{d}\tau \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\int_x^{+\infty}(s-x)^{-\alpha-1}s^{an}U^n\left(1 \pm c\frac{t}{s}\right)\mathrm{d}s \\ &= (-1)^{-\alpha}x^{an-\alpha}(1-\eta)^{1+an}\int_1^\eta(\eta-\psi)^{-1-\alpha}(1-\psi)^{-1+\alpha-an}U^n(\psi)\,\mathrm{d}\psi \end{aligned} \tag{8}$$

The Riesz fractional derivative is given by:

$$\begin{aligned} &D_x^\alpha u^n(x,t) \\ &= d_\alpha\left[1+(-1)^{-\alpha}\right]x^{an-\alpha}(1-\eta)^{1+an}\int_1^\eta(\eta-\psi)^{-1-\alpha}(1-\psi)^{-1+\alpha-an}U^n(\psi)\,\mathrm{d}\psi. \end{aligned} \tag{9}$$

Replacing, Equation (5) and Equation (9) in Equation (3), we get:

$$\begin{aligned} &(\pm c)x^{a-1}\frac{\partial}{\partial\eta}U(\eta) \\ &= d_\alpha\left[1+(-1)^{-\alpha}\right]x^{an-\alpha}(1-\eta)^{1+an}\int_1^\eta(\eta-\psi)^{-1-\alpha}(1-\psi)^{-1+\alpha-an}U^n(\psi)\,\mathrm{d}\psi \\ &\quad + x^{a-1}\left[a+(1-\eta)\right]\frac{\partial}{\partial\eta}U(\eta) \end{aligned} \tag{10}$$

Imposing invariance on the variable x for Equation (10):

$$a-1=an-\alpha \Rightarrow a=\frac{\alpha-1}{n-1} \tag{11}$$

We obtain the equation:

$$\begin{aligned}&(\pm c)\frac{\partial}{\partial \eta}U(\eta)\\&=-d_{\alpha}\left[1+(-1)^{-\alpha}\right](1-\eta)^{1+an}\int_{\eta}^{1}(\eta-\psi)^{-1-\alpha}(1-\psi)^{-1+\alpha-an}U^{n}(\psi)\mathrm{d}\psi\\&\quad+\left[a+(1-\eta)\frac{\partial}{\partial \eta}\right]U(\eta)\end{aligned} \tag{12}$$

5. Wavefronts

Look for a travelling wave, called wavefront [26] [27] [28]. Let us consider:

$$U(\eta)=\begin{cases}A\eta^{k}, & \eta<0\\0, & \eta\geq 0\end{cases} \tag{13}$$

where A and k are unknowns. To determine these variables, we replace this equation in Equation (12). The calculus of the integral is given by:

$$\begin{aligned}&d_{\alpha}\left[1+(-1)^{-\alpha}\right](1-\eta)^{1+an}\int_{\eta}^{1}(\eta-\psi)^{-1-\alpha}(1-\psi)^{-1+\alpha-an}U^{n}(\psi)\mathrm{d}\psi=\\&=d_{\alpha}\left[1+(-1)^{-\alpha}\right](1-\eta)^{1+an}\int_{\eta}^{0}(\eta-\psi)^{-1-\alpha}(1-\psi)^{-1+\alpha-an}A^{n}\psi^{\kappa n}\mathrm{d}\psi\\&=d_{\alpha}\left[1+(-1)^{-\alpha}\right]\eta^{\kappa n-\alpha}(1-\eta)^{1+an}A^{n}\int_{1}^{0}(1-\mu)^{-1-\alpha}(1-\eta\mu)^{-1+\alpha-an}\mu^{\kappa n}\mathrm{d}\mu\\&=d_{\alpha}\left[1+(-1)^{-\alpha}\right]A^{n}\eta^{kn-\alpha}\\&\quad\times(1-\eta)^{1+an}\frac{(-1)\Gamma(-\alpha)\Gamma(1+\kappa n)}{\Gamma(1-\alpha+\kappa n)}\,{}_{2}F_{1}(1-\alpha+an,1+\kappa n;1-\alpha+\kappa n;\eta),\end{aligned} \tag{14}$$

where ${}_{2}F_{1}$ is hypergeometric function. Thus, we obtain the relation:

$$\begin{aligned}&(\pm c)Ak\eta^{k-1}\\&=-d_{\alpha}\left[1+(-1)^{-\alpha}\right]A^{n}\eta^{kn-\alpha}\\&\quad\times(1-\eta)^{1+an}\frac{(-1)\Gamma(-\alpha)\Gamma(1+kn)}{\Gamma(1-\alpha+kn)}\,{}_{2}F_{1}(1-\alpha+an,1+kn;1-\alpha+kn;\eta)\\&\quad+A\left[(a-k)\eta^{k}+k\eta^{k-1}\right].\end{aligned} \tag{15}$$

Once again, considering $k=a$, we impose invariance on the variable η for Equation (15), that gives us:

$$(\pm c)kA=-d_{\alpha}\left[1+(-1)^{-\alpha}\right]A^{n}\frac{(-1)\Gamma(-\alpha)\Gamma(1+kn)}{\Gamma(1-\alpha+kn)}+kA. \tag{16}$$

Thus,

$$A^{n-1}=-\frac{\exp(i\alpha\pi/2)\Gamma(1-\alpha+kn)}{\Gamma(1+kn)}k\left[(\pm c)-1\right], \tag{17}$$

and using Equation (11), we have:

$$A=\left\{-\frac{\exp(i\alpha\pi/2)\Gamma(1+k)}{\Gamma(1+kn)}\left[(\pm c)-1\right]\right\}^{\frac{1}{n-1}}. \tag{18}$$

The values of A and a in Equation (4),

$$u(x,t)=\begin{cases}\left\{-\frac{\exp(i\alpha\pi/2)\Gamma(1+k)}{\Gamma(1+kn)}\left[(\pm c)-1\right](x\pm ct)^{\alpha-1}\right\}^{\frac{1}{n-1}}, & |x|<ct\\ 0, & |x|\geq ct.\end{cases} \tag{19}$$

The integer classical equation is recovered by taking the limit $\alpha\to 2$ in Equation (3):

$$\frac{\partial}{\partial t}u(x,t)=\frac{\partial^2}{\partial x^2}u^n(x,t)+\frac{\partial}{\partial x}u(x,t), \tag{20}$$

and its solution is given by:

$$u(x,t)=\begin{cases}\left\{\frac{(n-1)}{n}\left[(\pm c)-1\right](x\pm ct)\right\}^{\frac{1}{n-1}}, & |x|<ct\\ 0, & |x|\geq ct,\end{cases} \tag{21}$$

which is same result found by B.H. Gilding and R. Kersner in [27].

The condition $\eta<0$ gives us the relations: $x\in(0,\infty)$ associated to "$-c$" and $x\in(-\infty,0)$ associated to"$+c$", that is, $x\to -x$ implies the change of velocity $-c\to +c$. If we regard $u(x,t)$ in Equation (21) nonnegative, we have $c\geq 0\ x\in(0,\infty)$ and $c>1\ x\in(-\infty,0)$. Therefore, the fractional equation in Equation (3) has only stationary solution in $(0,\infty)$. The relation between the values A and c is establishes in Equation (18).

6. Concluding Remarks

We introduce the space-fractional model based on Riesz fractional derivative with nonlinear diffusion in conjunction linear convection. We obtain the explicit travelling wave solution, with finite propagation, wavefront; that is, Equation (3) admits solution in the region $x<ct$ such that $u(x,t)$ vanishes on the front $x=ct$. This property and the behaviour near the interface are in general described by wavefront; it may give rise to interfaces separating the regions that we have just mentioned. As a continuation of this paper, we can suppose fractional derivatives in time and nonlinear convection for Equation (3) and we search its travelling waves.

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