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Equivalence between Modulus of Smoothness and K-Functional on Rotation Group SO(3)*

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Abstract

In this paper we obtain the equivalence between modulus of smoothness and K-functional on rotation group SO(3).

Keywords

Rotation Group, Modulus of Smoothness, K-Functional, Equivalence

1. Introduction

Many results of approximation are based on Euclid spaces or their compact subsets. Periodic approximation is based on compact group $\{\exp(ix)\}$, whereas matrix group $U(n)$ is the generalization of $\{\exp(ix)\}$. We know homomorphism between SU(2) and rotation group SO(3), which has many applications in Physics and Chemistry. Some approximation problems on compact groups have been studied since in 1920s F. Peter and H. Weyl proved the approximation theorem on compact group, that is, the irreducible character generate a dense subspace of the space of continuous classes function. For instance, Gongsheng (see [1]) studied the basic problems of Fourier analysis on unitary and rotation groups, including the degree of convergence of Abel sum based on Poisson kernel. Xue-an Zheng (see [2] [3]) studied the polynomial approximation on compact Lie groups. D. I. Cartwright *et al.* studied Jackson's theorem for compact connected Lie groups (see [4]), and so on. In this paper, we study the modulus of smoothness and K-functional on rotation group SO(3) and as classical casein Euclid space we will obtain the equivalence between them.

Let $G = SO(3) = \left\{\mathbf{x} \in GL(3;\mathbb{R});\ \mathbf{x}^{\mathrm{T}}\mathbf{x} = \mathbf{E};\ \det\mathbf{x} = 1\right\}$ be the rotation group, where $GL(n,\mathbb{R})$ is the group of invertible real $(n \times n)$ matrices. For $1 \le p < +\infty$,

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$L_p(G)\left\{f:\left[\int_G |f(x)|^p \mathrm{d}\mu(x)\right]^{1/p}<+\infty\right\}$, where μ is the normalized Harr measure on G. For $D_i \in \mathbf{g}$, the Lie algebra of $G = SO(3)$, $i = 1, 2, 3$, Let $L_{p,i}^r(G)=\left\{f:D_i^r f\in L_p(G)\right\}$, $i = 1, 2, 3$, where $D_i^r g$ denote the r-order derivative of g in direction D_i.

We also write the difference of function f and modulus of smoothness in the direction D_i as follows

$$\Delta_{sD_i}^r f(x)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(x\exp jsD_i),$$

and

$$\omega_{r,i}(f,\delta)_p=\sup\left\{\left\|\Delta_{sD_i}^r f\right\|_p : 0<s<\delta, \|D_i\|=1\right\}$$

where $\|D_i\|$ is the norm induced by Killing inner product on $\mathbf{g}$.

We denote

$$W_r(f,\delta)_p=\sum_{i=1}^{3}\omega_{r,i}(f,\delta)_p.$$

Accordingly, we denote K-functional as follows

$$K_{r,i}(f,\delta)_p=\inf_{g\in L_{p,i}^r(G)}\left\{\|f-g\|_p+\delta^r\left\|D_i^r g\right\|_p\right\}$$

$$K_r(f,\delta)_p=\sum_{i=1}^{3}K_{r,i}(f,\delta)_p$$

Further, for the isotropic case.

Let multi-indice $\mathbf{r}=(r_1,r_2,r_3)\in N_+^3$, and $\mathbf{s}=(s_1,s_2,s_3)>\mathbf{0}$, $\mathbf{t}=(t_1,t_2,t_3)$, $\Lambda=\left\{\mathbf{a}=(\alpha_1,\alpha_2,\alpha_3):\mathbf{a}<\mathbf{r}\right\}$, $\partial\Lambda=\left\{r_i e_i, i=1,2,3\right\}$, here e_i is the unit vector in the i-th direction. Define

$$\Delta_s^r f(x)=\Delta_{s_1D_1}^{r_1}\Delta_{s_2D_2}^{r_2}\Delta_{s_3D_3}^{r_3} f(x),$$

and

$$\omega_{\mathbf{r}}(f,\mathbf{t})_p=\sup_{0<\mathbf{s}<\mathbf{t}}\left\|\Delta_{\mathbf{s}}^{\mathbf{r}} f\right\|_p=\sup\left\{\left\|\Delta_{s_1D_1}^{r_1}\Delta_{s_2D_2}^{r_2}\Delta_{s_3D_3}^{r_3} f\right\|_p : \mathbf{0}<\mathbf{s}<\mathbf{t}, \|D_i\|=1, i=1,2,3\right\},$$

and

$$W_\Lambda(f,\mathbf{t})_p=\sum_{\mathbf{r}\in\partial\Lambda}\omega_{\mathbf{r}}(f,\mathbf{t})_p.$$

The corresponding K-functional is defined by

$$K_\Lambda(f,\mathbf{t})_p=\inf_{g\in L_p^\Lambda}\left\{\|f-g\|_p+\sum_{\mathbf{r}\in\partial\Lambda}\mathbf{t}^{\mathbf{r}}\left\|\mathbf{D}^{\mathbf{r}} g\right\|_p\right\},$$

where $L_p^\Lambda=L_p^\Lambda(G)=\left\{f:\mathbf{D}^{\mathbf{r}} f\in L_p(G), \mathbf{r}\in\partial\Lambda\right\}$, $\mathbf{t}^{\mathbf{r}}=t_1^{r_1}t_2^{r_2}t_3^{r_3}$, $\mathbf{D}^{\mathbf{r}}=D_1^{r_1}D_2^{r_2}D_3^{r_3}$.

In the next paragraph we denote by C or C_i the positive constants but are not the same in the different formula. And $A \asymp B$ means there exist two positive constants C_1, C_2 satisfying $C_1A \le B \le C_2A$.

2. Theorems and Their Proofs

We will use the next lemma 1.

Lemma 1 [5] [6]. If $f \in L_1^r[t,t+s]$, then
$\Delta_s^r f(t) = s^r \int_0^{rs} \frac{f^{(r)}(t+u)N_r(u/s)}{s} \mathrm{d}u$, where N_r denotes the normalized B-spline of order r (degree r-1).

Theorem 1. If $f \in L_p(SO(3))$, $1 \le p < +\infty$, $r \in \mathbb{N}_+$, $0 < \delta \in \mathbb{R}$, then

$$W_r(f,\delta)_p \asymp K_r(f,\delta)_p .$$

Proof. For i = 1, 2, 3, we first construct the approximation operators as follows

$$\begin{aligned} g_i(x) &= g_{sD_i}(x) = I_i f(x) \\ &= \int_0^{rs}\left[f(x)+(-1)^{r-1}\Delta_{uD_i}^r f(x)\right]\frac{N_r(u/s)}{s}\mathrm{d}u \\ &= \int_0^{rs}\sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j} f(x\exp(juD_i))\frac{N_r(u/s)}{s}\mathrm{d}u \\ &= \sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}\int_0^{rs} f(x\exp(juD_i))\frac{N_r(u/s)}{s}\mathrm{d}u \\ &= \sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}\int_0^{jrs} f(x\exp(tD_i))\frac{N_r(t/js)}{js}\mathrm{d}t. \end{aligned}$$

By Lemma 1,

$$g_i(x) = \sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}(js)^{-r}\Delta_{jsD_i}^r F(x),$$

where $D^r F = f$.

Obviously, I_i is a bounded operator from L_p to L_p($1 \le p < +\infty$).

If we differentiate r times, then

$$D_i^r g_i(x) = \sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}(js)^{-r}\Delta_{jsD_i}^r f(x),$$

So,

$$s^r \left\|D_i^r g_i\right\|_p \le c_r \left\|\Delta_{sD_i}^r f\right\|_p \le c_r \omega_{r,i}(f,s)_p . \tag{1}$$

Clearly,

$$\left\|f-g_i\right\|_p \le C\omega_{r,i}(f,s)_p .$$

We get

$$K_{r,i}(f,s)_p \le C\omega_{r,i}(f,s)_p ,$$

and

$$K_r(f,\delta)_p \le CW_r(f,\delta)_p .$$

Conversely, for $g_i(x) \in L_{p,i}(G)$, using (see [7])

$$\Delta_{sD_i}^r f(x) = \int_o^s \mathrm{d}t_1 \cdots \int_o^s \mathrm{d}t_r D_i^r g_i\left(x\exp(t_1+\cdots+t_r)D_i\right),$$

we have

$$\begin{aligned} W_r(f,\delta)_p &= \sum_{i=1}^{3}\omega_{r,i}(f,\delta)_p \le \sum_{i=1}^{3}\left[\omega_{r,i}(f-g_i,\delta)_p + \omega_{r,i}(g_i,\delta)_p\right] \\ &\le C\sum_{i=1}^{3}\left[\left\|f-g_i\right\|_p + \delta^r\left\|D_i^r g_i\right\|_p\right]. \end{aligned}$$

Thus

$$W_r(f,\delta)_p \le CK_r(f,\delta)_p .$$

Theorem 2. For $f \in L_p(G), 1 \le p < +\infty$, $\mathbf{t} > 0$, then

$$W_\Lambda(f,\mathbf{t})_p \asymp K_\Lambda(f,\mathbf{t})_p .$$

Proof. Noting that for $g \in L_p^\Lambda$,

$$\omega_{\mathbf{r}}(g,\mathbf{t})_p \le \mathbf{t}^{\mathbf{r}} \left\|\mathbf{D}^{\mathbf{r}} g\right\|_p ,$$

we get

$$W_\Lambda(g,\mathbf{t})_p \le \sum_{\mathbf{r}\in\partial\Lambda} \mathbf{t}^{\mathbf{r}} \left\|\mathbf{D}^{\mathbf{r}} g\right\|_p .$$

Writing $f = f - g + g$ and using the last inequality will give

$$W_\Lambda(f,\mathbf{t})_p \le CK_\Lambda(f,\mathbf{t})_p .$$

Moreover, we construct the approximation operator as follows

$$\begin{aligned} g(x) &= I(f,x) = I_1 I_2 I_3(f,x) \\ &= \int_0^{r_1 s_1}\int_0^{r_2 s_2}\int_0^{r_3 s_3}\left[f(x) + \Delta_{\mathbf{s}}^{\mathbf{r}} f(x)\right] N_{\mathbf{r}}(\mathbf{s},\mathbf{u})\,\mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3, \end{aligned}$$

where

$$N_{\mathbf{r}}(\mathbf{s},\mathbf{u}) = \frac{N_{r_1}(u_1/s_1) N_{r_2}(u_2/s_2) N_{r_3}(u_3/s_3)}{s_1 s_2 s_3}.$$

It easy to see that by using the boundedness of I_i, $i = 1, 2, 3$.

$$\begin{aligned} &\left\|f-g\right\|_p \le \left\|f - I_1 f\right\|_p + \left\|I_1 f - I_1 I_2 f\right\|_p + \left\|I_1 I_2 f - I_1 I_2 I_3 f\right\|_p \\ &\le C\sum_{i=1}^{3} \omega_{r,i}(f,t_i)_p = CW_\Lambda(f,\mathbf{t})_p \end{aligned}$$

It is similarly to (1), we have

$$\mathbf{t}^{\mathbf{r}} \left\|D^{\mathbf{r}} g\right\|_p \le \left\|\Delta_{\mathbf{s}}^{\mathbf{r}} f\right\|_p \quad \text{and} \quad \sum_{\mathbf{r}\in\partial\Lambda} \mathbf{t}^{\mathbf{r}} \left\|D^{\mathbf{r}} g\right\|_p \le W_\Lambda(f,\mathbf{t}).$$

Thus $K_\Lambda(f,\mathbf{t})_p \le CW_\Lambda(f,\mathbf{t})_p$.

Remark: Theorem 1 and theorem 2 can be easily generalized to SO(n) ($n > 3$).

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