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Cordial Volterra Integral Equations with Vanishing Delays*

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Abstract

Cordial Volterra integral equations (CVIEs) from some applications models associated with a noncompact cordial Volterra integral operator are discussed in the recent years. A lot of real problems are effected by a delayed history information. In this paper we investigate some properties of cordial Volterra integral operators influenced by a vanishing delay. It is shown that to replicate all eigenfunctions t^{λ}, $\lambda = 0$ or $\Re\lambda > 0$, the vanishing delay must be a proportional delay. For such a linear delay, the spectrum, eigenvalues and eigenfunctions of the operators and the existence, uniqueness and solution spaces of solutions are presented. For a nonlinear vanishing delay, we show a necessary and sufficient condition such that the operator is compact, which also yields the existence and uniqueness of solutions to CVIEs with the vanishing delay.

Keywords

Cordial Volterra Integral Equations, Vanishing Delay, Propositional Delay, Compactness, Existence and Uniqueness

1. Introduction

A kind of Volterra integral equations with weakly singular kernels arisen in 1975 [1] from some heat condition problems with mixed-type boundary conditions is transformed by Watson transforms [2] and the convolution theorem [3]. In [4], the author generalizes such kind of equations into cordial Volterra integral equations (CVIEs) with the form

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$$\mu u(t) = f(t) + (\mathcal{V}_\varphi u)(t), \quad t \in I := [0,T], \tag{1}$$

where $T > 0$, the core $\varphi \in L^1(0,1)$ and the cordial Volterra integral operator is defined by

$$(\mathcal{V}_\varphi u)(t) := \int_0^t t^{-1}\varphi(s/t)u(s)\mathrm{d}s = \int_0^1 \varphi(x)u(tx)\mathrm{d}x.$$

CVIEs appear in a lot of application models, such as Diogo core $\varphi(x) = (1-x^2)^{-\frac{1}{2}}$, linear Lighthill's equation ($\varphi(x) = \left(1-x^{\frac{3}{2}}\right)^{-\frac{2}{3}}$), and so on.

It is shown that the cordial Volterra integral operator $\mathcal{V}_\varphi$ in the Banach space $C(I)$ is noncompact and its spectrum is a non-countable set, *i.e.*,

$$\sigma(\mathcal{V}_\varphi) = \{0\} \cup \{\hat{\varphi}(\lambda) : \Re\lambda \ge 0\},$$

where

$$\hat{\varphi}(\lambda) = \int_0^1 \varphi(x)x^\lambda \mathrm{d}x.$$

In [5], the author describes the eigenvalues and eigenfucntions of the operator $\mathcal{V}_\varphi$ on the space $C(I)$ when $\varphi \in L^p(0,1)$ with some $p > 1$:

1) the point spectrum of $\mathcal{V}_\varphi$ is exactly the set $\sigma^+(\mathcal{V}_\varphi) = \{\hat{\varphi}(\lambda) : \Re\lambda > 0\}$;

2) the dimension of the null space $\mathcal{N}(\mu\mathcal{I} - \mathcal{V}_\varphi)$ is the sum of the multiplicities of the roots of $\gamma_\mu(\lambda) := \mu - \hat{\varphi}(\lambda) = 0$ in the complex plane $\{\Re\lambda > 0 : \lambda \in \mathbb{C}\}$;

3) the linearly independent eigenfunctions are given by

$$t^\lambda, \ldots, t^\lambda (\ln t)^{k_\lambda},$$

where k_λ is the multiplicity of the root λ of $\gamma_\mu(\lambda) = 0$.

The pure Volterra integral equations with vanishing delay (VIEwND) are initially studied in [6] and a special form of VIEwND, proportional delay differential equations, is widely used in practical applications, for example, electrodynamics [7] [8], nonlinear dynamical systems [9] [10], and also the survey papers [11] [12]. In this paper, we consider the CVIEs with a vanishing delay,

$$\mu u(t) = f(t) + (\mathcal{V}_{\theta,\varphi} u)(t), \; t \in I, \tag{2}$$

where $\theta(t)$ is a continuous delay function such that $\theta(0) = 0$ and $\theta(t) < t$ for all $0 < t \in I$ and the operator with delay is similarly defined by

$$(\mathcal{V}_{\theta,\varphi} u)(t) := \int_{\theta(t)}^t t^{-1}\varphi(s/t)u(s)\mathrm{d}s. \tag{3}$$

Besides the existence and uniqueness of solutions to (2), it is more interesting how the eigenvalues and eigenfunctions of the operators are influenced by vanishing delays. In Section 2, we show that the proportional delay $\theta(t) = qt$, $0 < q < 1$, is the only one that replicates all eigenfunctions t^λ, $\lambda = 0$ or $\Re\lambda > 0$. For such a delay, we describe the spectrum, eigenvalues and eigenfunctions of the operator $\mathcal{V}_{\theta,\varphi}$. In Section 3, we present a necessary and sufficient condition for the compactness of the operator $\mathcal{V}_{\theta,\varphi}$ with a vanishing delay. Based on these discussions, we present the existence, uniqueness and the construction of solutions to (2).

2. Propositional Delays

For a vanishing delay $\theta(t)$ satisfying that

(D1) $\theta(0)=0$,

(D2) $0\le\theta(t)<t$ for all $0<t\in I$,

(D3) $\theta(t)$ is a continuous function in the interval I and $\theta'(0)$ exists,

the operator (3) is rewritten as the following form

$$(\mathcal{V}_{\theta,\varphi}u)(t)=(\mathcal{W}_{\xi,\varphi}u)(t):=\int_{\xi(t)}^{1}\varphi(x)u(tx)\mathrm{d}x, \tag{4}$$

where the function $\xi(t)=\frac{\theta(t)}{t}$ is a well-defined continuous function in the whole interval I. Obviously

$$0\le\xi(0)=\lim_{t\to0+}\xi(t)=\theta'(0)\le1,$$

and $\xi(t)\in[0,1]$ for all $t\in I$.

The cordial Volterra integral operator with a vanishing delay (3) is also written as a cordial Volterra integral operator with a variable kernel, *i.e.*,

$$(\mathcal{V}_{\theta,\varphi}u)(t)=(\mathcal{V}_{\varphi,a}u)(t):=\int_0^1\varphi(x)a(t,tx)u(tx)\mathrm{d}x,$$

where the discontinuous kernel $a(t,s)$ is defined by

$$a(t,s)=\begin{cases}0, & 0\le s\le\theta(t),\\ 1, & \theta(t)<s\le t.\end{cases}$$

The properties of the operator $\mathcal{V}_{\varphi,a}$ with continuous kernels are investigated in [13] such as it is compact if and only if $a(0,0)=0$. From the above definition, the discontinuous function a always satisfies $a(0,0)=0$, but the compactness of the operator $\mathcal{V}_{\theta,\varphi}$ is influenced not only by the core but also by the value of $\theta'(0)$ (see in Corollary 2.3 and Theorem 3.1).

Theorem 2.1. *Assume that the function* $\xi\in C(I)$.

1) *The operator* $\mathcal{V}_{\theta,\varphi}$ *is a bounded operator from* $C(I)$ *to* $C(I)$.

2) *If all power-functions* t^{λ}, $\lambda=0$ *or* $\Re\lambda>0$, *are eigenfunctions of* $\mathcal{V}_{\theta,\varphi}$, *then*

$$\int_{\xi(0)}^{\xi(t)}|\varphi(x)|\mathrm{d}x\equiv0$$

where for $\xi(t)<\xi(0)$, *the integration is defined by*

$$\int_{\xi(0)}^{\xi(t)}|\varphi(x)|\mathrm{d}x=-\int_{\xi(t)}^{\xi(0)}|\varphi(x)|\mathrm{d}x.$$

Proof. (i) For $\varphi\in L^1(0,1)$, $u\in C(I)$ and $\epsilon>0$, there exists a $\delta_1=\delta_1(\epsilon,u)>0$ such that

$$\int_a^{\min\{1,a+\delta_1\}}|\varphi(x)|\mathrm{d}x<\frac{1}{2\left(1+\|u\|_\infty\right)}\epsilon \text{ for all } a\in[0,1]$$

and for all $s_1,s_2\in I$ with $|s_1-s_2|\le\delta_1$

$$|u(s_1)-u(s_2)|\le\frac{1}{2\left(1+\|\varphi\|_1\right)}\epsilon,$$

since u is uniformly continuous on the closed interval. The uniform continuity of ξ implies that there exists a $\delta \le \delta_1$ such that $|\xi(t_1) - \xi(t_2)| \le \delta_1$ for all $t_1, t_2 \in I$ with $|t_1 - t_2| \le \delta$.

We, without loss of generality, assume that $\xi(t_1) \ge \xi(t_2)$ in the following estimation. Then

$$\begin{aligned}
&\left|\left(\mathcal{V}_{\theta,\varphi}u\right)(t_1) - \left(\mathcal{V}_{\theta,\varphi}u\right)(t_2)\right| = \left|\left(\mathcal{W}_{\xi,\varphi}u\right)(t_1) - \left(\mathcal{W}_{\xi,\varphi}u\right)(t_2)\right| \\
&\le \int_{\xi(t_1)}^{\xi(t_2)} |\varphi(x)||u(t_1x)| \mathrm{d}x + \int_{\xi(t_2)}^{1} |\varphi(x)||u(t_1x) - u(t_2x)| \mathrm{d}x \\
&\le \|u\|_\infty \int_{\xi(t_1)}^{\min\{1,\xi(t_1)+\delta_1\}} |\varphi(x)| \mathrm{d}x + \frac{1}{2\left(1+\|\varphi\|_1\right)} \epsilon \int_0^1 |\varphi(x)| \mathrm{d}x \le \epsilon.
\end{aligned}$$

Hence $\mathcal{V}_{\theta,\varphi}$ maps $C(I)$ to $C(I)$ and its boundedness comes from

$$\begin{aligned}
\max_{t\in[0,T]} \left|\left(\mathcal{V}_{\theta,\varphi}u\right)(t)\right| &= \max_{t\in[0,T]} \left|\left(\mathcal{W}_{\xi,\varphi}u\right)(t)\right| \\
&\le \max_{t\in[0,T]} \left|\int_{\xi(t)}^{1} \varphi(x) u(tx) \mathrm{d}x\right| \le \|\varphi\|_1 \|u\|_\infty .
\end{aligned}$$

2) Without loss of generality, suppose that $\xi(t^*) > \xi(0)$ and $\int_{\xi(0)}^{\xi(t^*)} |\varphi(x)| \mathrm{d}x > 0$ for some $t^* \in (0,T]$. Then similarly to the approach in [4], there exists a polynomial $p(t) = \sum_{i=0}^{n} p_i t^i$ such that

$$\int_{\xi(0)}^{\xi(t^*)} \varphi(x) p(t^*x) \mathrm{d}x \ge \frac{1}{2} \int_{\xi(0)}^{\xi(t^*)} |\varphi(x)| \mathrm{d}x > 0$$

Since t^i, $i = 0,1,\ldots$, is an eigenfunction of $\mathcal{W}_{\xi,\varphi}$,

$$\int_{\xi(t)}^{1} \varphi(x) x^i \mathrm{d}x = c_i$$

is also independent of t for $i = 0,1,\ldots$. Thus

$$\sum_{i=0}^{n} \int_{\xi(0)}^{1} \varphi(x) p_i (t^*x)^i \mathrm{d}x = \sum_{i=0}^{n} p_i (t^*)^i \int_{\xi(0)}^{1} \varphi(x) x^i \mathrm{d}x = \sum_{i=0}^{n} p_i c_i (t^*)^i,$$

$$\sum_{i=0}^{n} \int_{\xi(t^*)}^{1} \varphi(x) p_i (t^*x)^i \mathrm{d}x = \sum_{i=0}^{n} p_i (t^*)^i \int_{\xi(t^*)}^{1} \varphi(x) x^i \mathrm{d}x = \sum_{i=0}^{n} p_i c_i (t^*)^i,$$

and hence

$$\int_{\xi(0)}^{\xi(t^*)} \varphi(x) p(t^*x) \mathrm{d}x = 0.$$

This contradiction implies the proof is complete. □

Remark 2.2. In [4], the author shows that an operator $\mathcal{K}$ mapping $C(I)$ to $C(I)$ has the two properties:

1) $\mathcal{K}$ is a bounded operator;

2) all power-functions t^λ, $\lambda = 0$ or $\Re\lambda > 0$, are eigenfunctions of $\mathcal{K}$;

if and only if $\mathcal{K} = \mathcal{V}_\varphi$ is a cordial Volterra integral operator. While including vanishing delays, the two properties only hold for a proportional delay $\theta(t) = qt$, $0 < q < 1$.

For a core $\varphi \in L^1(0,1)$, we define an integration function of the core by

$$\Phi(x) = \int_0^x |\varphi(r)| \mathrm{d}r.$$

If $\Phi(x) \equiv 0$ for $x \in [0,q]$ with some $0 < q < 1$ (or $\operatorname{supp} \varphi \in [q,1]$), then CVIEs naturally reduce to a proportional delay form

$$\mu u(t) = f(t) + (\mathcal{W}_{q,\varphi} u)(t), \tag{5}$$

where the corresponding operator has the form $\mathcal{W}_{q,\varphi} = \mathcal{V}_{\varphi_q}$ with $\varphi_q(x) := \mathcal{X}_q(x)\varphi(x)$ and

$$\mathcal{X}_q(x) := \begin{cases} 0, & x \in [0,q), \\ 1, & x \in [q,1]. \end{cases}$$

Corollary 2.3. *Assume that* $\varphi \in L^1(0,1)$ *and* $\Phi(x)$ *is a strictly increasing function for* $x \in [0,1]$. *Then a cordial Volterra integral operator with vanishing delays opposites the two properties in Remark* 2.2 *if and only if the delay* $\theta(t) = qt$ *is a proportional delay. Of course it is a noncompact operator.*

Proof. By Theorem 2.1, one obtains that $\xi(t) \equiv \xi(0)$ is a constant. Thus the proof is completed by $\theta(t) = t\xi(t)$. □

Based on $\mathcal{W}_{q,\varphi} = \mathcal{V}_{\varphi_q}$, some more detailed properties on cordial Volterra integral operators with a proportional delay are presented in the following theorem.

Theorem 2.4. *Assume that a core* $\varphi \in L^p(0,1)$ *with some* $p > 1$, $\theta(t) = qt$, $0 < q < 1$ *and* $\Phi(x) > 0$ *for* $x \in (q,1]$. *Then*

1) *The spectrum of* $\mathcal{V}_{\theta,\varphi}$ *is given by*

$$\sigma\left(\mathcal{V}_{\theta,\varphi}\right) = \{0\} \cup \left\{\hat{\varphi}_q(\lambda) : \Re\lambda \ge 0\right\},$$

where $\hat{\varphi}_q(\lambda) := \int_0^1 \varphi_q(x) x^\lambda \mathrm{d}x = \int_q^1 \varphi(x) x^\lambda \mathrm{d}x$.

2) *The point spectrum of* $\mathcal{V}_{\theta,\varphi}$ *is exactly the set* $\sigma^+\left(\mathcal{V}_{\theta,\varphi}\right) = \left\{\hat{\varphi}_q(\lambda) : \Re\lambda > 0\right\}$.

3) *The dimension of the null space* $\mathcal{N}(\mu\mathcal{I} - \mathcal{V}_{\theta,\varphi})$ *is the sum of the multiplicities of the roots of* $\gamma_{q,\mu}(\lambda) := \mu - \hat{\varphi}_q(\lambda) = 0$ *in the complex plane* $\{\Re\lambda > 0 : \lambda \in \mathbb{C}\}$.

4) *The linearly independent eigenfunctions are given by*

$$t^\lambda, \ldots, t^\lambda \left(\ln t\right)^{k_\lambda},$$

where k_λ *is the multiplicity of the root* λ *of* $\gamma_{q,\mu}(\lambda) = 0$.

5) *The range of the operator* $\mu\mathcal{I} - \mathcal{V}_{\theta,\varphi}$ *is the whole space* $C(I)$ *if and only if* $\mu \notin \sigma^0\left(\mathcal{V}_{\theta,\varphi}\right) = \left\{\hat{\varphi}_q(\lambda) : \Re\lambda = 0\right\} \cup \{0\}$.

Both the existence and uniqueness of solutions to (5) are valid when the parameter μ does not lie in the spectrum of the corresponding operators. On the other hand, for μ lying in the spectrum, by the same technique in [5], we are also able to construct solutions to (5). For convenience, we review some notations in [5]:

1) $C_{c_1,\ldots,c_n}[0,T] = \left\{u \in C[0,T] : u(c_i) = 0, i = 1,\ldots,n\right\}$ with different parameters $c_1,\ldots,c_n \in (0,T]$

2) $C^\epsilon[0,T] := \left\{t^\epsilon w(t) : w \in C[0,T]\right\}$ with the norm

$$\|u\|_\epsilon := \|w\| \text{ for } u \in C^\epsilon[0,T].$$

Theorem 2.5. *Assume that* $\varphi \in L^p(0,1)$ *with some* $p > 1$ *and that* $\theta(t) = qt$, $0 < q < 1$, *and* $\Phi(x) > 0$ *for* $x \in (q,1]$. *Let* $n = \dim \mathcal{N}(\mu\mathcal{I} - \mathcal{V}_{\theta,\varphi})$. *Then there* *exist* $c_1,\ldots,c_n$ *distinct points in* $(0,T]$ *such that the following statements are true.*

1) *For* $\mu\in\sigma^+(\mathcal{V}_{\theta,\varphi})\setminus\sigma^0(\mathcal{V}_{\theta,\varphi})$, *there exists a unique solution* $u^*\in C_{c_1,\ldots,c_n}[0,T]$ *to* (5) *that continuously depends on* $f\in C[0,T]$, *and all solutions have the form*

$$u=u^*+u^\perp,$$

where $u^\perp$ *is a linear combination of functions fo functions* $t^{\lambda^*}(\ln t)^i$, $i=0,\ldots,k_{\lambda^*}-1$, *and* $\lambda^*\in\mathbb{C}^+$ *is a root of* $\gamma_{q,\mu}(\lambda)=0$ *with multiplicity* k_{λ^*}.

2) *For* $\mu\in\sigma^0(\mathcal{V}_{\theta,\varphi})\setminus\sigma^+(\mathcal{V}_{\theta,\varphi})$, *there exists at most one solution to* (5), *and there exists exactly one solution to* (5) *when* $f\in C^\epsilon[0,T]$ *for any* $\epsilon>0$.

3) *For* $\mu\in\sigma^0(\mathcal{V}_{\theta,\varphi})\cap\sigma^+(\mathcal{V}_{\theta,\varphi})$, *there exists at most one solution* u^* *belonging to* $C_{c_1,\ldots,c_n}[0,T]$, *and there exists a unique solution in* $u^*\in C_{c_1,\ldots,c_n}[0,T]$ *for any* $\epsilon>0$ *and* $f\in C^\epsilon[0,T]$. *All solutions have the form*

$$u=u^*+u^\perp,$$

where $u^\perp$ *is linearly combined by such functions* 1 (*if* $\mu=\hat{\varphi}_q(0)$) *and* $t^{\lambda^*}(\ln t)^i$, $i=0,1,\ldots,k_{\lambda^*}-1$, *and* $\lambda^*\in\mathbb{C}^+$ *is a root of* $\gamma_{q,\mu}(\lambda)=0$ *with multiplicity* k_{λ^*}.

3. General Vanishing Delays

For a more general vanishing delay, the compactness of the cordial Volterra integral operators is influenced by the value of $\theta'(0)$.

Theorem 3.1. *Assume that* $\varphi\in L^1(0,1)$ *and that the delay function* $\theta(t)$ *satisfies the assumptions* $(D1)$, $(D2)$, $(D3)$. *Then the operator* $\mathcal{V}_{\theta,\varphi}$ *is compact in* $C(I)$ *if and only if* $\operatorname{supp}\varphi\subseteq[0,\theta'(0)]$.

Proof. From the definition of the function ξ, it is known that $\xi(0)=\theta'(0)$. In Lemma 3.6, one obtains from $\operatorname{supp}\varphi\subseteq[0,\theta'(0)]$ that $(\mathcal{V}_{\theta,\varphi}u)(0)=0$ for all $u\in C(I)$. Hence by Ascoli-Arzela theorem, the compactness of the cordial Volterra integral operator $\mathcal{V}_{\theta,\varphi}$ with such a vanishing delay term is shown in Lemma 3.7. The proof will be completed, when the non-compactness of the operator is proved in Lemma 3.8. □

The simplest compact condition according to Theorem 3.1 is $\theta'(0)=1$.

Corollary 3.2. *Assume that* $\varphi\in L^1(0,1)$ *and that the delay function* $\theta(t)$ *satisfies the assumptions (D1), (D2), (D3). Then the operator* $\mathcal{V}_{\theta,\varphi}$ *is compact in* $C(I)$ *for any core* $\varphi\in L^1(0,1)$ *provided that* $\theta'(0)=1$.

Remark 3.3. Consider the constant core $\varphi(x)\equiv1$. Then

1) $\mathcal{V}_{q,1}$, $0<q<1$, are non-compact in $C(I)$.

2) For $\theta(t)=\sin t$, $\mathcal{V}_{\theta,1}$ is compact in $C([0,1])$.

The existence and uniqueness of solutions to (2) is similar to the classical second kind of VIEs when the corresponding operator is compact.

Theorem 3.4. *Assume that* $\varphi\in L^1(0,1)$ *and that the delay function* $\theta(t)$ *satisfies the assumptions* $(D1)$, $(D2)$, $(D3)$ *and that* $\operatorname{supp}\varphi\subseteq[0,\theta'(0)]$. *Then for all* $\mu\neq0$ *and all* $f\in C(I)$, *there exists a unique solution to* (2).

Proof. In Lemma 3.9, it is shown that the null space of the operator $\mu\mathcal{I}-\mathcal{V}_{\theta,\varphi}$

in $C(I)$ is $\{0\}$, which together with the compactness of $\mathcal{V}_{\theta,\varphi}$ implies that the operator $\mu\mathcal{I}-\mathcal{V}_{\theta,\varphi}$ has a bounded inverse in $C(I)$ (see in [14]). Hence the proof is complete. □

Example 3.5. *Consider the following CVIEs with a vanishing delay*

1) $\varphi(x)=\left(\frac{1}{2}-x\right)_+$ *and* $\theta(t)=\frac{1}{2}t-t^2$ for $t\in I=\left[0,\frac{1}{2}\right]$;

2) $\varphi(x)=\left(1-x^{\frac{3}{2}}\right)^{-\frac{2}{3}}$ (*the linear form of Lighthill's equations*) *and* $\theta(t)=\sin t$ for $t\in I=[0,1]$;

3) $\varphi(x)=\left(\frac{1}{2}-x\right)_+$ *and* $\theta(t)=\frac{1}{2}t+t^2$ *for* $t\in I=\left[0,\frac{1}{2}\right]$.

Then the corresponding operators are compact and there exists a unique solution to (2) for $\mu\neq 0$ and $f\in C(I)$.

Theorems 3.1 and 3.4 are proved by the following lemmas.

Lemma 3.6 *Assume that* $\varphi\in L^1(0,1)$ *and that* $\xi(t)\in[0,1]$ *is a continuous function in* I. *Then one obtains that* $(\mathcal{V}_{\theta,\varphi}u)(0)=0$ *for all* $u\in C(I)$ *if* $\operatorname{supp}\varphi\subseteq[0,\xi(0)]$.

Proof. In view of

$$(\mathcal{V}_{\theta,\varphi}u)(0)=(\mathcal{W}_{\xi,\varphi}u)(0)=u(0)\int_{\xi(0)}^{1}\varphi(x)\mathrm{d}x,$$

the condition in this lemma yields that for all $u\in C(I)$,

$$(\mathcal{V}_{\theta,\varphi}u)(0)=0.$$

The proof is complete. □

Lemma 3.7 *Assume that* $\varphi\in L^1(0,1)$, $\xi(t)\in[0,1]$ *is a continuous function in I and that* $\operatorname{supp}\varphi\subseteq[0,\xi(0)]$. *Then* $\mathcal{W}_{\xi,\varphi}$ *is a compact operator in* $C(I)$.

Proof. By Ascoli-Arzela theorem, the compactness will be proved by the equiv-continuity of $(\mathcal{W}_{\xi,\varphi}u)(t)$.

Since $\xi(t)$ is a continuous function of t and $\Phi(x)$ is a continuous function of x, for any given $\epsilon>0$ there exists an $T^*(\epsilon)>0$ such that

$$\left|\Phi(\xi(t))-\Phi(\xi(0))\right|\leq\frac{1}{2}\epsilon \text{ for } t\in\left[0,T^*(\epsilon)\right].$$

Therefore, for $u\in C(I)$ with $\|u\|_\infty=1$, $(\mathcal{W}_{\xi,\varphi}u)(0)=0$ by Lemma 3.6 and for $t\in[0,T^*(\epsilon)]$,

$$\left|(\mathcal{W}_{\xi,\varphi}u)(t)\right|\leq\left|\Phi(\xi(t))-\Phi(\xi(0))\right|\leq\frac{1}{2}\epsilon.$$

In the following, we let $\frac{1}{2}T^*\leq t_1<t_2\leq T$ and we choose $\delta=\delta(\epsilon)>0$ such that for all $|t_2-t_1|\leq\delta$ implies

$$\left|\Phi(\xi(t_1))-\Phi(\xi(t_2))\right|\leq\frac{1}{3}\epsilon,$$

$$\int_0^1\left|\varphi\left(\frac{t_1}{t_2}x\right)-\varphi(x)\right|\mathrm{d}x\leq\frac{1}{3}\epsilon,$$

$$\frac{2\left|t_2-t_1\right|}{T^*}\|\varphi\|_1\leq\frac{1}{3}\epsilon.$$

Therefore,

$$\begin{aligned}&\left|\left(\mathcal{W}_{\xi,\varphi}u\right)(t_2)-\left(\mathcal{W}_{\xi,\varphi}u\right)(t_1)\right|\\&\le\left|\int_{\xi(t_1)}^{\xi(t_2)}|\varphi(x)|\,\mathrm{d}x\right|+\int_0^1|\varphi(x)|\left|u(t_2x)-u(t_1x)\right|\mathrm{d}x\\&\le\frac{1}{3}\epsilon+\left|\frac{t_1}{t_2}-1\right|\|\varphi\|_1+\int_0^1\left|\varphi\left(\frac{t_1}{t_2}x\right)-\varphi(x)\right|\mathrm{d}x\le\epsilon.\end{aligned}$$

The proof is complete. □

Lemma 3.8. *Assume that* $\varphi\in L^1(0,1)$, $\xi(0)<1$ *and that* $\Phi(\xi(0))<\Phi(1)$. *Then* $\mathcal{W}_{\xi,\varphi}$ *is a noncompact operator in* $C(I)$.

Proof. Without loss of generality, we assume that $\xi(t)\ge\xi(0)$ (or $\xi(t)\le\xi(0)$) for all $t\in I$ and suppose that the operator $\mathcal{W}_{\xi,\varphi}$ is compact. Then the operator

$$\left(\mathcal{W}_{\xi(0),\varphi}u\right)(t)=\left(\mathcal{W}_{\xi,\varphi}u\right)(t)+\int_{\xi(0)}^{\xi(t)}\varphi(x)u(tx)\mathrm{d}x$$

or

$$\left(\mathcal{W}_{\xi(0),\varphi}u\right)(t)=\left(\mathcal{W}_{\xi,\varphi}u\right)(t)-\int_{\xi(t)}^{\xi(0)}\varphi(x)u(tx)\mathrm{d}x$$

is compact by Lemma 3.7. This contradicts to Corollary 2.3 and the proof is complete. □

Lemma 3.9 *Assume that* $\varphi\in L^1(0,1)$, $\xi(t)\in[0,1]$ *is a continuous function in I and that* $\operatorname{supp}\varphi\subseteq[0,\theta'(0)]$. *Then the null space of* $\mu\mathcal{I}-\mathcal{W}_{\xi,\varphi}$ *is trivial in* $C(I)$ *for all* $\mu\ne0$.

Proof. We suppose that $\mu\ne0$ and there exists a $u\in C(I)$ such that

$$\mathcal{W}_{\xi,\varphi}u=\mu u. \tag{6}$$

Then $u|_{[0,T^*(|\mu|)]}\equiv0$ by

$$\left|(\mathcal{W}_{\xi,\varphi}u)(t)\right|\le\frac{1}{2}|\mu|\,\|u\|_{\infty,[0,T^*(|\mu|)]}.$$

Thus, (6) reduces to

$$\mu u(t)=\int_{\xi(t)}^1\varphi(x)u(tx)\mathrm{d}x,\quad t>T^*\left(|\mu|\right).$$

For all $\lambda<0$ and $\delta>0$, it holds

$$\begin{aligned}e^{\lambda t}\int_{\xi(t)}^1|\varphi(x)||u(tx)|\mathrm{d}x&\le\int_{\xi(t)}^1|\varphi(x)|e^{\lambda t(1-x)}\mathrm{d}x\max_{t\in[T^*(|\mu|),T]}e^{\lambda t}|u(t)|\\&\le\left(e^{\lambda T^*(|\mu|)\delta}\Phi(1-\delta)+\left(\Phi(1)-\Phi(1-\delta)\right)\right)\max_{t\in[T^*(|\mu|),T]}e^{\lambda t}|u(t)|.\end{aligned}$$

Hence (6) yields for sufficiently small $\delta>0$ and sufficiently large $\lambda<0$,

$$|\mu|\max_{t\in[T^*(|\mu|),T]}e^{\lambda t}|u(t)|\le\frac{1}{2}|\mu|\max_{t\in[T^*(|\mu|),T]}e^{\lambda t}|u(t)|.$$

This implies that $\|u\|_{\infty,[T^*(|\mu|),T]}=0$ and the proof is complete. □

4. Concluding Remarks

In this paper, we consider CVIEs with a vanishing delay:

1) a proportional delay,

2) a nonlinear vanishing delay $\theta(t)$.

The first case reduces to a classical CVIE with a core limited to a subinterval. Hence these results are trivial from [4] [5]. For case 2), we present the compactness of the operators, *i.e.*, $\text{supp } \varphi \subseteq [0, \theta'(0)]$. In subsequent work, we will investigate the spectrum, eigenvalues and eigenfunctions when $\Phi(\theta'(0)) < \Phi(1)$ and also numerical methods for CVIEs with vanishing delays.

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