Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2017, 5, 194-223
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Periodic Solutions of Some Polynomial Differential Systems in $\mathbb{R}^4$

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How to cite this paper: Amar, M. and Lilia, B. (2017) Periodic Solutions of Some Polynomial Differential Systems in $\mathbb{R}^4$. *Journal of Applied Mathematics and Physics*, **5**, 194-223.
http://dx.doi.org/10.4236/jamp.2017.51019

Received: December 20, 2016
Accepted: January 23, 2017
Published: January 26, 2017

Abstract

We provide sufficient conditions for the existence of periodic solutions of the polynomial fourth order differential system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{u}\\ \dot{v}\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}x\\ y\\ u\\ v\end{pmatrix} + \begin{pmatrix}h_1(t)\\ h_2(t)\\ h_3(t)\\ h_4(t)\end{pmatrix} + \varepsilon\begin{pmatrix}P_1(x,y,u,v)\\ P_2(x,y,u,v)\\ P_3(x,y,u,v)\\ P_4(x,y,u,v)\end{pmatrix},$$

where $\boldsymbol{A}$ is a 4×4 constant matrix, P_1, P_2, P_3 and P_4 are polynomials in the variables x, y, u, v of degrees n, $h_i(t) = h_i(t+2\pi)$ with $i = 1,2,3,4$ being periodic functions and ε is a small parameter.

Keywords

Periodic Solution, Averaging Theory, Differential System

1. Introduction

One of the main problems in the theory of differential systems is the study of their periodic orbits, their existence, their number and their stability. As usual a limit cycle of a differential system is a periodic orbit isolated in the set of all periodic orbits of the differential system.

The goal of this paper is to study the existence of the periodic orbits of the polynomial fourth order differential system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{u}\\ \dot{v}\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}x\\ y\\ u\\ v\end{pmatrix} + \begin{pmatrix}h_1(t)\\ h_2(t)\\ h_3(t)\\ h_4(t)\end{pmatrix} + \varepsilon\begin{pmatrix}P_1(x,y,u,v)\\ P_2(x,y,u,v)\\ P_3(x,y,u,v)\\ P_4(x,y,u,v)\end{pmatrix}, \tag{1.1}$$

where $\boldsymbol{A}$ is 4×4 a constant matrix, P_1, P_2, P_3 and P_4 are polynomials in the variables x, y, u, v of degrees n, $h_i(t) = h_i(t+2\pi)$ with $i = 1,2,3,4$ being periodic functions and ε is a small parameter.

There are many papers studying the periodic orbits of the fourth order differential systems and equations (see for instance [1]-[11]). But our main tool for studying the periodic orbits of the system (1.1) is completely different to the tools of the mentioned papers, and consequently the results obtained are distinct and new. We shall use the averaging theory, more precisely Theorem 6 and 7. Many of the quoted papers dealing with the peiodic orbits of four-order differential equations use Schauder's or Leray-Schauder's fixed point theorem, or the nonlocal reduction method or variational methods.

To obtain analytically periodic solutions is in general a very difficult work, usually impossible. Here with the averaging theory we reduce this difficult problem for the differential system (1.1) to find the zeros of a nonlinear system of four equations with four unknowns. It is known that in general the averaging theory for finding periodic solutions does not provide all the periodic solutions of the system. To explain this idea, there are two main reasons. First, the averaging theory for studying the periodic solutions of a differential system is based on the so-called displacement function, whose zeros provide periodic solutions. This displacement function in general is not global and consequently it cannot control all the periodic solution, only the ones which are in its domain of definition and that are hyperbolic. Second, the displacement function is expanded in power series of a small parameter ε, and the averaging theory only controls the zeros of the dominant term of this displacement function. When the dominant term is ε^k, we talk about the averaging theory of order k. For more details, see for instance [12] and the references quoted there.

The method of averaging is a classical tool that allows studying the dynamics of the nonlinear differential systems under periodic forcing. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace, who provided an intuitive justification of the method. The first formalization of this theory was done in 1928 by Fatou [13]. Important practical and theoretical contributions to the averaging theory were made in the 1930's by Bogoliubov and Krylov [14], in 1945 by Bogoliubov [15], and by Bogoliubov and Mitropolsky [16] (English version 1961). For a more modern exposition of the averaging theory see the book of Sanders, Verhulst and Murdock [17]. For more information about averaging theory, see Section 2 of this paper.

In [18], the authors studied the bifurcation of limit cycles from the periodic orbits of a linear differential system in $\mathbb{R}^4$ in resonance 1:n perturbed inside a class of piecewise linear differential systems which appear in a natural way in control theory. In [19], the authors studied the limit cycles of the fourth-order differential equation

$$\ddot{\ddot{x}} - (\lambda+\mu)\dddot{x} + (1+\lambda\mu)\ddot{x} - (\lambda+\mu)\dot{x} + \lambda\mu x = \varepsilon F(x,\dot{x},\ddot{x},\dddot{x},t),$$

where ε is a small enough parameter and $F \in C^2$ is a 2π-periodic function in the variable t. In [20], the authors studied the autonomous case of the previous equation, (*i.e.* F does not depend on t) using another approach. In [21], the authors provide sufficient conditions for the existence of periodic solutions of the fourth-order differential equation

$$\ddddot{x}+\left(1+p^2\right)\ddot{x}+p^2x=\varepsilon F\left(x,\dot{x},\ddot{x},\dddot{x}\right),$$

where p is a rational number different from 0, ε is small and F is a nonlinear function. In [22], the authors provide sufficient conditions for the existence of periodic solutions of the fourth-order differential equation

$$\ddddot{x}+q\ddot{x}+px=\varepsilon F\left(t,x,\dot{x},\ddot{x},\dddot{x}\right),$$

where p, q and ε are real parameters, ε is small and F is a nonlinear non-autonomous periodic function with respect to t. The five previous cited papers used averaging method.

In [23] we studied the system (1.1) in dimension 3 using averaging method, *i.e.* the following system

$$\begin{aligned}\dot{x}&=-y+\varepsilon P(x,y,z)+h_1(t),\\ \dot{y}&=x+\varepsilon Q(x,y,z)+h_2(t),\\ \dot{z}&=az+\varepsilon R(x,y,z)+h_3(t),\end{aligned}$$

where a is a real number, P, Q and R are polynomials in the variables x, y, z of degrees n, $h_i(t)=h_i(t+2\pi)$ with $i=1,2,3$ being periodic functions and ε is a small parameter. In this paper our objective is to provide the existence of periodic solutions of system (1.1).

Our main results on the periodic solutions of the differential system (1.1) are the following theorems.

One considers system (1.1) with $\boldsymbol{A}=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$, our result is the following.

Theorem 1. *One defines*

$$\mathcal{F}_1(x_0,y_0,u_0,v_0)=\frac{1}{2\pi}\int_0^{2\pi}\left(\cos(t)P_1\left(a(t),b(t),c(t),d(t)\right)+\sin(t)P_2\left(a(t),b(t),c(t),d(t)\right)\right)\mathrm{d}t,$$

$$\mathcal{F}_2(x_0,y_0,u_0,v_0)=\frac{1}{2\pi}\int_0^{2\pi}\left(-\sin(t)P_1\left(a(t),b(t),c(t),d(t)\right)+\cos(t)P_2\left(a(t),b(t),c(t),d(t)\right)\right)\mathrm{d}t,$$

$$\mathcal{F}_3(x_0,y_0,u_0,v_0)=\frac{1}{2\pi}\int_0^{2\pi}\left(\cos(t)P_3\left(a(t),b(t),c(t),d(t)\right)+\sin(t)P_4\left(a(t),b(t),c(t),d(t)\right)\right)\mathrm{d}t,$$

$$\mathcal{F}_4(x_0,y_0,u_0,v_0)=\frac{1}{2\pi}\int_0^{2\pi}\left(-\sin(t)P_3\left(a(t),b(t),c(t),d(t)\right)+\cos(t)P_4\left(a(t),b(t),c(t),d(t)\right)\right)\mathrm{d}t,$$

where

$$a(t)=\cos(t)x_0-\sin(t)y_0+\int_0^t\left(\cos(t-s)h_1(s)-\sin(t-s)h_2(s)\right)\mathrm{d}s,$$

$$b(t)=\sin(t)x_0+\cos(t)y_0+\int_0^t(\sin(t-s)h_1(s)+\cos(t-s)h_2(s))\mathrm{d}s,$$

$$c(t)=\cos(t)u_0-\sin(t)v_0+\int_0^t(\cos(t-s)h_3(s)-\sin(t-s)h_4(s))\mathrm{d}s,$$

$$d(t)=\sin(t)u_0+\cos(t)v_0+\int_0^t(\sin(t-s)h_3(s)+\cos(t-s)h_4(s))\mathrm{d}s.$$

If

$$\begin{aligned}&\int_0^{2\pi}(\cos(s)h_1(s)-\sin(s)h_2(s))\mathrm{d}s=0,\\&\int_0^{2\pi}(-\sin(s)h_1(s)+\cos(s)h_2(s))\mathrm{d}s=0,\\&\int_0^{2\pi}(\cos(s)h_3(s)-\sin(s)h_4(s))\mathrm{d}s=0,\\&\int_0^{2\pi}(-\sin(s)h_3(s)+\cos(s)h_4(s))\mathrm{d}s=0,\end{aligned}\tag{1.2}$$

then for every $(x_0^*,y_0^*,u_0^*,v_0^*)$ *solution of the system*

$$\mathcal{F}_k(x_0,y_0,u_0,v_0)=0,\quad k=1,2,3,4,$$

satisfying

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4)}{\partial(x_0,y_0,u_0,v_0)}\right|_{(x_0,y_0,u_0,v_0)=(x_0^*,y_0^*,u_0^*,v_0^*)}\right)\neq 0,$$

the differential system (1.1) *has a periodic solution* $\begin{pmatrix}x(t,\varepsilon)\\y(t,\varepsilon)\\u(t,\varepsilon)\\v(t,\varepsilon)\end{pmatrix}$, *which tends to the periodic solution given by*

$$\begin{pmatrix}x(t)\\y(t)\\u(t)\\v(t)\end{pmatrix}=\begin{pmatrix}\cos(t)x_0^*-\sin(t)y_0^*+\int_0^t(\cos(t-s)h_1(s)-\sin(t-s)h_2(s))\mathrm{d}s\\\sin(t)x_0^*+\cos(t)y_0^*+\int_0^t(\sin(t-s)h_1(s)+\cos(t-s)h_2(s))\mathrm{d}s\\\cos(t)u_0^*-\sin(t)v_0^*+\int_0^t(\cos(t-s)h_3(s)-\sin(t-s)h_4(s))\mathrm{d}s\\\sin(t)u_0^*+\cos(t)v_0^*+\int_0^t(\sin(t-s)h_3(s)+\cos(t-s)h_4(s))\mathrm{d}s\end{pmatrix}$$

of the differential system

$$\begin{aligned}\dot{x}&=-y+h_1(t),\\\dot{y}&=x+h_2(t),\\\dot{u}&=-v+h_3(t),\\\dot{v}&=u+h_4(t),\end{aligned}$$

when $\varepsilon\to 0$.

Note that this solution is periodic of period 2π.

One considers system (1.1) with $\boldsymbol{A}=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&\lambda&0\\0&0&0&\mu\end{pmatrix}$. We distinguish three cases for different parameter values λ and μ:

Case 1: $\lambda \neq 0$ and $\mu \neq 0$.

Case 2: $\lambda = 0$ and $\mu \neq 0$. (Or $\lambda \neq 0$ and $\mu = 0$).

Case 3: $\lambda = \mu = 0$.

Our results for these three cases are the following ones.

Theorem 2. *Case* 1

One defines

$$\mathcal{F}_1(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi}\left(\cos(t)P_1(a(t),b(t),c(t),d(t)) + \sin(t)P_2(a(t),b(t),c(t),d(t))\right)\mathrm{d}t,$$

$$\mathcal{F}_2(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi}\left(-\sin(t)P_1(a(t),b(t),c(t),d(t)) + \cos(t)P_2(a(t),b(t),c(t),d(t))\right)\mathrm{d}t,$$

where

$$a(t) = \cos(t)x_0 - \sin(t)y_0 + \int_0^t\left(\cos(t-s)h_1(s) - \sin(t-s)h_2(s)\right)\mathrm{d}s,$$

$$b(t) = \sin(t)x_0 + \cos(t)y_0 + \int_0^t\left(\sin(t-s)h_1(s) + \cos(t-s)h_2(s)\right)\mathrm{d}s,$$

$$c(t) = e^{\lambda t}u_0 + \int_0^t e^{\lambda(t-s)}h_3(s)\mathrm{d}s,$$

$$d(t) = e^{\mu t}v_0 + \int_0^t e^{\mu(t-s)}h_4(s)\mathrm{d}s.$$

If

$$\int_0^{2\pi}\left(\cos(s)h_1(s) + \sin(s)h_2(s)\right)\mathrm{d}s = 0, \tag{1.3}$$

$$\int_0^{2\pi}\left(-\sin(s)h_1(s) + \cos(s)h_2(s)\right)\mathrm{d}s = 0,$$

$$u_0 = \frac{1}{1-e^{2\pi\lambda}}\int_0^{2\pi}e^{\lambda(2\pi-s)}h_3(s)\mathrm{d}s,$$

$$v_0 = \frac{1}{1-e^{2\pi\mu}}\int_0^{2\pi}e^{\mu(2\pi-s)}h_4(s)\mathrm{d}s,$$

then for every (x_0^*, y_0^*) *solution of the system*

$$\mathcal{F}_k(x_0, y_0) = 0, \quad k = 1, 2,$$

satisfying

$$\det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)}\right) \neq 0,$$

the differential system (1.1) *has a periodic solution* $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \\ u(t,\varepsilon) \\ v(t,\varepsilon) \end{pmatrix}$, *which tends to the periodic solution given by*

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t)x_0^* - \sin(t)y_0^* + \int_0^t (\cos(t-s)h_1(s) - \sin(t-s)h_2(s))\mathrm{d}s \\ \sin(t)x_0^* + \cos(t)y_0^* + \int_0^t (\sin(t-s)h_1(s) + \cos(t-s)h_2(s))\mathrm{d}s \\ \frac{e^{\lambda t}}{1-e^{2\pi\lambda}} \int_0^{2\pi} e^{\lambda(2\pi-s)} h_3(s)\mathrm{d}s + \int_0^t e^{\lambda(t-s)} h_3(s)\mathrm{d}s \\ \frac{e^{\mu t}}{1-e^{2\pi\mu}} \int_0^{2\pi} e^{\mu(2\pi-s)} h_3(s)\mathrm{d}s + \int_0^t e^{\mu(t-s)} h_4(s)\mathrm{d}s \end{pmatrix}$$

of the differential system

$$\dot{x} = -y + h_1(t),$$
$$\dot{y} = x + h_2(t),$$
$$\dot{u} = \lambda u + h_3(t),$$
$$\dot{v} = \mu v + h_4(t),$$

when $\varepsilon \to 0$.

Note that this solution is periodic of period 2π.

Theorem 3. *Case 2* $(\lambda = 0)$

One defines

$$\mathcal{F}_1(x_0, y_0, u_0) = \frac{1}{2\pi}\int_0^{2\pi} \big(\cos(t)P_1(a(t),b(t),c(t),d(t)) + \sin(t)P_2(a(t),b(t),c(t),d(t))\big)\mathrm{d}t,$$

$$\mathcal{F}_2(x_0, y_0, u_0) = \frac{1}{2\pi}\int_0^{2\pi} \big(-\sin(t)P_1(a(t),b(t),c(t),d(t)) + \cos(t)P_2(a(t),b(t),c(t),d(t))\big)\mathrm{d}t,$$

$$\mathcal{F}_3(x_0, y_0, u_0) = \frac{1}{2\pi}\int_0^{2\pi} P_3(a(t),b(t),c(t),d(t))\mathrm{d}t,$$

where

$$a(t) = \cos(t)x_0 - \sin(t)y_0 + \int_0^t (\cos(t-s)h_1(s) - \sin(t-s)h_2(s))\mathrm{d}s,$$

$$b(t) = \sin(t)x_0 + \cos(t)y_0 + \int_0^t (\sin(t-s)h_1(s) + \cos(t-s)h_2(s))\mathrm{d}s,$$

$$c(t) = u_0 + \int_0^t h_3(s)\mathrm{d}s,$$

$$d(t) = e^{\mu t}v_0 + \int_0^t e^{\mu(t-s)} h_4(s)\mathrm{d}s.$$

If

$$\begin{aligned} &\int_0^{2\pi} (\cos(s)h_1(s) + \sin(s)h_2(s))\mathrm{d}s = 0, \\ &\int_0^{2\pi} (-\sin(s)h_1(s) + \cos(s)h_2(s))\mathrm{d}s = 0, \\ &\int_0^{2\pi} h_3(s)\mathrm{d}s = 0, \\ &v_0 = \frac{1}{1-e^{2\pi\mu}} \int_0^{2\pi} e^{\mu(2\pi-s)} h_4(s)\mathrm{d}s, \end{aligned} \tag{1.4}$$

then for every (x_0^*, y_0^*, u_0^*) *solution of the system*

$$\mathcal{F}_k(x_0, y_0, u_0) = 0, \quad k = 1,2,3,$$

satisfying

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)}{\partial(x_0,y_0,u_0)}\right|_{(x_0,y_0,u_0)=(x_0^*,y_0^*,u_0^*)}\right)\neq 0,$$

the differential system (1.1) *has a periodic solution* $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \\ u(t,\varepsilon) \\ v(t,\varepsilon) \end{pmatrix}$, *which tends to the periodic solution given by*

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix}=\begin{pmatrix} \cos(t)x_0^*-\sin(t)y_0^*+\int_0^t\left(\cos(t-s)h_1(s)-\sin(t-s)h_2(s)\right)\mathrm{d}s \\ \sin(t)x_0^*+\cos(t)y_0^*+\int_0^t\left(\sin(t-s)h_1(s)+\cos(t-s)h_2(s)\right)\mathrm{d}s \\ u_0^*+\int_0^t h_3(s)\mathrm{d}s \\ \frac{e^{\mu t}}{1-e^{2\pi\mu}}\int_0^{2\pi}e^{\mu(2\pi-s)}h_4(s)\mathrm{d}s+\int_0^t e^{\mu(t-s)}h_4(s)\mathrm{d}s \end{pmatrix}$$

of the differential system

$$\dot{x}=-y+h_1(t),$$

$$\dot{y}=x+h_2(t),$$

$$\dot{u}=h_3(t),$$

$$\dot{v}=\mu v+h_4(t),$$

when $\varepsilon\to 0$.

Note that this solution is periodic of period 2π.

Theorem 4. *Case* 3

One defines

$$\mathcal{F}_1(x_0,y_0,u_0,v_0)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(\cos(t)P_1(a(t),b(t)c(t),d(t))+\sin(t)P_2(a(t),b(t),c(t),d(t))\right)\mathrm{d}t,$$

$$\mathcal{F}_2(x_0,y_0,u_0,v_0)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(-\sin(t)P_1(a(t),b(t),c(t),d(t))+\cos(t)P_2(a(t),b(t),c(t),d(t))\right)\mathrm{d}t,$$

$$\mathcal{F}_3(x_0,y_0,u_0,v_0)=\frac{1}{2\pi}\int_0^{2\pi}\left(P_3(a(t),b(t),c(t),d(t))\right)\mathrm{d}t,$$

$$\mathcal{F}_3(x_0,y_0,u_0,v_0)=\frac{1}{2\pi}\int_0^{2\pi}\left(P_4(a(t),b(t),c(t),d(t))\right)\mathrm{d}t,$$

where

$$a(t)=\cos(t)x_0-\sin(t)y_0+\int_0^t\left(\cos(t-s)h_1(s)-\sin(t-s)h_2(s)\right)\mathrm{d}s,$$

$$b(t)=\sin(t)x_0+\cos(t)y_0+\int_0^t\left(\sin(t-s)h_1(s)+\cos(t-s)h_2(s)\right)\mathrm{d}s,$$

$$c(t)=u_0+\int_0^t h_3(s)\mathrm{d}s,$$

$$d(t)=v_0+\int_0^t h_4(s)\,\mathrm{d}s.$$

If

$$\begin{aligned}&\int_0^{2\pi}\left(\cos(s)h_1(s)+\sin(s)h_2(s)\right)\mathrm{d}s=0,\\&\int_0^{2\pi}\left(-\sin(s)h_1(s)+\cos(s)h_2(s)\right)\mathrm{d}s=0,\\&\int_0^{2\pi}h_3(s)\,\mathrm{d}s=0,\\&\int_0^{2\pi}h_4(s)\,\mathrm{d}s=0,\end{aligned}\tag{1.5}$$

then for every $\left(x_0^*,y_0^*,u_0^*,v_0^*\right)$ *solution of the system*

$$\mathcal{F}_k\left(x_0,y_0,u_0,v_0\right)=0,\quad k=1,2,3,4,$$

satisfying

$$\det\left(\left.\frac{\partial\left(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4\right)}{\partial\left(x_0,y_0,u_0,v_0\right)}\right|_{\left(x_0,y_0,u_0,v_0\right)=\left(x_0^*,y_0^*,u_0^*,v_0^*\right)}\right)\neq 0,$$

the differential system (1.1) *has a periodic solution* $\begin{pmatrix}x(t,\varepsilon)\\y(t,\varepsilon)\\u(t,\varepsilon)\\v(t,\varepsilon)\end{pmatrix}$, *which tends to the periodic solution given by*

$$\begin{pmatrix}x(t)\\y(t)\\u(t)\\v(t)\end{pmatrix}=\begin{pmatrix}\cos(t)x_0^*-\sin(t)y_0^*+\int_0^t\left(\cos(t-s)h_1(s)-\sin(t-s)h_2(s)\right)\mathrm{d}s\\\sin(t)x_0^*+\cos(t)y_0^*+\int_0^t\left(\sin(t-s)h_1(s)+\cos(t-s)h_2(s)\right)\mathrm{d}s\\u_0^*+\int_0^t h_3(s)\,\mathrm{d}s\\v_0^*+\int_0^t h_4(s)\,\mathrm{d}s\end{pmatrix}$$

of the differential system

$$\begin{aligned}\dot{x}&=-y+h_1(t),\\\dot{y}&=x+h_2(t),\\\dot{u}&=h_3(t),\\\dot{v}&=h_4(t),\end{aligned}$$

when $\varepsilon\to 0$.

Note that this solution is periodic of period 2π.

One considers system (1.1) with $\boldsymbol{A}=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&\lambda&1\\0&0&0&\lambda\end{pmatrix}$. Our result are the following.

Theorem 5. *One defines*

$$\mathcal{F}_1\left(x_0,y_0\right)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(\cos(t)P_1\left(a(t),b(t),c(t),d(t)\right)+\sin(t)P_2\left(a(t),b(t),c(t),d(t)\right)\right)\mathrm{d}t,$$

$$\mathcal{F}_2(x_0, y_0)$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(-\sin(t)P_1(a(t), b(t), c(t), d(t)) + \cos(t)P_2(a(t), b(t), c(t), d(t))\right)\mathrm{d}t,$$

where

$$a(t) = \cos(t)x_0 - \sin(t)y_0 + \int_0^t(\cos(t-s)h_1(s) - \sin(t-s)h_2(s))\mathrm{d}s,$$

$$b(t) = \sin(t)x_0 + \cos(t)y_0 + \int_0^t(\sin(t-s)h_1(s) + \cos(t-s)h_2(s))\mathrm{d}s,$$

$$c(t) = e^{\lambda t}u_0 + te^{\lambda t}v_0 + \int_0^t e^{\lambda(t-s)}(h_3(s) + (t-s)h_4(s))\mathrm{d}s,$$

$$d(t) = e^{\lambda t}v_0 + \int_0^t e^{\lambda(t-s)}h_4(s)\mathrm{d}s.$$

If

$$\begin{aligned}
&\int_0^{2\pi}(\cos(s)h_1(s) + \sin(s)h_2(s))\mathrm{d}s = 0,\\
&\int_0^{2\pi}(-\sin(s)h_1(s) + \cos(s)h_2(s))\mathrm{d}s = 0,\\
&u_0 = \frac{2\pi}{(1-e^{2\pi\lambda})^2}\int_0^{2\pi}e^{\lambda(2\pi-s)}h_4(s)\mathrm{d}s + \frac{1}{1-e^{2\pi\lambda}}\int_0^{2\pi}e^{\lambda(2\pi-s)}(h_3(s) - sh_4(s))\mathrm{d}s,\\
&v_0 = \frac{1}{1-e^{2\pi\lambda}}\int_0^{2\pi}e^{\lambda(2\pi-s)}h_4(s)\mathrm{d}s,
\end{aligned} \tag{1.6}$$

then for every (x_0^*, y_0^*) *solution of the system*

$$\mathcal{F}_k(x_0, y_0) = 0, \quad k = 1, 2,$$

satisfying

$$\det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)}\right) \neq 0,$$

the differential system (1) *has a periodic solution* $\begin{pmatrix} x(t,\varepsilon)\\ y(t,\varepsilon)\\ u(t,\varepsilon)\\ v(t,\varepsilon)\end{pmatrix}$, *which tends to the periodic solution given by*

$$\begin{pmatrix} x(t)\\ y(t)\\ u(t)\\ v(t)\end{pmatrix} = \begin{pmatrix} \cos(t)x_0^* - \sin(t)y_0^* + \int_0^t(\cos(t-s)h_1(s) - \sin(t-s)h_2(s))\mathrm{d}s\\ \sin(t)x_0^* + \cos(t)y_0^* + \int_0^t(\sin(t-s)h_1(s) + \cos(t-s)h_2(s))\mathrm{d}s\\ \int_0^t e^{\lambda(t-s)}(h_3(s) + (t-s)h_4(s))\mathrm{d}s + \dfrac{e^{2\pi\lambda}\int_0^{2\pi}e^{\lambda(t-s)}(h_3(s) - sh_4(s))\mathrm{d}s}{e^{2\pi\lambda}-1} + \dfrac{2\pi e^{2\pi\lambda}\int_0^{2\pi}e^{\lambda(t-s)}h_4(s)\mathrm{d}s}{(e^{2\pi\lambda}-1)^2}\\ \dfrac{e^{2\pi\lambda}}{1-e^{2\pi\lambda}}\int_0^{2\pi}e^{\lambda(t-s)}h_4(s)\mathrm{d}s + \int_0^t e^{\lambda(t-s)}h_4(s)\mathrm{d}s \end{pmatrix}$$

of the differential system

$$\dot{x} = -y + h_1(t),$$
$$\dot{y} = x + h_2(t),$$
$$\dot{u} = \lambda u + v + h_3(t),$$
$$\dot{v} = \lambda v + h_4(t),$$

when $\varepsilon \to 0$.

Note that this solution is periodic of period 2π.

2. Basic Results on Averaging Theory

In this section we present the basic results on the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T-periodic solutions from differential systems of the form

$$\dot{x} = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon). \tag{2.1}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ being sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\boldsymbol{x}} = F_0(t, \boldsymbol{x}), \tag{2.2}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory. For a general introduction to the averaging theory see the books of Sanders and Verhulst [17], and of Verhulst [24].

Let $\boldsymbol{x}(t,\boldsymbol{z},\varepsilon)$ be the solution of the system (2.2) such that $\boldsymbol{x}(0,\boldsymbol{z},\varepsilon) = \boldsymbol{z}$. We write the linearization of the unperturbed system along a periodic solution $\boldsymbol{x}(t,\boldsymbol{z},0)$ as

$$\dot{\boldsymbol{y}} = D_x F_0(t, \boldsymbol{x}(t,\boldsymbol{z},0))\,\boldsymbol{y}. \tag{2.3}$$

In what follows we denote by $M_{\boldsymbol{z}}(t)$ some fundamental matrix of the linear differential system (2.2), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; *i.e.* $\xi(x_1,\cdots,x_n) = (x_1,\cdots,x_k)$. We assume that there exists a k-dimensional submanifold $\boldsymbol{Z}$ of Ω filled with T-periodic solutions of (2.2). Then an answer to the problem of bifurcation of T-periodic solutions from the periodic solutions contained in $\boldsymbol{Z}$ for system (2.1) is given in the following result.

Theorem 6. *Let* W *be an open and bounded subset of* $\mathbb{R}^k$, *and let* $\beta : Cl(W) \to \mathbb{R}^{n-k}$ *be a* C^2 *function. We assume that*

(i) $\boldsymbol{Z} = \{\boldsymbol{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in Cl(W)\} \subset \Omega$ *and that for each* $\boldsymbol{z}_\alpha \in \boldsymbol{Z}$ *the solution* $\boldsymbol{x}(t,\boldsymbol{z}_\alpha)$ *of* (8) *is T-periodic;*

(ii) *for each* $\boldsymbol{z}_\alpha \in \boldsymbol{Z}$ *there is a fundamental matrix* $\boldsymbol{M}_{\boldsymbol{z}_\alpha}(t)$ *of* (9) *such that the matrix* $\boldsymbol{M}^{-1}_{\boldsymbol{z}_\alpha}(0) - \boldsymbol{M}^{-1}_{\boldsymbol{z}_\alpha}(T)$ *has in the upper right corner the* $k \times (n-k)$ *zero matrix, and in the lower right corner a* $(n-k)\times(n-k)$ *matrix* Δ_α *with* $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{F}: Cl(W) \to \mathbb{R}^k$

$$\mathcal{F}(\boldsymbol{\alpha}) = \xi\left(\frac{1}{T}\int_0^T \boldsymbol{M}_{z_\alpha}^{-1}(t) F_1(t, \boldsymbol{x}(t, \boldsymbol{z}_\alpha))\,\mathrm{d}t\right). \tag{2.4}$$

If there exists $a \in W$ *with* $\mathcal{F}(a) = 0$ *and* $\det((\mathrm{d}\mathcal{F}/\mathrm{d}\alpha)(a)) \neq 0,$ *then there is a T-periodic solution* $\varphi(t,\varepsilon)$ *of system* (2.1) *such that* $\varphi(0,\varepsilon) \to \boldsymbol{z}_a$ *as* $\varepsilon \to 0$.

Theorem 6 goes back to Malkin [25] *and Roseau* [26], *for a shorter proof see* [27].

We assume that there exists an open set V with $Cl(V) \subset \Omega$ such that for each $\boldsymbol{z} \in Cl(V)$, $\boldsymbol{x}(t,\boldsymbol{z},0)$ is T-periodic, where $\boldsymbol{x}(t,\boldsymbol{z},0)$ denotes the solution of the unperturbed system (2.2) with $\boldsymbol{x}(0,\boldsymbol{z},0) = \boldsymbol{z}$. The set $Cl(V)$ is isochronous for the system (2.1); *i.e.* it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solutions $\boldsymbol{x}(t,\boldsymbol{z},0)$ contained in $Cl(V)$ is given in the following result.

Theorem 7. (*Perturbations of an isochronous set*)

We assume that there exists an open and bounded set V *with* $Cl(V) \subset \Omega$ *such that for each* $\boldsymbol{z} \in Cl(V)$, *the solution* $\boldsymbol{x}(t,\boldsymbol{z},0)$ *is T-periodic, considering a function* $\mathcal{F}: Cl(V) \to \mathbb{R}^n$ *defined by*

$$\mathcal{F}(\boldsymbol{z}) = \int_0^T M_z^{-1}(t) F_1(t, \boldsymbol{x}(t,\boldsymbol{z}))\,\mathrm{d}t. \tag{2.5}$$

If there exists an $a \in V$ *with* $\mathcal{F}(a) = 0$ *and* $\det\left(\left(\frac{\partial \mathcal{F}}{\partial \boldsymbol{z}}\right)(a)\right) \neq 0,$ *then there exists a T-periodic solution* $\varphi(t,\varepsilon)$ *to system* (2.1) *such that* $\varphi(0,\varepsilon) \to a$ *as* $\varepsilon \to 0.$

For the proof of theorem 7 *please see Corollary* 1 *of* [27].

3. Proof of Theorems

3.1. Proof of Theorem 1

We shall apply Theorem 7 to the differential system (1.1). It can be written as system (2.1) taking

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}, t = t, F_0(t,\boldsymbol{x}) = \begin{pmatrix} -y + h_1(t) \\ x + h_2(t) \\ -v + h_3(t) \\ u + h_4(t) \end{pmatrix} \text{ and } F_1(t,\boldsymbol{x}) = \begin{pmatrix} P_1(x,y,u,v) \\ P_2(x,y,u,v) \\ P_3(x,y,u,v) \\ P_4(x,y,u,v) \end{pmatrix}.$$

We shall study the periodic solutions of system (2.2) in our case the system $(1.1)_{\varepsilon=0}$.

By using

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \\ u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} h_1(s) \\ h_2(s) \\ h_3(s) \\ h_4(s) \end{pmatrix} \mathrm{d}s,$$

we obtain

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t)x_0 - \sin(t)y_0 + \int_0^t \left(\cos(t-s)h_1(s) - \sin(t-s)h_2(s)\right)\mathrm{d}s \\ \sin(t)x_0 + \cos(t)y_0 + \int_0^t \left(\sin(t-s)h_1(s) + \cos(t-s)h_2(s)\right)\mathrm{d}s \\ \cos(t)u_0 - \sin(t)v_0 + \int_0^t \left(\cos(t-s)h_3(s) - \sin(t-s)h_4(s)\right)\mathrm{d}s \\ \sin(t)u_0 + \cos(t)v_0 + \int_0^t \left(\sin(t-s)h_3(s) + \cos(t-s)h_4(s)\right)\mathrm{d}s \end{pmatrix}, \quad (3.1)$$

it can be written as

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \\ 0 \end{pmatrix} x_0 + \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \\ 0 \end{pmatrix} y_0 + \begin{pmatrix} 0 \\ 0 \\ \cos(t) \\ \sin(t) \end{pmatrix} u_0 + \begin{pmatrix} 0 \\ 0 \\ -\sin(t) \\ \cos(t) \end{pmatrix} v_0$$

$$+ \begin{pmatrix} \int_0^t \left(\cos(t-s)h_1(s) - \sin(t-s)h_2(s)\right)\mathrm{d}s \\ \int_0^t \left(\sin(t-s)h_1(s) + \cos(t-s)h_2(s)\right)\mathrm{d}s \\ \int_0^t \left(\cos(t-s)h_3(s) - \sin(t-s)h_4(s)\right)\mathrm{d}s \\ \int_0^t \left(\sin(t-s)h_3(s) + \cos(t-s)h_4(s)\right)\mathrm{d}s \end{pmatrix}$$

These solutions are $2\pi-$periodic if and only if

$$\begin{pmatrix} x(2\pi) \\ y(2\pi) \\ u(2\pi) \\ v(2\pi) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{pmatrix}.$$

We obtain the periodicity conditions given in the theorem 1 by (1.2).

The set of periodic solutions has dimension 4. To look for the periodic solutions of our system (1.1) we must calculate the zeros $\boldsymbol{z} = (x_0, y_0, u_0, v_0)$ of the system $\mathcal{F}(\boldsymbol{z}) = 0$, where $\mathcal{F}(\boldsymbol{z})$ is given by (2.5). The fundamental matrix $\boldsymbol{M}(t)$ of the differential system (2.3) is

$$\boldsymbol{M}(t) = \boldsymbol{M}_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}.$$

Now computing the function $\mathcal{F}(\boldsymbol{z})$ we find the system

$$\begin{cases} \mathcal{F}_1(x_0, y_0, u_0, v_0) = 0, \\ \mathcal{F}_2(x_0, y_0, u_0, v_0) = 0, \\ \mathcal{F}_3(x_0, y_0, u_0, v_0) = 0, \\ \mathcal{F}_4(x_0, y_0, u_0, v_0) = 0, \end{cases} \quad (3.2)$$

where $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ have been defined in the statement of Theorem 1. The zeros $(x_0^*, y_0^*, u_0^*, v_0^*)$ of the system (3.2) with respect to the variables x_0, y_0, u_0 and v_0 provide periodic solutions of system (1.1) with $\varepsilon \neq 0$ being sufficiently small if they are simple, *i.e.* if

$$\det\left(\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4)}{\partial(x_0,y_0,u_0,v_0)}\bigg|_{(x_0,y_0,u_0,v_0)=(x_0^*,y_0^*,u_0^*,v_0^*)}\right)\neq 0.$$

For simple zero $(x_0^*,y_0^*,u_0^*,v_0^*)$ of system (3.2) we obtain a 2π-periodic solution $\begin{pmatrix} x(t,\varepsilon)\\ y(t,\varepsilon)\\ u(t,\varepsilon)\\ v(t,\varepsilon)\end{pmatrix}$ of the differential system (1.1), for $\varepsilon\neq 0$ being sufficiently small which tends to the periodic solution given in the statement of theorem 1 of the differential system $(1.1)_{\varepsilon=0}$ when $\varepsilon\to 0$.

This completes the proof of Theorem 1.

3.2. Proof of Theorem 2

We shall apply Theorem 6 to the differential system (1.1). It can be written as system (2.1) taking

$$\boldsymbol{x}=\begin{pmatrix} x\\ y\\ u\\ v\end{pmatrix}, t=t, F_0(t,\boldsymbol{x})=\begin{pmatrix} -y+h_1(t)\\ x+h_2(t)\\ -v+h_3(t)\\ u+h_4(t)\end{pmatrix} \text{ and } F_1(t,\boldsymbol{x})=\begin{pmatrix} P_1(x,y,u,v)\\ P_2(x,y,u,v)\\ P_3(x,y,u,v)\\ P_4(x,y,u,v)\end{pmatrix}.$$

We shall study the periodic solutions of system (2.2) in our case the system $(1.1)_{\varepsilon=0}$.

By using

$$\begin{pmatrix} x(t)\\ y(t)\\ u(t)\\ v(t)\end{pmatrix}=e^{At}\begin{pmatrix} x_0\\ y_0\\ u_0\\ v_0\end{pmatrix}+\int_0^t e^{A(t-s)}\begin{pmatrix} h_1(s)\\ h_2(s)\\ h_3(s)\\ h_4(s)\end{pmatrix}\mathrm{d}s,$$

we obtain

$$\begin{pmatrix} x(t)\\ y(t)\\ u(t)\\ v(t)\end{pmatrix}=\begin{pmatrix} \cos(t)x_0-\sin(t)y_0+\int_0^t\left(\cos(t-s)h_1(s)-\sin(t-s)h_2(s)\right)\mathrm{d}s\\ \sin(t)x_0+\cos(t)y_0+\int_0^t\left(\sin(t-s)h_1(s)+\cos(t-s)h_2(s)\right)\mathrm{d}s\\ e^{\lambda t}u_0+\int_0^t e^{\lambda(t-s)}h_3(s)\mathrm{d}s\\ e^{\mu t}v_0+\int_0^t e^{\mu(t-s)}h_4(s)\mathrm{d}s\end{pmatrix}, \quad (3.3)$$

it can be written as

$$\begin{pmatrix} x(t)\\ y(t)\\ u(t)\\ v(t)\end{pmatrix}=\begin{pmatrix} \cos(t)\\ \sin(t)\\ 0\\ 0\end{pmatrix}x_0+\begin{pmatrix} -\sin(t)\\ \cos(t)\\ 0\\ 0\end{pmatrix}y_0+\begin{pmatrix} 0\\ 0\\ e^{\lambda t}\\ 0\end{pmatrix}u_0$$

$$+\begin{pmatrix} 0\\ 0\\ 0\\ e^{\mu t}\end{pmatrix}v_0+\begin{pmatrix} \int_0^t\left(\cos(t-s)h_1(s)-\sin(t-s)h_2(s)\right)\mathrm{d}s\\ \int_0^t\left(\sin(t-s)h_1(s)+\cos(t-s)h_2(s)\right)\mathrm{d}s\\ \int_0^t e^{\lambda(t-s)}h_3(s)\mathrm{d}s\\ \int_0^t e^{\mu(t-s)}h_4(s)\mathrm{d}s\end{pmatrix}$$

These solutions are 2π – periodic if and only if

$$\begin{pmatrix} x(2\pi) \\ y(2\pi) \\ u(2\pi) \\ v(2\pi) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{pmatrix}.$$

We obtain the periodicity conditions given in the theorem 2 by (1.3). Since u_0 and v_0 are now fixed then the set of periodic solutions has dimension 2. To look for the periodic solutions of our system (1.1) we must calculate the zeros $\mathbf{z} = (x_0, y_0)$ of the system $\mathcal{F}(\mathbf{z}) = 0$, where $\mathcal{F}(\mathbf{z})$ is given by (2.4). The fundamental matrix $\boldsymbol{M}(t)$ of the differential system (2.3) is

$$\boldsymbol{M}(t) = \boldsymbol{M}_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & e^{\lambda t} & 0 \\ 0 & 0 & 0 & e^{\mu t} \end{pmatrix}.$$

Now computing the function $\mathcal{F}(\mathbf{z})$ we find the system

$$\begin{cases} \mathcal{F}_1(x_0, y_0) = 0, \\ \mathcal{F}_2(x_0, y_0) = 0, \end{cases} \tag{3.4}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ have been defined in the statement of Theorem 2.

The zeros (x_0^*, y_0^*) of the system (3.4) with respect to the variables x_0, y_0 provide periodic solutions of system (1.1) with $\varepsilon \neq 0$ being sufficiently small if they are simple, *i.e.* if

$$\det\left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)} \right|_{(x_0, y_0) = (x_0^*, y_0^*)} \right) \neq 0.$$

For simple zeros (x_0^*, y_0^*) of system (3.2) we obtain a 2π-periodic solution $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \\ u(t,\varepsilon) \\ v(t,\varepsilon) \end{pmatrix}$ of the differential system (1.1), for $\varepsilon \neq 0$ being sufficiently small which tends to the periodic solution given in the statement of theorem 2 of the differential system $(1.1)_{\varepsilon=0}$ when $\varepsilon \to 0$.

This completes the proof of Theorem 2.

3.3. Proof of Theorem 3

We shall apply Theorem 6 to the differential system (1.1). It can be written as system (2.1) taking

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}, t = t, F_0(t, \mathbf{x}) = \begin{pmatrix} -y + h_1(t) \\ x + h_2(t) \\ h_3(t) \\ \mu v + h_4(t) \end{pmatrix} \text{ and } F_1(t, \mathbf{x}) = \begin{pmatrix} P_1(x, y, u, v) \\ P_2(x, y, u, v) \\ P_3(x, y, u, v) \\ P_4(x, y, u, v) \end{pmatrix}.$$

We shall study the periodic solutions of system (2.2) in our case the system $(1.1)_{\varepsilon=0}$.

By using

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = e^{At}\begin{pmatrix} x_0 \\ y_0 \\ u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{A(t-s)}\begin{pmatrix} h_1(s) \\ h_2(s) \\ h_3(s) \\ h_4(s) \end{pmatrix}\mathrm{d}s,$$

we obtain

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t)x_0 - \sin(t)y_0 + \int_0^t \left(\cos(t-s)h_1(s) - \sin(t-s)h_2(s)\right)\mathrm{d}s \\ \sin(t)x_0 + \cos(t)y_0 + \int_0^t \left(\sin(t-s)h_1(s) + \cos(t-s)h_2(s)\right)\mathrm{d}s \\ u_0 + \int_0^t h_3(s)\mathrm{d}s \\ e^{\mu t}v_0 + \int_0^t e^{\mu(t-s)}h_4(s)\mathrm{d}s \end{pmatrix}, \quad (3.5)$$

it can be written as

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \\ 0 \end{pmatrix} x_0 + \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \\ 0 \end{pmatrix} y_0 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_0$$
$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{\mu t} \end{pmatrix} v_0 + \begin{pmatrix} \int_0^t \left(\cos(t-s)h_1(s) - \sin(t-s)h_2(s)\right)\mathrm{d}s \\ \int_0^t \left(\sin(t-s)h_1(s) + \cos(t-s)h_2(s)\right)\mathrm{d}s \\ \int_0^t h_3(s)\mathrm{d}s \\ \int_0^t e^{\mu(t-s)}h_4(s)\mathrm{d}s \end{pmatrix}$$

These solutions are $2\pi-$periodic if and only if

$$\begin{pmatrix} x(2\pi) \\ y(2\pi) \\ u(2\pi) \\ v(2\pi) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{pmatrix}.$$

We obtain the periodicity conditions given in the theorem 3 by (1.4). Since v_0 is now fixed then the set of periodic solutions has dimension 3. To look for the periodic solutions of our system (1.1) we must calculate the zeros $\boldsymbol{z} = (x_0, y_0, u_0)$ of the system $\mathcal{F}(\boldsymbol{z}) = 0$, where $\mathcal{F}(\boldsymbol{z})$ is given by (2.4). The fundamental matrix $\boldsymbol{M}(t)$ of the differential system (2.3) is

$$\boldsymbol{M}(t) = \boldsymbol{M}_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\mu t} \end{pmatrix}.$$

Now computing the function $\mathcal{F}(\boldsymbol{z})$ we find the system

$$\begin{cases} \mathcal{F}_1(x_0, y_0, u_0) = 0, \\ \mathcal{F}_2(x_0, y_0, u_0) = 0, \\ \mathcal{F}_3(x_0, y_0, u_0) = 0, \end{cases} \tag{3.6}$$

where $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ have been defined in the statement of Theorem 3.

The zeros (x_0^*, y_0^*, u_0^*) of the system (3.5) with respect to the variables x_0, y_0 and u_0 provide periodic solutions of system (1.1) with $\varepsilon \neq 0$ being sufficiently small if they are simple, *i.e.* if

$$\det\left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)}{\partial(x_0, y_0, u_0)} \right|_{(x_0, y_0, u_0) = (x_0^*, y_0^*, u_0^*)} \right) \neq 0.$$

For simple zeros (x_0^*, y_0^*, u_0^*) of system (3.5) we obtain a 2π-periodic solution $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \\ u(t,\varepsilon) \\ v(t,\varepsilon) \end{pmatrix}$ of the differential system (1.1), for $\varepsilon \neq 0$ being sufficiently small which tends to the periodic solution given in the statement of theorem 3 of the differential system $(1.1)_{\varepsilon=0}$ when $\varepsilon \to 0$.

This completes the proof of Theorem 3.

3.4. Proof of Theorem 4

We shall apply Theorem 7 to the differential system (1.1). It can be written as system (2.1) taking

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}, t = t, F_0(t, \boldsymbol{x}) = \begin{pmatrix} -y + h_1(t) \\ x + h_2(t) \\ h_3(t) \\ h_4(t) \end{pmatrix} \text{ and } F_1(t, \boldsymbol{x}) = \begin{pmatrix} P_1(x, y, u, v) \\ P_2(x, y, u, v) \\ P_3(x, y, u, v) \\ P_4(x, y, u, v) \end{pmatrix}.$$

We shall study the periodic solutions of system (2.2) in our case the system $(1.1)_{\varepsilon=0}$.

By using

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \\ u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} h_1(s) \\ h_2(s) \\ h_3(s) \\ h_4(s) \end{pmatrix} \mathrm{d}s,$$

we obtain

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) x_0 - \sin(t) y_0 + \int_0^t \left(\cos(t-s) h_1(s) - \sin(t-s) h_2(s) \right) \mathrm{d}s \\ \sin(t) x_0 + \cos(t) y_0 + \int_0^t \left(\sin(t-s) h_1(s) + \cos(t-s) h_2(s) \right) \mathrm{d}s \\ u_0 + \int_0^t h_3(s) \mathrm{d}s \\ v_0 + \int_0^t h_4(s) \mathrm{d}s \end{pmatrix}, \tag{3.7}$$

it can be written as

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \\ 0 \end{pmatrix} x_0 + \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \\ 0 \end{pmatrix} y_0 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_0$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v_0 + \begin{pmatrix} \int_0^t \left(\cos(t-s) h_1(s) - \sin(t-s) h_2(s)\right) ds \\ \int_0^t \left(\sin(t-s) h_1(s) + \cos(t-s) h_2(s)\right) ds \\ u_0 + \int_0^t h_3(s) ds \\ v_0 + \int_0^t h_4(s) ds \end{pmatrix}$$

These solutions are $2\pi-$periodic if and only if

$$\begin{pmatrix} x(2\pi) \\ y(2\pi) \\ u(2\pi) \\ v(2\pi) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{pmatrix}.$$

We obtain the periodicity conditions given in the theorem 4 by (1.5).

The set of periodic solutions has dimension 4. To look for the periodic solutions of our system (1.1) we must calculate the zeros $\boldsymbol{z} = (x_0, y_0, u_0, v_0)$ of the system $\mathcal{F}(\boldsymbol{z}) = 0$, where $\mathcal{F}(\boldsymbol{z})$ is given by (2.5). The fundamental matrix $\boldsymbol{M}(t)$ of the differential system (2.3) is

$$\boldsymbol{M}(t) = \boldsymbol{M}_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now computing the function $\mathcal{F}(\boldsymbol{z})$ we find the system

$$\begin{cases} \mathcal{F}_1(x_0, y_0, u_0, v_0) = 0, \\ \mathcal{F}_2(x_0, y_0, u_0, v_0) = 0, \\ \mathcal{F}_3(x_0, y_0, u_0, v_0) = 0, \\ \mathcal{F}_4(x_0, y_0, u_0, v_0) = 0, \end{cases} \tag{3.8}$$

where $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ have been defined in the statement of Theorem 4.

The zeros $(x_0^*, y_0^*, u_0^*, v_0^*)$ of the system (3.8) with respect to the variables x_0, y_0, u_0 and v_0 provide periodic solutions of system (1.1) and $\varepsilon \neq 0$ being sufficiently small if they are simple, *i.e.* if

$$\det\left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)}{\partial(x_0, y_0, u_0, v_0)} \right|_{(x_0, y_0, u_0, v_0) = (x_0^*, y_0^*, u_0^*, v_0^*)} \right) \neq 0.$$

For simple zeros $(x_0^*, y_0^*, u_0^*, v_0^*)$ of system (3.8) we obtain a 2π-periodic solution $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \\ u(t,\varepsilon) \\ v(t,\varepsilon) \end{pmatrix}$ of the differential system (1.1), for $\varepsilon \neq 0$ being sufficiently small which tends to the periodic solution given in the statement of theorem 4 of the differential system $(1.1)_{\varepsilon=0}$ when $\varepsilon \to 0$.

This completes the proof of Theorem 4.

3.5. Proof of Theorem 5

We shall apply Theorem 6 to the differential system (1.1). It can be written as system (2.1) taking

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}, t = t, F_0(t, \boldsymbol{x}) = \begin{pmatrix} -y + h_1(t) \\ x + h_2(t) \\ \lambda u + v + h_3(t) \\ \lambda v + h_4(t) \end{pmatrix} \text{ and } F_1(t, \boldsymbol{x}) = \begin{pmatrix} P_1(x, y, u, v) \\ P_2(x, y, u, v) \\ P_3(x, y, u, v) \\ P_4(x, y, u, v) \end{pmatrix}.$$

We shall study the periodic solutions of system (2.2) in our case the system $(1.1)_{\varepsilon=0}$.

By using

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \\ u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} h_1(s) \\ h_2(s) \\ h_3(s) \\ h_4(s) \end{pmatrix} \mathrm{d}s,$$

we obtain

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) x_0 - \sin(t) y_0 + \int_0^t \left(\cos(t-s) h_1(s) - \sin(t-s) h_2(s)\right) \mathrm{d}s \\ \sin(t) x_0 + \cos(t) y_0 + \int_0^t \left(\sin(t-s) h_1(s) + \cos(t-s) h_2(s)\right) \mathrm{d}s \\ e^{\lambda t} u_0 + t e^{\lambda t} v_0 + \int_0^t e^{\lambda(t-s)} \left(h_3(s) + (t-s) h_4(s)\right) \mathrm{d}s \\ e^{\lambda t} v_0 + \int_0^t e^{\lambda(t-s)} h_4(s) \mathrm{d}s \end{pmatrix}, \quad (3.9)$$

it can be written as

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \\ 0 \end{pmatrix} x_0 + \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \\ 0 \end{pmatrix} y_0 + \begin{pmatrix} 0 \\ 0 \\ e^{\lambda t} \\ 0 \end{pmatrix} u_0$$

$$+ \begin{pmatrix} 0 \\ 0 \\ t e^{\lambda t} \\ e^{\lambda t} \end{pmatrix} v_0 + \begin{pmatrix} \int_0^t \left(\cos(t-s) h_1(s) - \sin(t-s) h_2(s)\right) \mathrm{d}s \\ \int_0^t \left(\sin(t-s) h_1(s) + \cos(t-s) h_2(s)\right) \mathrm{d}s \\ \int_0^t e^{\lambda(t-s)} \left(h_3(s) + (t-s) h_4(s)\right) \mathrm{d}s \\ \int_0^t e^{\lambda(t-s)} h_4(s) \mathrm{d}s \end{pmatrix}$$

These solutions are $2\pi-$periodic if and only if

$$\begin{pmatrix} x(2\pi) \\ y(2\pi) \\ u(2\pi) \\ v(2\pi) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{pmatrix}.$$

We obtain the periodicity conditions given in the theorem 5 by (1.6). Since u_0 and v_0 are now fixed then the set of periodic solutions has dimension 2. To look for the periodic solutions of our system (1.1) we must calculate the zeros $\mathbf{z}=(x_0,y_0)$ of the system $\mathcal{F}(\mathbf{z})=0$, where $\mathcal{F}(\mathbf{z})$ is given by (2.4). The fundamental matrix $\mathbf{M}(t)$ of the differential system (2.3) is

$$\mathbf{M}(t)=\mathbf{M}_z(t)=\begin{pmatrix}\cos(t) & -\sin(t) & 0 & 0\\ \sin(t) & \cos(t) & 0 & 0\\ 0 & 0 & e^{\lambda t} & te^{\lambda t}\\ 0 & 0 & 0 & e^{\lambda t}\end{pmatrix}.$$

Now computing the function $\mathcal{F}(\mathbf{z})$ we find the system

$$\begin{cases}\mathcal{F}_1(x_0,y_0)=0,\\ \mathcal{F}_2(x_0,y_0)=0,\end{cases} \tag{3.10}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ have been defined in the statement of Theorem 5. The zeros (x_0^*,y_0^*) of the system (3.10) with respect to the variables x_0,y_0 provide periodic solutions of system (1.1) and $\varepsilon\neq 0$ being sufficiently small if they are simple, *i.e.* if

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(x_0,y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)}\right)\neq 0.$$

For simple zeros (x_0^*,y_0^*) of system (3.10) we obtain a 2π-periodic solution $\begin{pmatrix}x(t,\varepsilon)\\ y(t,\varepsilon)\\ u(t,\varepsilon)\\ v(t,\varepsilon)\end{pmatrix}$ of the differential system (1.1), for $\varepsilon\neq 0$ being sufficiently small which tends to the periodic solution given in the statement of theorem 5 of the differential system $(1.1)_{\varepsilon=0}$ when $\varepsilon\to 0$.

This completes the proof of Theorem 5.

4. Applications

4.1. Application of Theorem 1

Consider the differential system (1) where

$$\mathbf{A}=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix},$$

$$\begin{pmatrix}h_1(t)\\ h_2(t)\\ h_3(t)\\ h_4(t)\end{pmatrix}=\begin{pmatrix}\sin(t)\\ \cos(t)\\ \sin(t)\\ \cos(t)\end{pmatrix}$$

and

$$\begin{pmatrix} P_1(x,y,u,v) \\ P_2(x,y,u,v) \\ P_3(x,y,u,v) \\ P_4(x,y,u,v) \end{pmatrix} = \begin{pmatrix} x+y-yx^2 \\ x+y-xy^2 \\ x+y+u+v \\ x+y+u+v \end{pmatrix}.$$

We can easily verify conditions (1.2)

$$\int_0^{2\pi} \left(\cos(s)\sin(s)-\sin(s)\cos(s)\right)\mathrm{d}s = 0,$$

$$\int_0^{2\pi} \left(-\sin^2(s)+\cos^2(s)\right)\mathrm{d}s = 0,$$

$$\int_0^{2\pi} \left(\cos(s)\sin(s)-\sin(s)\cos(s)\right)\mathrm{d}s = 0,$$

$$\int_0^{2\pi} \left(-\sin^2(s)+\cos^2(s)\right)\mathrm{d}s = 0,$$

Computing the functions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ we find

$$\mathcal{F}_1(x_0,y_0,u_0,v_0) = \frac{1}{2}+x_0+\frac{3}{8}y_0+\frac{3}{4}x_0y_0,$$

$$\mathcal{F}_2(x_0,y_0,u_0,v_0) = -\frac{1}{2}-\frac{1}{8}x_0-\frac{1}{8}x_0^2+y_0+\frac{5}{8}y_0^2,$$

$$\mathcal{F}_3(x_0,y_0,u_0,v_0) = x_0+u_0+1,$$

$$\mathcal{F}_4(x_0,y_0,u_0,v_0) = y_0+v_0-1.$$

The stability of the periodic solutions associated to a simple zero of $\mathcal{F}$ is controlled by the eigenvalues of the jacobian matrix.

The system $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3=\mathcal{F}_4=0$ has four solutions $\left(x_0^*,y_0^*,u_0^*,v_0^*\right)$ given by

$$\left(-\frac{1}{2},-\frac{4}{5}+\frac{\sqrt{139}}{10},-\frac{1}{2},\frac{9}{5}-\frac{\sqrt{139}}{10}\right),\ \left(-\frac{1}{2},-\frac{4}{5}-\frac{\sqrt{139}}{10},-\frac{1}{2},\frac{9}{5}-\frac{\sqrt{139}}{10}\right),$$

$$\left(-\frac{1}{2},-\frac{4}{5}+\frac{\sqrt{139}}{10},-\frac{1}{2},\frac{9}{5}+\frac{\sqrt{139}}{10}\right),\ \left(-\frac{1}{2},-\frac{4}{5}-\frac{\sqrt{139}}{10},-\frac{1}{2},\frac{9}{5}+\frac{\sqrt{139}}{10}\right)$$

and the eigenvalues of the jacobian matrix of $\begin{pmatrix} \mathcal{F}_1(x_0,y_0,u_0,v_0) \\ \mathcal{F}_2(x_0,y_0,u_0,v_0) \\ \mathcal{F}_3(x_0,y_0,u_0,v_0) \\ \mathcal{F}_4(x_0,y_0,u_0,v_0) \end{pmatrix}$ at these solutions are $\begin{pmatrix} \frac{\sqrt{139}}{8} \\ \frac{2}{5}+\frac{3\sqrt{139}}{40} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{139}}{8} \\ \frac{2}{5}+\frac{3\sqrt{139}}{40} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{2}{5}-\frac{3\sqrt{139}}{40} \\ \frac{-\sqrt{139}}{8} \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{2}{5}-\frac{3\sqrt{139}}{40} \\ \frac{-\sqrt{139}}{8} \\ 1 \\ 1 \end{pmatrix}$, which have all at least two positive real parts. Since

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4)}{\partial(x_0,y_0,u_0,v_0)}\right|_{(x_0,y_0,u_0,v_0)=\left(x_0^*,y_0^*,u_0^*,v_0^*\right)}\right)$$

at these four solutions $\left(x_0^*,y_0^*,u_0^*,v_0^*\right)$ is $1,89,0,71,1,89,0,71$, respectively, then the differential system (1.1) has four periodic unstable solutions $\begin{pmatrix} x_k(t,\varepsilon) \\ y_k(t,\varepsilon) \\ u_k(t,\varepsilon) \\ v_k(t,\varepsilon) \end{pmatrix}$ with $k=1,2,3,4$, tending to the unstable periodic solutions $\begin{pmatrix} x_k(t) \\ y_k(t) \\ u_k(t) \\ v_k(t) \end{pmatrix}$ where

$$\begin{pmatrix} x_1(t) \\ y_1(t) \\ u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\cos(t)+\left(\frac{4}{5}-\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)+\left(-\frac{4}{5}+\frac{\sqrt{139}}{10}\right)\cos(t) \\ -\frac{1}{2}\cos(t)-\left(\frac{9}{5}+\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)+\left(\frac{9}{5}+\frac{\sqrt{139}}{10}\right)\cos(t) \end{pmatrix}$$

$$\begin{pmatrix} x_2(t) \\ y_2(t) \\ u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\cos(t)+\left(\frac{4}{5}-\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)+\left(-\frac{4}{5}+\frac{\sqrt{139}}{10}\right)\cos(t) \\ -\frac{1}{2}\cos(t)-\left(\frac{9}{5}-\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)+\left(\frac{9}{5}-\frac{\sqrt{139}}{10}\right)\cos(t) \end{pmatrix}$$

$$\begin{pmatrix} x_3(t) \\ y_3(t) \\ u_3(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\cos(t)+\left(\frac{4}{5}+\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)-\left(\frac{4}{5}+\frac{\sqrt{139}}{10}\right)\cos(t) \\ -\frac{1}{2}\cos(t)-\left(\frac{9}{5}+\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)+\left(\frac{9}{5}+\frac{\sqrt{139}}{10}\right)\cos(t) \end{pmatrix}$$

$$\begin{pmatrix} x_4(t) \\ y_4(t) \\ u_4(t) \\ v_4(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\cos(t)+\left(\frac{4}{5}+\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)-\left(\frac{4}{5}+\frac{\sqrt{139}}{10}\right)\cos(t) \\ -\frac{1}{2}\cos(t)-\left(\frac{9}{5}-\frac{\sqrt{139}}{10}\right)\sin(t) \\ \frac{1}{2}\sin(t)+\left(\frac{9}{5}-\frac{\sqrt{139}}{10}\right)\cos(t) \end{pmatrix}$$

of the differential system

$$\dot{x} = -y + \sin(t),$$
$$\dot{y} = x + \cos(t),$$
$$\dot{u} = -v + \sin(t),$$
$$\dot{v} = u + \cos(t)$$

when $\varepsilon \to 0.$

4.2. Application of Theorem 2

Consider the differential system (1.1) where

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix},$$

$$\begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ h_4(t) \end{pmatrix} = \begin{pmatrix} \sin(t)\cos(s) \\ \cos^2(t) \\ \cos(t) \\ \cos(t) \end{pmatrix}$$

and

$$\begin{pmatrix} P_1(x,y,u,v) \\ P_2(x,y,u,v) \\ P_3(x,y,u,v) \\ P_4(x,y,u,v) \end{pmatrix} = \begin{pmatrix} x-y+xy \\ x+y-xy \\ x+y+u \\ x+y \end{pmatrix}.$$

We can easily verify conditions (1.3)

$$\int_0^{2\pi} \left(\cos^2(s)\sin(s) - \sin(s)\cos^2(s)\right) \mathrm{d}s = 0,$$

$$\int_0^{2\pi} \left(-\sin^2(s)\cos(s) + \cos^3(s)\right) \mathrm{d}s = 0,$$

$$u_0 = \frac{-3}{10},$$

$$v_0 = -\frac{5}{26}.$$

Computing the functions $\mathcal{F}_1$, $\mathcal{F}_2$ we find

$$\mathcal{F}_1(x_0, y_0) = \frac{7}{9} + \frac{7}{6}x_0 - \frac{4}{3}y_0,$$

$$\mathcal{F}_2(x_0, y_0) = \frac{7}{9} + \frac{7}{6}x_0 + \frac{4}{3}y_0,$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = 0$ has one solution (x_0^*, y_0^*) given by $\left(-\frac{2}{3}, 0\right)$ and the eigenvalues of the jacobian matrix of $\begin{pmatrix} \mathcal{F}_1(x_0, y_0) \\ \mathcal{F}_2(x_0, y_0) \end{pmatrix}$ at this solution are

$\begin{pmatrix} \frac{5}{4}+\frac{1}{12}I\sqrt{233} \\ \frac{5}{4}-\frac{1}{12}I\sqrt{233} \end{pmatrix}$, which have two positive real parts. Since

$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(x_0,y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)} \right) = \frac{28}{9}$, then the differential system (1.1) has an unstable periodic solution $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \\ u(t,\varepsilon) \\ v(t,\varepsilon) \end{pmatrix}$, tending to the unstable periodic solution

$$\begin{pmatrix} x(t) \\ y(t) \\ u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{-1}{3}-\frac{1}{3}\cos^2(t) \\ \frac{1}{3}\sin(t)\cos(t) \\ \frac{-3}{10}\cos(t)+\frac{1}{10}\sin(t) \\ \frac{-5}{26}\cos(t)+\frac{1}{26}\sin(t) \end{pmatrix}$$

of the differential system

$$\dot{x} = -y+\sin(t)\cos(t),$$
$$\dot{y} = x+\cos^2(t),$$
$$\dot{u} = 3u+\cos(t),$$
$$\dot{v} = 5v+\cos(t),$$

when $\varepsilon \to 0$.

4.3. Application of Theorem 3

Consider the differential system (1.1) where

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ h_4(t) \end{pmatrix} = \begin{pmatrix} \sin(t)\cos(s) \\ \sin^3(t)\cos(t) \\ \cos(t) \\ \sin(t) \end{pmatrix}$$

and

$$\begin{pmatrix} P_1(x,y,u,v) \\ P_2(x,y,u,v) \\ P_3(x,y,u,v) \\ P_4(x,y,u,v) \end{pmatrix} = \begin{pmatrix} x-y+xy \\ x+y-xy \\ x+y-u^2+3u \\ x+y \end{pmatrix}.$$

We can easily verify conditions (1.4)

$$\int_0^{2\pi}\left(\cos^2(s)\sin(s)+\sin^4(s)\cos(s)\right)ds=0,$$
$$\int_0^{2\pi}\left(-\sin^2(s)\cos(s)+\cos^2(s)\sin^3(s)\right)ds=0,$$
$$\int_0^{2\pi}\cos(s)ds=0,$$
$$v_0=-\frac{1}{10}.$$

Computing the functions $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ we find

$$\mathcal{F}_1(x_0,y_0,u_0)=\frac{15}{16}x_0-\frac{49}{48}y_0+\frac{127}{720},$$
$$\mathcal{F}_2(x_0,y_0,u_0)=\frac{47}{48}x_0+\frac{17}{16}y_0+\frac{737}{720},$$
$$\mathcal{F}_3(x_0,y_0,u_0)=-u_0^3+3u_0-\frac{1}{2},$$

The system $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3=0$ has two solutions (x_0^*,y_0^*,u_0^*) given by $\left(-\frac{1}{3},-\frac{2}{15},\frac{1}{2}(3-\sqrt{7})\right)$, $\left(-\frac{1}{3},-\frac{2}{15},\frac{1}{2}(3+\sqrt{7})\right)$ and the eigenvalues of the jacobian matrix of $\begin{pmatrix}\mathcal{F}_1(x_0,y_0,u_0)\\ \mathcal{F}_2(x_0,y_0,u_0)\\ \mathcal{F}_3(x_0,y_0,u_0)\end{pmatrix}$ at these solutions are $\begin{pmatrix}1+I\frac{\sqrt{2294}}{48}\\ 1-I\frac{\sqrt{2294}}{48}\\ \sqrt{7}\end{pmatrix}$, $\begin{pmatrix}1+I\frac{\sqrt{2294}}{48}\\ 1-I\frac{\sqrt{2294}}{48}\\ -\sqrt{7}\end{pmatrix}$, which have all at least two positive real parts. Since $\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)}{\partial(x_0,y_0,u_0)}\right|_{(x_0,y_0,u_0)=(x_0^*,y_0^*,u_0^*)}\right)$ is $\frac{2299\sqrt{7}}{1152}$, $\frac{-2299\sqrt{7}}{1152}$ respectively, then the differential system (1.1) has two unstable periodic solutions $\begin{pmatrix}x_k(t,\varepsilon)\\ y_k(t,\varepsilon)\\ u_k(t,\varepsilon)\\ v_k(t,\varepsilon)\end{pmatrix}$ with $k = 1, 2$, tending to the unstable periodic solutions

$$\begin{pmatrix}x_1(t)\\ y_1(t)\\ u_1(t)\\ v_1(t)\end{pmatrix}=\begin{pmatrix}\frac{1}{3}+\frac{1}{5}\sin(t)\cos(t)-\frac{2}{3}\cos^2(t)-\frac{1}{15}\sin(t)\cos^3(t)\\ \frac{1}{5}-\frac{1}{3}\sin(t)\cos(t)-\frac{3}{5}\cos^2(t)+\frac{4}{15}\cos^4(t)\\ \frac{1}{2}(3-\sqrt{7})+\sin(t)\\ \frac{1}{10}\cos(t)+\frac{3}{10}\sin(t)\end{pmatrix}$$

$$\begin{pmatrix} x_2(t) \\ y_2(t) \\ u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{3}+\frac{1}{5}\sin(t)\cos(t)-\frac{2}{3}\cos^2(t)-\frac{1}{15}\sin(t)\cos^3(t) \\ \frac{1}{5}-\frac{1}{3}\sin(t)\cos(t)-\frac{3}{5}\cos^2(t)+\frac{4}{15}\cos^4(t) \\ \frac{1}{2}\left(3+\sqrt{7}\right)-\sin(t) \\ \frac{1}{10}\cos(t)+\frac{3}{10}\sin(t) \end{pmatrix}$$

of the differential system

$$\dot{x} = -y + \sin(t)\cos(t),$$
$$\dot{y} = x + \sin^3(t)\cos(t),$$
$$\dot{u} = \cos(t),$$
$$\dot{v} = 3v + \sin(t)$$

when $\varepsilon \to 0.$

4.4. Application of Theorem 4

Consider the differential system (1.1) where

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ h_4(t) \end{pmatrix} = \begin{pmatrix} \sin(t) \\ \cos(t) \\ \sin(t) \\ \sin(t)\cos(t) \end{pmatrix}$$

and

$$\begin{pmatrix} P_1(x,y,u,v) \\ P_2(x,y,u,v) \\ P_3(x,y,u,v) \\ P_4(x,y,u,v) \end{pmatrix} = \begin{pmatrix} 3x^2 - xy^2 + y^2 \\ 3y^2 - yx^2 + x^2 \\ u \\ v \end{pmatrix}.$$

We can easily verify conditions (1.5)

$$\int_0^{2\pi} \left(2\sin(s)\cos(s)\right) \mathrm{d}s = 0,$$
$$\int_0^{2\pi} \left(-\sin^2(s)+\cos^2(s)\right) \mathrm{d}s = 0,$$
$$\int_0^{2\pi} \sin(s) \mathrm{d}s = 0,$$
$$\int_0^{2\pi} \sin(s)\cos(s) \mathrm{d}s = 0.$$

Computing the functions $\mathcal{F}_1$ $\mathcal{F}_2$ $\mathcal{F}_3$ and $\mathcal{F}_4$ we find

$$\mathcal{F}_1(x_0,y_0,u_0,v_0) = -\frac{1}{8}x_0 - \frac{3}{8}x_0^2 - \frac{1}{8}y_0^2 - \frac{1}{4}x_0^3 - \frac{1}{4}x_0 y_0^2,$$

$$\mathcal{F}_2(x_0,y_0,u_0,v_0) = -\frac{1}{4}x_0 y_0 - \frac{1}{4}x_0^2 y_0 - \frac{1}{4}y_0^3 - \frac{3}{8}y_0,$$

$$\mathcal{F}_3(x_0, y_0, u_0, v_0) = u_0 + 1,$$

$$\mathcal{F}_4(x_0, y_0, u_0, v_0) = v_0 + \frac{1}{4}.$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0$ has three solutions $(x_0^*, y_0^*, u_0^*, v_0^*)$ given by $\left(0, 0, -1, \frac{-1}{4}\right)$, $\left(-\frac{1}{2}, 0, -1, -\frac{1}{4}\right)$, $\left(-1, 0, -1, -\frac{1}{4}\right)$ and the eigenvalues of the jacobian matrix of $\begin{pmatrix} \mathcal{F}_1(x_0, y_0, u_0, v_0) \\ \mathcal{F}_2(x_0, y_0, u_0, v_0) \\ \mathcal{F}_3(x_0, y_0, u_0, v_0) \\ \mathcal{F}_4(x_0, y_0, u_0, v_0) \end{pmatrix}$ at these solutions are $\begin{pmatrix} \frac{-1}{8} \\ \frac{-3}{8} \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{16} \\ \frac{-5}{16} \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} \frac{-1}{8} \\ \frac{-3}{8} \\ 1 \\ 1 \end{pmatrix}$, which have all at least two positive real parts. Since

$$\det\left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)}{\partial(x_0, y_0, u_0, v_0)} \right|_{(x_0, y_0, u_0, v_0) = (x_0^*, y_0^*, u_0^*, v_0^*)} \right)$$

at these three solutions $(x_0^*, y_0^*, u_0^*, v_0^*)$ is $\frac{3}{64}, -\frac{5}{256}, \frac{3}{64}$, respectively, then this differential system has three unstable periodic solutions $\begin{pmatrix} x_k(t, \varepsilon) \\ y_k(t, \varepsilon) \\ u_k(t, \varepsilon) \\ v_k(t, \varepsilon) \end{pmatrix}$, where $k = 1$, 2, 3 tending to the unstable periodic solutions

$$\begin{pmatrix} x_1(t) \\ y_1(t) \\ u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} -\cos(t) \\ 0 \\ -\cos(t) \\ \frac{1}{4} - \frac{1}{2}\cos^2(t) \end{pmatrix}$$

$$\begin{pmatrix} x_2(t) \\ y_2(t) \\ u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \sin(t) \\ -\cos(t) \\ \frac{1}{4} - \frac{1}{2}\cos^2(t) \end{pmatrix}$$

$$\begin{pmatrix} x_3(t) \\ y_3(t) \\ u_3(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} \frac{-1}{2}\cos(t) \\ \frac{1}{2}\sin(t) \\ -\cos(t) \\ \frac{1}{4} - \frac{1}{2}\cos^2(t) \end{pmatrix}$$

of the differential system

$$\dot{x} = -y + \sin(t),$$

$$\dot{y} = x + \cos(t),$$

$$\dot{u} = \sin(t),$$

$$\dot{v} = \sin(t)\cos(t),$$

when $\varepsilon \to 0$.

4.5. Application of Theorem 5

Consider the differential system (1.1) where

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ h_4(t) \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ -\cos(t) \\ \sin(t) \\ \sin(t) \end{pmatrix}$$

and

$$\begin{pmatrix} P_1(x,y,u,v) \\ P_2(x,y,u,v) \\ P_3(x,y,u,v) \\ P_4(x,y,u,v) \end{pmatrix} = \begin{pmatrix} y - x^2 y \\ -xy^2 \\ x + y + u \\ x + y \end{pmatrix}.$$

We can easily verify conditions (1.6)

$$\int_0^{2\pi} \left(-2\cos(s)\sin(s)\right) \mathrm{d}s = 0,$$

$$\int_0^{2\pi} \left(\sin^2(s) - \cos^2(s)\right) \mathrm{d}s = 0,$$

$$u_0 = -\frac{1}{25},$$

$$v_0 = -\frac{1}{5}.$$

Computing the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ we find

$$\mathcal{F}_1(x_0, y_0) = -\frac{7}{8} y_0 - \frac{6}{8} x_0 y_0,$$

$$\mathcal{F}_2(x_0, y_0) = \frac{1}{2} - \frac{5}{8} x_0 + \frac{1}{8} x_0^2 - \frac{5}{8} y_0^2,$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = 0$ has two solutions (x_0^*, y_0^*) given by $(1,0),(4,0)$ and the eigenvalues of the jacobian of $\begin{pmatrix} \mathcal{F}_1(x_0, y_0) \\ \mathcal{F}_2(x_0, y_0) \end{pmatrix}$ at these solutions are

$\begin{pmatrix} \frac{1}{8} I\sqrt{3} \\ \frac{-1}{8} I\sqrt{3} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{8} I\sqrt{51} \\ \frac{-1}{8} I\sqrt{51} \end{pmatrix}$, which have all zero real parts. Since

$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\Big|_{(x_0,y_0)=(x_0^*,y_0^*)} \right)$ for these solutions (x_0^*, y_0^*) is $\frac{3}{64}, \frac{51}{64}$, respectively,

then this differential system has two periodic solutions $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \\ u(t,\varepsilon) \\ v(t,\varepsilon) \end{pmatrix}$, tending to

the two periodic solutions

$$\begin{pmatrix} x_1(t) \\ y_1(t) \\ u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ 0 \\ \frac{-1}{25}\cos(t) - \frac{7}{25}\sin(t) \\ -\frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) \end{pmatrix}$$

$$\begin{pmatrix} x_2(t) \\ y_2(t) \\ u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ \frac{-1}{25}\cos(t) - \frac{7}{25}\sin(t) \\ -\frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) \end{pmatrix}$$

of the differential system

$$\dot{x} = -y - \sin(t),$$

$$\dot{y} = x - \cos(t),$$

$$\dot{u} = 2u + v + \sin(t),$$

$$\dot{v} = 2v + \sin(t)$$

when $\varepsilon \to 0$.

In this case we can say nothing about the stability of these solutions.

5. Conclusion

This study leads us to consider the general case when A is an $n \times n$ matrix, $P_1 \cdots P_n$ are polynomials in the variables $x_1, \cdots, x_n$ of degree n and $h_i(t+2\pi) = h_i(t)$, with $i = 1 \cdots n$. In the next work, we shall generalize the studied system (1.1) in $\mathbb{R}^n$.

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