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Homological Properties in Relation to Nakayama Algebras

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Abstract

The goal of this paper is to investigate whether the Ext-groups of all pairs (M, N) of modules over Nakayama algebras of type (n,n,n) satisfy the condition $Ext^n_\Lambda(M,N)=0$ for $n \gg 0 \Leftrightarrow Ext^n_\Lambda(N,M)=0$ for $n \gg 0$. We achieve that by discussing the Ext-groups of Nakayama algebra with projectives of lengths $3n+1$ and $3n+2$ using combinations of modules of different lengths.

Keywords

Quivers, Path algebras, Ext-Groups, Projective Resolutions

1. Introduction

In this paper, we describe homological properties of Nakayama algebras. The algebra Λ is a Nakayama algebra if every projective indecomposable and every injective indecomposable Λ-module is uniserial. In other words, these modules have a unique composition series, (see Schröer [1]). Nakayama algebras are finite dimensional and representation-finite algebras that have a nice representation theory in the sense that the finite-dimensional indecomposable modules are easy to describe.

The main contribution of this paper is to investigate whether the Ext-groups of all pairs (M, N) of modules over Nakayama algebras of type (n,n,n) satisfy the condition $Ext^n_\Lambda(M,N)=0$ for $n \gg 0 \Leftrightarrow Ext^n_\Lambda(N,M)=0$ for $n \gg 0$.

2. Preliminary Notes

This section will briefly discuss the properties of Nakayama algebra and some related propositions. We consider also proofs of some of the propositions. All the information presented here can be found in deeper details in the book from

Rotman [2].

Definition: Let Λ be an artin algebra. A Λ-module A is called a uniserial module if the set of submodules is totally ordered by inclusion.

Proposition 1: The following are equivalent for Λ-module A

1. A is uniserial.

2. There is only one composition series for A.

3. The radical filtration of A is a composition series for A.

4. The socle filtration of A is a composition series for A.

5. $l(A)=rl(A)$, where $l(A)$ is the length of A and $rl(A)$ is the radical length of A.

Proof: See Rotman [2] for the details.

Proposition 2: The following are equivalent for an artin algebra Λ.

1. Λ is a sum of uniserial modules .

2. Λ/a is a sum of uniserial modules for all ideals a of Λ.

3. Λ/r^2 is a sum uniserial modules.

Proof: $1\Rightarrow 2$ and $2\Rightarrow 3$ are trivial. If Λ is the sum of uniserial modules, then Λ/a and Λ/r^2 which are factors of Λ are also a sum of uniserial modules.

$$3\Rightarrow 1$$

Let P be an indecomposable projective Λ-module. We show that P/r^nP is uniserial by induction on n when $n\geq 2$. When $n=2$, there is nothing to prove. Suppose $n>2$. Let the radical filtration of P be;

$P\supset rP\supset r^2P\supset\cdots\cdots r^{n-1}P\supset r^nP=0$ such that $r^iP/r^{i+1}P$ is simple for $i=0,1,\cdots,n-1$.

When $n=3$, we have $r^3P\subset r^2P\subset rP\subset P$. Hence by induction hypothesis, $P/r^{n-1}P$ is uniserial. Considering the exact sequence $0\to rP\to P\to P/rP\to 0$, which also implies that P/rP is uniserial, hence $P/r^{n-1}P$ is also uniserial.

If $r^{n-1}P=0$, then P/r^nP is clearly uniserial, so we have to assume that $r^{n-1}P\neq 0$. From proposition 1, it follows that $r^iP/r^{i+1}P$ is simple for $i=0,1,\cdots,n-2$. To show that P/r^nP is uniserial, then it is sufficient by proposition 1 to prove that $r^{n-1}P/r^nP$ is also simple.

Let $Q\to r^{n-2}P$ be a projective cover. Since $r^{n-2}P/r^{n-1P}$ is simple, Q must be indecomposable and so Q/r^2Q is uniserial. But we have an epimorphism $rQ/r^2Q\to r^{n-1}P/r^nP$ which shows that $r^{n-1}P/r^nP$ is simple.

Proposition3: Proposition 2 (1).

Let φ be a D Tr-orbit of $\operatorname{ind}\Lambda$. Suppose there is a projective module P in φ. Then we have the following

1. φ of non-zero objects in $\left\{P,(DT_r)^{-1}P,\cdots,(DT_r)^{-i}P,\cdots\right\}_{i\in N}$.

2. φ is finite if and only if $(DT_r)^{-n}P=(T_rD)^n$ is injective for some n in N. Moreover, if $(T_rD)^nP$ is injective, then $\varphi=\left\{P,(DT_r)^{-1}P,\cdots,(DT_r)^{-n}P\right\}$.

Proof: By proposition 1, $DTrP=0$ if and only if P is projective. Since P is projective module in $\varphi,(DTr)^iP=0$ for all $i>0$. Hence the claim in 2 (1).

We claim that if $(DTr)^{-i}P\simeq(TDr)^{-(i+j)}P\neq 0$ with $j>0$ we have $(DTr)^i(DTr)^{-(i+j)}P\simeq(DTr)(DTr)^{-i}P$ which implies that

$P \simeq (DTr)^j P = (TrD)^j P$ which is not possible since $j > 0$. φ can therefore be finite if $(DTr)^{-(n+1)} P = 0$ for some $n \geq 0$. Since $(DTr)^{-n} P = (TrD)^n P$, then P is injective in φ. We know therefore that if $(DTr)^{-n} P$ is injective, then $\varphi = \{P, (DTr)^{-1} P, \cdots, (DTr)^{-n} P\}$.

3. Results

The main goal of this work is to investigate whether the Ext-groups of all pairs (M, N) of modules over Nakayama algebras of type (n, n, n) satisfy the condition: $Ext_\Lambda^n(M,N) = 0$ for $n \gg 0 \Leftrightarrow Ext_\Lambda^n(N,M) = 0$ for $n \gg 0$, where n is a positive integer. We discuss the Ext-groups of Nakayama algebras with projectives of lengths $3n+1$ and $3n+2$ using combinations of modules of different lengths. Reader may refer to Auslander, M *et al.* [3] for ideas illustrated in this section.

We begin with the Ext-groups of Nakayama algebra with projectives of length $3n+1$ using the combinations $(3i, 3j)$, and $(3i, 3j+2)$ where $3i$ and $3j$ are modules of lengths $3i$ and $3j$ respectively, $3j+2$ is also module of length $3j+2$.

Let Γ be a path with the relations

$$\alpha\gamma\beta\cdots\gamma\beta\alpha, \beta\alpha\gamma\cdots\alpha\gamma\beta \quad \text{and} \quad \gamma\beta\alpha\cdots\beta\alpha\gamma$$

where the length of each relation is $3n+1$. Let

$$\Lambda = k\Gamma / \langle \alpha\gamma\beta\cdots\gamma\beta\alpha, \beta\alpha\gamma\cdots\alpha\gamma\beta, \gamma\beta\alpha\cdots\beta\alpha\gamma \rangle.$$

The projectives of the above path algebra are as follows:

$$P_1 = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} S_2 \\ S_3 \\ S_1 \\ \vdots \\ S_2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} S_3 \\ S_1 \\ S_2 \\ \vdots \\ S_3 \end{pmatrix}.$$

The above projectives P_1, P_2 and P_3 each has length $3n+1$. The minimal projective resolution of the module $M = (S_1, S_2, \cdots, S_3)^t$ of length $3i, i = 1, 2, \cdots, n$ is given as;

$$\cdots Q_4 \xrightarrow{d_5} Q_3 \xrightarrow{d_4} Q_3 \xrightarrow{d_3} Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \to 0$$

where $Q_{6i} = Q_{6i+1} = P_1, Q_{6i+2} = Q_{6i+3} = P_2$ and $Q_{6i+4} = Q_{6i+5} = P_3$ for $i \geq 0$.

The combinations have been reduced to four class because in the above minimal projective resolution of the module of length $3i$, we see that it is in the same group as the module of length $3i+1$. The modules of lengths $3i$ and $3i+1$ therefore have the same properties with respect to the conditions; $Ext_\Lambda^n(M,N) = 0$ for $n \gg 0 \Leftrightarrow Ext_\Lambda^n = 0$ for $n \gg 0$.

From the above resolution, we have;

$$\cdots \Lambda e_1 \xrightarrow{d_6} \Lambda e_3 \xrightarrow{d_5} \Lambda e_3 \xrightarrow{d_4} \Lambda e_2 \xrightarrow{d_3} \lambda e_2$$
$$\xrightarrow{d_2} \Lambda e_1 \xrightarrow{d_1} \Lambda e_1 \xrightarrow{d_0} (S_1, S_2, \cdots, S_3) \to 0.$$

Let $A = (S_1, S_2, \cdots, S_3)^t$, the pd $A = \infty$ since the resolution is periodic. The period is 6. The truncation of the resolution is given as;

$$P\cdots\Lambda e_1 \xrightarrow{d_6} \Lambda e_3 \xrightarrow{d_5} \Lambda e_3 \xrightarrow{d_4} \Lambda e_2 \xrightarrow{d_3} \lambda e_2 \xrightarrow{d_2} \Lambda e_1 \xrightarrow{d_1} \Lambda e_1 \xrightarrow{d_0} 0.$$

The map d_{6i+1} is the multiplication by $(\gamma\beta\alpha)^i, d_{6i+2}$ is the multiplication by $\alpha(\gamma\beta\alpha)^{n-i}, d_{6i+3}$ is multiplication by $(\alpha\gamma\beta)^i, d_{6i+4}$ is the multiplication by $\beta(\alpha\gamma\beta)^{n-i}, d_{6i+5}$ is the multiplication by $(\beta\alpha\gamma)^i$ and d_{6i} is the multiplication by $\gamma(\beta\alpha\gamma)^{n-i}$.

Applying $Hom_\Lambda(\ ,M)$ where M is the module

$$\begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_3 \end{pmatrix}$$

of length $3j$, we have;

$$0 \xrightarrow{d_0^\times} Hom_\Lambda(\Lambda e_1, M) \xrightarrow{d_1^\times} Hom_\Lambda(\Lambda e_1, M) \xrightarrow{d_2^\times} Hom_\Lambda(\Lambda e_2, M)$$

$$\xrightarrow{d_3^\times} Hom_\Lambda(\Lambda e_2, M) \xrightarrow{d_4^\times} Hom_\Lambda(\Lambda e_3, M) \xrightarrow{d_5^\times} Hom_\Lambda(\Lambda e_1, M),$$

where $Hom_\Lambda(\Lambda e_1, M) \simeq e_1 M, Hom_\Lambda(\Lambda e_2, M) \simeq e_2 M$ and $Hom_\Lambda(\Lambda e_3, M) \simeq e_3 M$.

We have the following Ext-groups;

$$Ext_\Lambda^1(N,M) = \ker\left(e_1 M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2 M\right) \Big/ \Im\left(e_1 M \xrightarrow{(\gamma\beta\alpha)^i} e_1 M\right),$$

where N and M are modules of length $3i$ and $3j$ respectively. We first compute the kernel for

$$d_2^\times : \left(e_1 M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2 M\right).$$

We have

$$d_2^\times\left((a_1, a_2, \cdots, a_s, a_{s+1}, \cdots)^t, 0, 0\right) = \left(0, (0,0,0,\cdots,0,a_1,a_2,\cdots,a_s)^t, 0\right)$$

where a_1 is in the coordinate number $n-i+1$, therefore a_s is in the coordinate number $n-i+s$. Hence $s = j+i-n$ and consequently $s+1 = j+i+1-n$,

$$\ker\left(e_1 M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2 M\right) = \left\{\left((0,0,\cdots,0,a_{s+1},\cdots,a_j))^t, 0, 0\right) \middle| a_r \in k, r = s+1, \cdots, j\right\}.$$

Next we compute the image of

$$d_1^\times : \left(e_1 M \xrightarrow{(\gamma\beta\alpha)^i} e_1 M\right).$$

We have that

$$d_1^\times\left((b_1, b_2, \cdots, b_j)^t, 0, 0\right) = \left((0,0,\cdots,0,b_1,\cdots,b_{j-1})^t, 0, 0\right)$$

where b_1 is in the coordinate number $i+1$. This shows that

$$\Im\left(e_1 M \xrightarrow{(\gamma\beta\alpha)^i} e_1 M\right) = \left\{\left((0,0,\cdots,0,b_1,\cdots,b_{j-1})^t, 0, 0\right) \middle| b_t \in k, t = 1, \cdots, j-1\right\}.$$

Hence $\dim_k Ext_\Lambda^1\left((S_1, S_2, \cdots, S_3)^t, M\right) = (i+1) - (j+i+1-n) = n-j$, for

$j \le n$. We compute

$$Ext_\Lambda^2\left(\left(S_1, S_2, \cdots, S_3\right)^t, M)\right) = \ker\left(e_2M \xrightarrow{(\alpha\gamma\beta)^i} e_2M\right) \Big/ \Im\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right).$$

We compute the kernel for

$$d_3^\times : \left(e_2M \xrightarrow{(\alpha\gamma\beta)^i} e_2M\right),$$

we have

$$d_3^\times\left(0, \left(a_1, a_2, \cdots, a_j\right)^t, 0\right) = \left(0, \left(0, 0, \cdots, 0, a_1, \cdots, a_{j-i}\right)^t, 0\right)$$

where a_1 is the coordinate number $i+1$. This shows that

$$\ker\left(e_2M \xrightarrow{(\alpha\gamma\beta)^i} e_2M\right) = \left\{\left(0, \left(0, 0, \cdots, 0, a_1, \cdots, a_j\right)^t, 0\right) \middle| a_r \in k, r = 1, \cdots, j\right\},$$

where a_1 is in the coordinate number $j-i+1$. Next we compute the image of

$$d_2^\times : \left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_1M\right).$$

We see that

$$d_2^\times\left(\left(b_1, b_{2,\cdots,b_s,\cdots,b_j}\right)^t, 0\right) = \left(0, \left(0, 0, \cdots, 0, b_1, b_2, \cdots, b_s\right)^t, 0\right)$$

where b_1 is in the coordinate number $n-i+1$ and therefore b_s is in the coordinate number $n-i+s$. This implies $n-i+s = j$ and hence $s = j+i-n$,

$$\Im\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right) = \left\{\left(0, \left(0, 0, \cdots, 0, b_1, \cdots, b_s\right)^t, 0\right) \middle| b_t \in k, t = 1, \cdots, s\right\}.$$

Hence $\dim_k Ext_\Lambda^2\left(\left(S_1, S_2, \cdots, S_3\right)^t, M\right) = (n-i+1) - (j-i+1) = n-j$ for $j \le n$.

$$\dim_k Ext_\Lambda^2\left(\left(S_1, \cdots, S_3\right)^t, M\right) = n-j$$

We compute

$$Ext_\Lambda^3\left(\left(S_1, \cdots, S_3\right)^t, M\right) = \ker\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right) \Big/ \Im\left(e_2M \xrightarrow{(\alpha\gamma\beta)^t} e_2M\right).$$

First we compute the kernel for

$$d_4^\times : \left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right).$$

We have

$$d_4^\times\left(0, \left(a_1, a_2, \cdots, a_s, \cdots, a_j\right)^t, 0\right) = \left(0, 0, \left(0, 0, \cdots, 0, a_1, a_2, \cdots, a_s\right)^t, \right),$$

where a_1 is in the coordinate number $n-i+1$ and therefore a_s is in the coordinate number $n-i+s$. Hence $s = j+i-n$ and consequently $s+1 = j+i+1-n$. This shows that

$$\ker\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right)$$
$$= \left\{\left(0, \left(0, 0, \cdots, 0, a_{s+1}, a_{s+2}, \cdots, a_j\right)^t, 0\right) \middle| a_r \in k, r = s+1, \cdots, j\right\}.$$

We compute the image of

$$d_3^{\times}:\left(e_2M \xrightarrow{(\alpha\gamma\beta)^i} e_2M\right)$$

to have

$$d_3^{\times}\left(0,\left(b_1,b_2,\cdots,b_j\right)^t,0\right)=\left(0,\left(0,0,\cdots,0,b_1,b_2,\cdots,b_{j-i}\right)^t,0\right)$$

where b_1 is in the coordinate number $i+1$. This shows that

$$\Im\left(e_2M \xrightarrow{(\alpha\gamma\beta)^i} e_2M\right)=\left\{\left(0,\left(0,0,\cdots,b_1,\cdots,b_{j-i}\right)^t,0\right)\middle| b_t\in k,t=1,\cdots,j-i\right\}.$$

Hence

$$\dim_k Ext_\Lambda^3\left(\left(S_1,\cdots,S_3\right)^t,M\right)=i+1-\left(j+i+1-n\right)=n-j$$

for $j\le n$.

Next we have

$$Ext_\Lambda^4\left(\left(S_1,\cdots,S_3\right)^t,M\right)=\ker\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right)\Big/\Im\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right)$$

Similarly, we have

$$\ker\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right)=\left\{\left(0,0,\left(0,0,\cdots,0,a_1,a_2,\cdots,a_j\right)^t,0\right)\middle| a_r\in k,r=1,\cdots,j\right\}$$

where a_1 is in the coordinate number $j+1-i$, then

$$\Im\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right)=\left\{\left(0,0,\left(0,0,\cdots,0,b_1,b_2\cdots,b_s\right)^t\right)\middle| b_t\in k,t=1,\cdots,s\right\},$$

$s=j+i-n$ where b_1 is in the coordinate number $n-i+1$. Hence

$$\dim_k Ext_\Lambda^4\left(\left(S_1,\cdots,S_3\right)^t,M\right)=n-i+1-\left(j+1-i\right)=n-j,\ j\le n.$$

Again we have

$$Ext_\Lambda^5\left(\left(S_1,\cdots,S_3\right)^t,M\right)=\ker\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{n-i}} e_1M\right)\Big/\Im\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right).$$

We compute the kernel for

$$d_6^{\times}:\left(e_3M\to e_1M\right)$$

to have

$$d_6^{\times}\left(0,0,\left(a_1,a_2,\cdots,a_s,\cdots,a_j\right)^t\right)=\left(\left(0,0,\cdots,0,a_1,a_2,\cdots,a_s\right)^t,0,0\right)$$

where a_1 is in the coordinate number $n-i+2$ and therefore a_s is in the coordinate number $n-i+1+s, n-i+s+1=js=j+i-n-1$ and consequently $s+1=j+i-n.$ This show that

$$\ker\left(e_3M\to e_1M\right)=\left\{\left(0,0,\left(0,0,\cdots,a_{s+1},\cdots,a_j\right)^t\right)\middle| a_r\in k,r=s+1,\cdots,j\right\}.$$

We compute the image for

$$d_5^{\times}:\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right)$$

to get

$$d_5^{\times}\left(0,0,\left(b_1,b_2,\cdots,b_{j-i}\right)^t\right)=\left(0,0,\left(0,0,\cdots,0,b_1,b_2,\cdots,b_{j-i}\right)^t\right)$$

where b_1 is in the coordinate number $i+1$. This shows that

$$\Im\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right)=\left\{\left(0,0,\left(0,0,\cdots,,b_1,\cdots,b_{j-i}\right)^t\right)\middle| b_t\in k,t=1,\cdots,j-1\right\}.$$

Hence $\dim_k Ext_\Lambda^5\left(\left(S_1,\cdots,S_3\right)^t,M\right)=i+1-\left(j+i-n\right)=n+1-j$. This shows that $\dim_k Ext_\Lambda^t\left(3i,3j\right)=n-j$ for $t=1,2,3,4$ and $n-j+1$ for $t=5, n-j=0, n-j+1=0$ which implies $n=j$ and $n+1=j$.

It follows from the computation of the fifth Ext-group that the condition $Ext_\Lambda^n\left(M,N\right)=0$ for $n\gg 0 \Leftrightarrow Ext_\Lambda^n\left(N,M\right)=0$ for $n\gg 0$ holds.

We calculate the Ext-groups for the combination $\left(3i,3j+2\right)$.

$$Ext_\Lambda^1\left(\left(S_1,\cdots,S_3\right)^t,M\right)$$

where M is the module of length $3j+2$

$$Ext_\Lambda^1\left(\left(S_1,\cdots,S_3\right)^t,M\right)=\ker\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right)\Big/\Im\left(e_1M \xrightarrow{(\gamma\beta\alpha)^i} e_1M\right).$$

We compute the kernel for

$$d_2^{\times}:\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right),$$

we have

$$d_2^{\times}\left(\left(a_1,a_2,\cdots,a_s,\cdots,a_{j+1}\right)^t,0,0\right)=\left(0,\left(0,0,\cdots,0,a_1,a_2,\cdots,a_s\right)^t,0\right)$$

where a_1 is in the coordinate number $n-i+1$ and therefore a_s is in the coordinate number $n-i+s=j+1$. Hence $s=j+i-n$ and consequently $s+1=j+i+2-n$. This shows that

$$\ker\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right)$$
$$=\left\{\left(\left(0,0,\cdots,0,a_{s+1},\cdots,a_{j+1}\right)^t,0,0\right)\middle| a_r\in k,r=s+1,\cdots,j+1\right\}.$$

Computing the image for

$$d_1^{\times}:\left(e_1M \xrightarrow{(\gamma\beta\alpha)^i} e_1M\right)$$

we have

$$d_1^{\times}\left(\left(b_1,b_2,\cdots,b_{j+1}\right)^t,0,0\right)=\left(\left(0,0,\cdots,0,b_1,\cdots,b_{j+1-i}\right)^t,0,0\right),$$

where b_1 is in the coordinate number $i+1$. This shows that

$$\Im\left(e_1M \xrightarrow{(\gamma\beta\alpha)^i} e_1M\right)=\left\{\left(\left(0,0,\cdots,b_1,\cdots,b_{j+1-i}\right)^t,0,0\right)\middle| b_t\in k,t=1,\cdots,j+1-i\right\}.$$

We therefore have

$$\dim_k Ext_\Lambda^1\left(\left(S_1,\cdots,S_3\right)^t,M\right)=i+1-\left(j+i+2-n\right)=n-\left(j+1\right)$$

for $j+1\le n$.

Next we have

$$Ext_\Lambda^2\left(\left(S_1,\cdots,S_3\right)^t,M\right)=\ker\left(e_2M\xrightarrow{(\alpha\gamma\beta)^i}e_2M\right)\Big/\Im\left(e_1M\ v\ e_2M\right).$$

We compute the kernel for

$$d_3^\times:\left(e_2M\xrightarrow{(\alpha\gamma\beta)^i}e_2M\right),$$

we have

$$d_3^\times\left(0,\left(a_1,a_2,\cdots,a_{j+1}\right)^t,0\right)=\left(0,\left(0,0,\cdots,0,a_1,\cdots,a_{j+1-i}\right)^t\right)$$

where a_1 is in the coordinate number $i+1$. This shows that

$$\ker\left(e_2M\xrightarrow{(\alpha\gamma\beta)^i}e_2M\right)=\left\{\left(0,\left(0,0,\cdots,0,a_1,\cdots,a_{j+1}\right)^t,0\right)\Big|a_r\in k,r=1,\cdots,j+1\right\},$$

where a_1 is in the coordinate number $j+2-i$. We compute the image for

$$d_2^\times:\left(e_1M\xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}}e_2M\right).$$

We have

$$d_2^\times\left(\left(b_1,b_2,\cdots,b_s,\cdots,b_{j+1}\right)^t,0,0\right)=\left(0,\left(0,0,\cdots,b_1,b_2,\cdots,b_s\right)^t,0\right)$$

where b_1 is in the coordinate number $n-i+1$ and therefore b_s is in the coordinate number $n-i+s=j+1$. Hence $s=j+1+i-n$. This shows that

$$\Im\left(e_1M\xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}}e_2M\right)=\left\{\left(0,\left(0,0,\cdots,0,b_1,\cdots,b_{j+1}\right)^t,0\right)\Big|b_t\in k,t=1,\cdots,j+1\right\}.$$

We therefore have
$\dim_k Ext_\Lambda^2\left(\left(S_1,\cdots,S_3\right)^t,M\right)=n-i+1-\left(j+2-i\right)=n-\left(j+1\right)$ for $j+1\le n$.
Next we have

$$Ext_\Lambda^3\left(\left(S_1,\cdots,S_3\right)^t,M\right)=\ker\left(e_2M\xrightarrow{\beta(\alpha\gamma\beta)^{n-i}}e_3M\right)\Big/\Im\left(e_2M\xrightarrow{(\alpha\gamma\beta)^i}e_2M\right).$$

Similarly, we have;

$$\ker\left(e_2M\xrightarrow{\beta(\alpha\gamma\beta)^{n-i}}e_3M\right)=\left\{\left(0,\left(0,0,\cdots,a_1,\cdots,a_{j+1}\right)^t,0\right)\Big|a_r\in k,r=1,\cdots,j+1\right\},$$

where a_1 is in the coordinate number $j+i+2-n$. We also have

$$\Im\left(e_2M\to e_2M\right)=\left\{\left(0,\left(0,0,\cdots,0,b_1,\cdots,b_{j+1-i}\right)^t,0\right)\Big|b_t\in k,t=1,\cdots,j+1-i\right\},$$

where b_1 is in the coordinate number $i+1$. We therefore have
$\dim_k Ext_\Lambda^3\left(\left(S_1,\cdots,S_3\right)^t,M\right)=i+1-\left(j+i+2-n\right)=n-\left(j+1\right)$ for $j+1\le n$.
Next we have

$$Ext_\Lambda^4\left(\left(S_1,\cdots,S_3\right)^t,M\right)=\ker\left(e_3M\xrightarrow{(\beta\alpha\gamma)^i}e_3M\right)/\Im\left(e_2M\xrightarrow{\beta(\alpha\gamma\beta)^{n-i}}e_3M\right).$$

We compute the kernel for

$$d_5^\times:\left(e_3M\xrightarrow{(\beta\alpha\gamma)^i}e_3M\right).$$

We have

$$d_5^{\times}\left(0,0,\left(a_1,a_2,\cdots,a_j\right)^t\right)=\left(0,0,\left(0,0,\cdots,0,a_1,\cdots,a_{j-i}\right)^t\right)$$

where a_1 is in the coordinate number $i+1$. This shows that

$$\ker\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right)=\left\{\left(0,0,\left(0,0,\cdots,a_1,\cdots,a_j\right)^t,\right)\middle| a_r\in k,r=1,\cdots,j\right\},$$

where a_1 is in the coordinate number $j+1-i$. We compute the image for

$$d_4^{\times}:\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right).$$

We have

$$d_4^{\times}\left(0,\left(b_1,b_2,\cdots,b_s,b_{j+1}\right)^t,0,\right)\left(0,0(\left(0,0,\cdots,b_1,b_2\cdots,b_s\right)^t\right)$$

where b_1 is in the coordinate number $n-i+1$ and therefore b_s is in the coordinate number $n-i+s=j+1$ and therefore $s=j+i-n$,

$$\Im\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^i} e_3M\right)=\left\{\left(0,0,\left(0,0,\cdots,b_1,\cdots,b_{j-i}\right)^t\right)\middle| b_t\in k,t=1,\cdots,j\right\}.$$

We therefore have $\dim_k Ext_\Lambda^4\left(\left(S_1,\cdots,S_3\right)^t,M\right)=n-i+1-\left(j+1-i\right)n-j$ for $j\le n$. Finally, we have

$$Ext_\Lambda^5\left(\left(S_1,\cdots,S_3\right)^t,M\right)=\ker\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{n-i}} e_1M\right)\Big/\Im\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right).$$

We compute the kernel for

$$d_6^{\times}:\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{n-i}} e_1M\right),$$

we have

$$d_6^{\times}\left(0,0,\left(a_1,a_2,\cdots,a_s,\cdots a_j\right)^t\right)=\left(\left(0,0,\cdots,0,a_1,\cdots,a_s\right)^t,0,0\right)$$

where a_1 is in the coordinate number $n-i+2$ and therefore a_s is in the coordinate number $n-i+1+s=j,s=j+i-n-1$ and consequently $s+1=j+i-n$. This shows that

$$\ker\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{n-i}} e_1M\right)=\left\{\left(0,0,\left(0,0,\cdots,a_1,\cdots,a_j\right)^t\right)\middle| a_r\in k,r=1,\cdots,j\right\}.$$

We compute the image for

$$d_5^{\times}:\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right),$$

we have

$$d_5^{\times}\left(0,0,\left(b_1,b_2,\cdots,b_j\right)^t\right)=\left(0,0,\left(0,0,\cdots,0,b_1,b_2,\cdots,b_{j-i}\right)^t\right)$$

where b_1 is the ith coordinate number $i+1$ and hence

$$\Im\left(e_3M \xrightarrow{(\beta\alpha\gamma)^i} e_3M\right)=\left\{\left(0,0,\left(b_1,\cdots,b_{j-i}\right)^t\right)\middle| b_t\in k,t=1,\cdots,j-i\right\}.$$

We therefore have

$$\dim_k Ext_\Lambda\left(\left(S_1,\cdots,S_3\right)^t,M\right)=i+1-\left(j+i-n\right)=n+1-j \quad \text{for} \quad j\le n+1$$

$\dim_k Ext_\Lambda^t(3i,3j+2)=n-(j+1)$ for $t=1,2,3$, $n-j$ for $t=4$ and $n+1-j$ for $t=5$. We therefore have $n=j+1, n=j$ and $n=j-1$.

It follows from the computations of the forth and fifth Ext-groups that the condition $Ext_\Lambda^n(M,N)=0$ for $n \gg 0 \Leftrightarrow Ext_\Lambda^n(N,M)=0$ for $n \gg 0$ holds.

We now discuss the Ext-groups of the Nakayama algebras with the projectives of length $3n+2$ using the combination $(3i+1,3j+1)$, where $3i+1$ and $3j+1$ are also modules of length $3i+1$ and $3j+1$ respectively.

Let Γ be a path with the relations $\beta\alpha\gamma\cdots\gamma\beta\alpha, \gamma\beta\alpha\cdots\alpha\gamma\beta$ and $\alpha\gamma\beta\cdots\beta\alpha\gamma$ where the length of each relation is $3n+2, n$ is a positive integer. Let

$$\Lambda = k\Gamma/\langle \beta\alpha\gamma\cdots\gamma\beta\alpha, \gamma\beta\alpha\cdots\alpha\gamma\beta, \alpha\gamma\beta\cdots\beta\alpha\gamma\rangle.$$

The projectives of the above path algebra are as follows:

$$P_1=(S_1,S_2,\cdots,S_1,S_2)^t, P_2=(S_2,S_3,\cdots,S_2,S_3)^t, P_3=(S_3,S_1,\cdots,S_3,S_1)^t.$$

The above projectives $P_i, i=1,2,3$ each has length $n+2$. The minimal projective resolution of module $(S_1,\cdots,S_3)^i$ of length $3i$ is given as;

$$\cdots Q_6 \xrightarrow{d_6} Q_5 \xrightarrow{d_5} Q_4 \xrightarrow{d_4} Q_3 \xrightarrow{d_3} Q_2$$
$$\xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} N \to 0.$$

where N is the module of length $3i, Q_{6i}=Q_{6i+1}=P_1, Q_{6i+2}=Q_{6i+3}=P_3$ and $Q_{6i+4}=Q_{6i+5}=P_2$ for $i \ge 0$ and d_{6i+1} is a multiplication by $(\gamma\beta\alpha)^i, d_{6i+2}$ is a multiplication by $\beta\alpha(\gamma\beta\alpha)^{n-i}, d_{6i+3}$ is a multiplication by $(\beta\alpha\gamma)^i, d_{6i+4}$ is a multiplication by $\alpha\gamma(\beta\alpha\gamma)^{n-i}, d_{6i+5}$ is a multiplication by $(\alpha\gamma\beta)^i$ and d_{6i} is multiplication by $\gamma\beta(\alpha\gamma\beta)^{n-i}$.

The combinations have been reduced to four class because in the above resolution, we see that the module of length $3i$ is in the same group with the module of length $3i+2$. The two modules therefore have the same properties with respect to the condition $Ext_\Lambda^n(M,N)=0$ for $n \gg 0 \Leftrightarrow Ext_\Lambda^n(N,M)=0$ for $n \gg 0$.

From the above resolution we have,

$$\cdots \Lambda e_1 \xrightarrow{d_7} \Lambda e_2 \xrightarrow{d_6} \Lambda e_2 \xrightarrow{d_5} \Lambda e_3 \xrightarrow{d_4} \Lambda e_3$$
$$\xrightarrow{d_3} \Lambda e_1 \xrightarrow{d_2} \Lambda e_1 \xrightarrow{d_1} N \xrightarrow{d_0} 0.$$

The $pdN=\infty$ since the resolution is periodic. The period is 6. The truncation of the above resolution is given as:

$$P\cdots \Lambda e_1 \xrightarrow{d_6} \Lambda e_2 \xrightarrow{d_5} \Lambda e_2 \xrightarrow{d_4} \Lambda e_3$$
$$\xrightarrow{d_3} \Lambda e_3 \xrightarrow{d_2} \Lambda e_1 \xrightarrow{d_1} \Lambda e_1 \xrightarrow{d_0} 0.$$

Applying $Hom(\ ,M)$ where M is the module of length $3j$, we have

$$0 \xrightarrow{d_0^\times} Hom_\Lambda(\Lambda e_1,M) \xrightarrow{d_1^\times} Hom_\Lambda(\Lambda e_1,M) \xrightarrow{d_2^\times} Hom_\Lambda(\Lambda e_3,M)$$
$$\xrightarrow{d_3^\times} Hom_\Lambda(\Lambda e_3,M) \xrightarrow{d_4^\times} Hom_\Lambda(\Lambda e_2,M) \xrightarrow{d_5^\times} Hom_\Lambda(\lambda e_2,M)$$
$$\xrightarrow{d_6^\times} Hom_\Lambda(\Lambda e_1,M),$$

where $Hom_\Lambda(\Lambda e_1,M) \simeq e_1M, Hom_\Lambda(\Lambda e_2,M) \simeq e_2M$ and

$Hom_{\Lambda}\left(\lambda e_3, M\right) \simeq e_3 M$.

We discuss the Ext-groups of the Nakayama algebras with projectives of length $3n+2$ by considering the case: $(3i+1, 3j+1)$. $3i+1$ is the module $\left(S_1,\cdots,S_1\right)^{i+1}$ of length $3i+1$ and $3j+1$ is the module $\left(S_1,\cdots,S_1\right)^{i+1}$ of length $3j+1$.

The minimal projective resolution of the module $\left(S_1,\cdots,S_1\right)^{i+1}$ of length $3i+1$ is given as;

$$\cdots \to Q_6 \xrightarrow{d_6} Q_5 \xrightarrow{d_5} Q_4 \xrightarrow{d_4} Q_3 \xrightarrow{d_3} Q_2$$
$$\xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} \to A \to 0$$

where $Q_{6i} = Q_{6i+3} = P_1, Q_{6i+1} = Q_{6i+4} = P_2$ and $Q_{6i+2} = Q_{6i+5} = P_3$ for $i \geq 0, d_{6i+1}$ is a multiplication by $\alpha\left(\gamma\beta\alpha\right)^{i+1}, d_{6i+2}$ is a multiplication by $\beta\left(\alpha\gamma\beta\right)^{n-i}, d_{6i+3}$ is a multiplication by $\gamma\left(\beta\alpha\gamma\right)^{i+1}, d_{6i+4}$ is a multiplication by $\alpha\left(\gamma\beta\alpha\right)^{n-i}, d_{6i+5}$ is a multiplication by $\beta\left(\alpha\gamma\beta\right)^{i+1}$ and d_{6i+6} is a multiplication by $\gamma\left(\beta\alpha\gamma\right)^{n-i}$ and A is the module of length $3i+1$.

Considering the above resolution, we have

$$\cdots \to \Lambda e_1 \xrightarrow{d_6} \Lambda e_3 \xrightarrow{d_5} \Lambda e_2 \xrightarrow{d_4} \Lambda e_1 \xrightarrow{d_3} \Lambda e_3$$
$$\xrightarrow{d_2} \Lambda e_2 \xrightarrow{d_1} \Lambda e_1 \xrightarrow{d_0} A \to 0.$$

The $pdA = \infty$ since the resolution is periodic. The period ic 6. The truncation of the above resolution is given as;

$$P: \cdots \Lambda e_1 \xrightarrow{d_6} \Lambda e_3 \xrightarrow{d_5} \Lambda e_2 \xrightarrow{d_4} \Lambda e_1$$
$$\xrightarrow{d_3} \Lambda e_3 \xrightarrow{d_2} \Lambda e_2 \xrightarrow{d_1} \Lambda e_1 \xrightarrow{d_0} 0.$$

Applying $Hom_{\Lambda}\left(, M\right)$, where M is in the module $\left(S_1,\cdots,S_j\right)^j$ of length $3j$, we have

$$0 \xrightarrow{d_0^{\times}} Hom_{\Lambda}\left(\Lambda e_1, M\right) \xrightarrow{d_1^{\times}} Hom_{\Lambda}\left(\Lambda e_2, M\right)$$
$$\xrightarrow{d_2^{\times}} Hom_{\Lambda}\left(\Lambda e_3, M\right) \xrightarrow{d_3^{\times}} Hom_{\Lambda}\left(\Lambda e_1, M\right) \xrightarrow{d_4^{\times}} Hom_{\Lambda}\left(\Lambda e_2, M\right)$$
$$\xrightarrow{d_5^{\times}} Hom_{\Lambda}\left(\lambda e_3, M\right) \xrightarrow{d_6^{\times}} Hom_{\Lambda}\left(\Lambda e_1, M\right),$$

where $Hom_{\Lambda}\left(\Lambda e_1, M\right) \simeq e_1 M, Hom_{\Lambda}\left(\Lambda e_2, M\right) \simeq e_2 M$ and
$Hom_{\Lambda}\left(\lambda e_3, M\right) \simeq e_3 M$.

We compute the Ext-groups for $(3i+1, 3j+1)$. The first Ext-group is;

$$Ext_{\Lambda}^1\left(\left(S_1,\cdots,S_1\right)^{i+1}, M\right) = \ker\left(e_2 M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3 M\right) \Big/ \Im\left(e_1 M \xrightarrow{\alpha(\gamma\beta\alpha)^{i+1}} e_2 M\right),$$

where M is the module of length $3j+1$. We compute the kernel for

$$d_2^{\times}: \left(e_2 M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3 M\right).$$

We have

$$d_2^{\times}\left(0, \left(a_1, a_2, \cdots, a_s, \cdots, a_j\right)^t, 0\right) = \left(0, 0, \left(0, 0, \cdots, 0, a_1, \cdots, a_s\right)^t\right),$$

where a_1 is in the coordinate number $n-i+1$ and therefore a_s is in the coordinate number $n-i+s = j, s = j+i-n$ and consequently, $s+1 = j+i+1-n$.

This shows that

$$\ker\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right)=\left\{\left(0,\left(0,0,\cdots,0,a_{s+1},\cdots,a_j\right)^t,0\right)\middle| a_r\in k, r=s+1,\cdots,j\right\}.$$

We compute the image

$$d_1^{\times}:\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{i+1}} e_2M\right).$$

We have

$$d_1^{\times}\left(\left(b_1,b_2,\cdots,b_{j+1}\right)^t,0,0\right)=\left(0,\left(0,0,\cdots,0,b_1,\cdots,b_{j-(i+1)}\right)^t,0\right),$$

where b_1 is in the coordinate number $i+2$. Hence

$$\Im\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{i+1}} e_2M\right)=\left\{\left(0,\left(0,0,\cdots,0,b_1,\cdots,b_{j-(i+1)}\right)^t,0\right)\middle| b_t\in k, t=1,\cdots,j-(i+1)\right\}.$$

We have $\dim_k Ext_\Lambda^1\left(\left(S_1,\cdots,S_1\right)^t,M\right)=i+2-(j+i+1-n)=n+1-j$ for $j\le n+1$. Next, we compute the second Ext-group.

$$Ext_\Lambda^2\left(\left(S_1,\cdots,S_1\right)^t,M\right)=\ker\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{i+1}} e_1M\right)\Big/\Im\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right).$$

We compute the kernel for

$$d_3^{\times}:\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{i+1}} e_1M\right).$$

We have

$$d_3^{\times}\left(0,0,\left(a_1,a_2,\cdots,a_j\right)^t\right)=\left(\left(0,0,\cdots,0,a_1,\cdots,a_{j-(i+1)}\right)^t,0,0\right),$$

where a_1 is in the coordinate number $i+2$. This shows that

$$\ker\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{i+1}} e_1M\right)=\left\{\left(0,0,\left(0,0,\cdots,0,a_1,\cdots,a_j\right)^t\right)\middle| a_r\in k, r=1,\cdots,j\right\},$$

where a_1 is in the coordinate number $j-i$. We compute the image for

$$d_2^{\times}:\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right).$$

We have

$$d_2^{\times}\left(0,\left(b_1,b_2,\cdots,b_s,\cdots,b_j\right)^t,0\right)=\left(0,0,\left(0,0,\cdots,0,b_1,\cdots,b_s\right)^t\right),$$

where b_1 is in the coordinate number $n-i+1$ and therefore b_s is the coordinate number $n-i+s=j$ and $s=j+i-n$. This implies that

$$\Im\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{n-i}} e_3M\right)=\left\{\left(0,0,\left(0,0,\cdots,0,b_1,\cdots,b_j\right)^t\right)\middle| b_t\in k, t=1,\cdots,j\right\},$$

where b_1 is in the coordinate number $n-i+1$. We have

$$\dim_k Ext_\Lambda^2\left(\left(S_1,\cdots,S_1\right)^t,M\right)=n-i+1-(j-i)=n+1-j$$

for $j\le n+1$. We compute the third Ext-group.

$$Ext_\Lambda^3\left(\left(S_1,\cdots,S_1\right)^t,M\right)=\ker\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right)\Big/\Im\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{i+1}} e_1M\right).$$

We compute the kernel for

$$d_4^{\times}:\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right).$$

We have

$$d_4^{\times}\left(\left(a_1,a_2,\cdots,a_s,\cdots,a_{j+1}\right)^t,0,0\right)=\left(0,\left(0,0,\cdots,0,a_1,\cdots,a_s\right)^t,0\right),$$

where a_1 is in the coordinate number $n-i+1$ and therefore a_s is in the coordinate number $n-i+s=j+1, s=j+i+1-n$ and consequently, $s+1=j+i+2-n$. Hence

$$\ker\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right)=\left\{\left(\left(0,0,\cdots,0,a_{s+1},\cdots,a_{j+1}\right)^t,0,0\right)\middle| a_r\in k, r=s+1,\cdots,j+1\right\}.$$

We compute the image for

$$d_3^{\times}:\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{i+1}} e_1M\right).$$

We have

$$d_3^{\times}\left(0,0,\left(b_1,b_2,\cdots,b_j\right)^t\right)=\left(\left(0,0,\cdots,0,b_1,\cdots,b_{j-i}\right)^t,0,0\right),$$

where b_1 is in the coordinate number $i+3$. This implies that

$$\Im\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{i+1}} e_1M\right)=\left\{\left(\left(0,0,\cdots,0,b_1,\cdots,b_{j-i}\right)^t,0,0\right)\middle| b_t\in k, t=1,\cdots,j-i\right\}.$$

We therefore have $\dim_k Ext_{\Lambda}^3\left(\left(S_1,\cdots,S_1\right)^t,M\right)i+3-\left(j+i+2-n\right)=n+1-j$ for $j\le n+1$. We discuss the fourth Ext-group.

$$Ext_{\Lambda}^4\left(\left(S_1,\cdots,S_1\right)^t,M\right)=\ker\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{i+1}} e_3M\right)\Big/\Im\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right).$$

We compute the kernel for

$$d_5^{\times}:\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{i+1}} e_3M\right).$$

We have

$$d_5^{\times}\left(0,\left(a_1,a_2,\cdots,a_j\right)^t,0\right)=\left(0,0,\left(0,0,\cdots,0,a_1,\cdots,a_{j-i}\right)^t\right),$$

where a_1 is in the coordinate number $i+2$. Hence

$$\ker\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{i+1}} e_3M\right)$$
$$=\left\{\left(0,\left(0,0,\cdots,0,a_1,\cdots,a_{j-(i+1)}\right)^t,0\right)\middle| a_r\in k, r=1,\cdots,j-(i+1)\right\},$$

where a_1 is in the coordinate number $j-i$. We compute the image for

$$d_4^{\times}:\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right)$$

to have

$$d_4^{\times}\left(\left(b_1,b_2,\cdots,b_s,\cdots,b_j\right)^t,0,0\right)=\left(0,\left(0,0,\cdots,0,b_1,\cdots,b_s\right)^t,0\right),$$

where b_1 is in the coordinate number $n-i+1$ and therefore b_s is in the coordinate number $n-i+s=j, s=j+i-n$. Consequently, $s+1=j+i+1-n$.

This shows that

$$\Im\left(e_1M \xrightarrow{\alpha(\gamma\beta\alpha)^{n-i}} e_2M\right)=\left\{\left(0,\left(0,0,\cdots,0,b_1,\cdots,b_{j-i}\right)^t,0\right)\middle| b_t\in k,t=1,\cdots,j\right\},$$

where b_1 is in the coordinate number $n-i+1$. We have

$$\dim_k Ext_\Lambda^4\left(\left(S_1,\cdots,S_1\right)^t,M\right)=n-i+1-\left(j-i\right)=n+1-j$$

for $j\le n+1$. We discuss the fifth Ext-group.

$$Ext_\Lambda^5\left(\left(S_1,\cdots,S_1\right)^t,M\right)=\ker\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{n-i}} e_1M\right)\Big/\Im\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{i+1}} e_3M\right).$$

We compute the kernel for

$$d_6^\times:\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{n-i}} e_1M\right).$$

We have

$$d_6^\times\left(0,0,\left(a_1,a_2,\cdots,a_s,\cdots,a_j\right)^t\right)=\left(\left(0,0,\cdots,0,a_1,\cdots,a_s\right)^t,0,0\right),$$

where a_1 is in the coordinate number $n-i+2$ and therefore a_s is in the coordinate number $n-i+1+s=j,s=j+i-1-n$ and consequently, $s+1=j+i-n$. This shows that

$$\ker\left(e_3M \xrightarrow{\gamma(\beta\alpha\gamma)^{n-i}} e_1M\right)=\left\{\left(0,0,\left(0,0,\cdots,0,a_{s+1},\cdots,a_j\right)^t\right)\middle| a_r\in\left(k,r=s+1,\cdots,j\right)\right\}.$$

We compute the image for

$$d_5^\times:\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{i+1}} e_3M\right).$$

We have

$$d_5^\times\left(0,\left(b_1,b_2,\cdots,b_j\right)^t,0\right)=\left(0,0,\left(0,0,\cdots,0,b_1,\cdots,b_{j-i}\right)^t,0,0\right),$$

where b_1 is in the coordinate number $i+2$. Hence

$$\Im\left(e_2M \xrightarrow{\beta(\alpha\gamma\beta)^{i+1}} e_3M\right)=\left\{\left(0,0,\left(0,0,\cdots,0,b_1,\cdots,b_{j-i}\right)^t\right)\middle| b_t\in k,t=1,\cdots,j\right\}.$$

We therefore have

$$\dim_k Ext_\Lambda^5\left(\left(S_1,\cdots,S_1\right)^t,M\right)=i+2-\left(j+i-n\right)=n+2-j \quad\text{for}\quad j\le n+2.$$

$\dim_k Ext_\Lambda^t\left(3i+1,3j+1\right)=n+1-j$ for $t=1,2,3$ and 4, and $n+2-j$ for $t=5$. We therefore have $n+1=j$ and $n+2=j$.

It follows from the fifth Ext-group that the condition $Ext_\Lambda^n\left(M,N\right)=0$ for $n\gg 0\Leftrightarrow Ext_\Lambda^n\left(N,M\right)=0$ for $n\gg 0$ holds.

4. Conclusion

The study found that the Ext-groups of pairs $\left(M,N\right)$ of modules over Nakayama algebras of type $\left(n,n,n\right)$ satisfy the condition $Ext_\Lambda^n\left(M,N\right)=0$ for $n\gg 0\Leftrightarrow Ext_\Lambda^n\left(N,M\right)=0$ for $n\gg 0$ and justified the claim with projectives of lengths $3n+1$ and $3n+2$ by using combinations of modules of different lengths.

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