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The Prolongation Structures for the System of the Reaction-Diffusion Type

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Abstract

Equations of the reaction-diffusion type are very well known and have been extensively studied in many research areas. In this paper, the prolongation structures for the system of the reaction-diffusion type are investigated from theory of coverings. The realizations and the classifications of the one-dimensional coverings of the system are researched. And the corresponding conservation law of the one-dimensional Abelian coverings is concluded, which is closely connected with the symmetry of the system by Noether theorem.

Keywords

Prolongation Structure, Reaction-Diffusion, Realization, Abelian Covering

1. Introduction

Equations of the reaction-diffusion type have been widely studied in many research areas [1] [2]. One of the reaction-diffusion types has received considerable attention,

$$\begin{cases} u_t - u_{xx} + 2u^2v - 2ku = 0 \\ v_t + v_{xx} - 2v^2u + 2kv = 0 \end{cases} \tag{1}$$

which has been applied in biological systems, chemical autocatalysis, and also in the gauge formulation of the (1 + 1)-dimensional gravity [3]. Moreover, the geometrical equivalent counterpart of the system (1) is the modified Heisenberg Ferromagnetic (HF) equation [4] [5].

About the prolongation structure of the system (1), some results have been obtained. Alfinito *et al.* [1] used Wahlquist-Estabrook (WE) prolongation structure theory proposed by Wahlquist and Estabrook [6] [7] and carried out the detailed integrable analysis. They showed that (1) allowed an incomplete prolongation algebra admitting an infinite-dimensional realization of the Kac-

Moody type. Later following my advisor Zhao W Z's suggestion, we investigated the inhomogeneous extension of the system (1) by the covariant prolongation structures theory [2]. We constructed $sl(2,R) \times R(\rho(t))$ prolongation structure and gave the corresponding Ablowitz-Kaup-Newell-Segur (AKNS-) type equations and Bäcklund transformation. In 2012 Krasil'shchik and Verbovetsky [8] gave an overview of the recent results on the geometry of partial differential equations (PDEs) in application to integrable systems. These results are essentially based on the geometrical approach to partial differential equations developed since 1970s by A.M. Vinogradov and his school [9] [10] [11] [12] [13]. The approach is called the theory of coverings, which treats a PDE as an (infinite-dimensional) submanifold in the space $J^\infty(\pi)$ of infinite jets for a bundle $\pi : E \to M$ whose sections play the role of unknown functions (fields). This attitude allows applying to PDEs powerful techniques of differential geometry and homological algebra. Readers can refer [8] for more information. It is noticed that the WE prolongation structures are an essentially special type of coverings [13] [15]. Cheng and He successfully gave the realizations and classifications of one-dimensional coverings of the MB (modified Boussinesq) system by using the theory. Moreover, they also gave the sufficient and necessary conditions for a vector to be a nonlocal symmetry of the MB system. Hence we want to apply the theory of coverings to the system (1) to give some new integrable information.

The paper is organized as follows. In Section 2, we review some basic notations and theorems due to [13] [14] [15]. In Section 3, we apply the theory to the system (1) and obtain the realization and classifications of one-dimensional coverings of the system. Also we give the corresponding conservation law for one-dimensional Abelian coverings. In Section 4, we give a conclusion.

2. Basic Definitions and Statements

In this section, we mainly recall some definitions and theorems in [14] [15]. For an equation ε in n independent variables x_i and m unknown dependent functions u^j, we consider the jet space $J^\infty(n,m)$ with the coordinates x_i, u_σ^j, where $(x_1, \cdots, x_n, j = 1, \cdots, m)$, and $\sigma = (i_1, i_2, \cdots, |i_\sigma|), 0 \le i_k \le n$ is a multi-index of finite, but unlimited length $|i_\sigma|$. Denote by $\pi : J^\infty(n,m) \to \mathbb{R}^n$ the projection to the space of space of independent variables $(x_1, \cdots, x_n)$.

The vector fields

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u_{\sigma+1}^j \frac{\partial}{\partial u_\sigma^j} \tag{2}$$

are called *total derivatives*. They commute each other, *i.e.*, $[D_i, D_j] = 0$ on $J^\infty(m,n)$ everywhere. $D_i, i = 1, \cdots, n$ span a distribution on $J^\infty(m,n)$ which is called *Cartan distribution* and is denoted by $\mathcal{C}$.

Let a system of PDEs be given by

$$F_\alpha(x_i, \cdots, u_\sigma^j, \cdots) = 0, \quad 1 \le \alpha \le s, \quad 1 \le i \le n, \quad 1 \le j \le m. \tag{3}$$

Then we consider all its differential consequences, or prolongation of (3)

$$D_\sigma F_\alpha = 0, \quad 1 \le \alpha \le s, \quad |\sigma| \ge 0, \tag{4}$$

where $D_\sigma = D_{(i_1,i_2,\cdots,i_k)}$ for $\sigma = (i_1,i_2,\cdots,i_k)$. The hyper surface defined by (4) is denoted by $\varepsilon \subset J^\infty(n,m)$. The vector fields D_i are tangent to ε, and their restriction to ε will be denoted by the same symbol D_i. They span a distribution on ε which is called *Cartan distribution* of ε and is denoted by $\mathcal{C}_\varepsilon$.

Consider another submanifold $\tilde{\varepsilon} \subset J^\infty(n,m)$ with the same independent variables $(x_1,\cdots,x_n)$. And $\tilde{\varepsilon}$ has an integrable distribution $\tilde{\mathcal{C}}$, *i.e.*, $\left[\tilde{\mathcal{C}},\tilde{\mathcal{C}}\right] = 0$. A smooth surjection $\tau : \tilde{\varepsilon} \to \varepsilon$ is called a *differential covering* (or simply a covering) of ε by $\tilde{\varepsilon}$ if its differential takes the Cartan distribution on $\tilde{\varepsilon}$ to that on $\tilde{\varepsilon}$, *i.e.*, $\tau_*\left(\tilde{\mathcal{C}}_\theta\right) = \varepsilon_{\tau(\theta)}$ for any $\theta \in \tilde{\varepsilon}$. Coordinates in the fiber of τ are called *nonlocal variables*.

In this paper we consider equations possessing two independent variables x,t and two dependent variables u,v, *i.e.*, $m = n = 2$ in (3). The WE prolongation structures correspond to the cases when τ is a trivial bundle, *i.e.*,

$$\tau : \varepsilon \times W \to \varepsilon, \tag{5}$$

where W is a finite dimensional manifold. Set $\dim W = l$, then the local coordinates $w^1,\cdots,w^l$ in W correspond to *pseudopotentials* in the WE prolongations approach [6] [7].

In coordinates, the above definition means that the total derivatives on $\tilde{\varepsilon} \subset J^\infty(2,2)$ are of the form

$$\tilde{D}_x = D_x + X, \quad \tilde{D}_t = D_t + T, \tag{6}$$

where X and T are τ−vertical vector fields:

$$X = \sum_{i=1}^l X_i \frac{\partial}{\partial w^i}, \quad T = \sum_{i=1}^l T_i \frac{\partial}{\partial w^i} \tag{7}$$

Since the distribution on $\tilde{\varepsilon}$ is integrable, we have

$$\left[\tilde{D}_x,\tilde{D}_t\right] = D_x(T) - D_t(X) + [X,T] = 0 \tag{8}$$

If the coefficients X^i, T^j in the vertical vector fields X and T are independent of nonlocal variables $w^i, i = 1,\cdots,l$, then the condition (8) reduces to

$$D_x(T) - D_t(X) = 0, \tag{9}$$

the corresponding covering is called *Abelian*.

3. The Prolongation Structures of the Reaction-Diffusion System

We have introduced the covering theory for prolongation structure of nonlinear evolution equation in the previous section. Based upon this theory, we will discuss the corresponding prolongation structure for the system (1) in this section. The system have two independent variables x,t and two dependent variables u,v, *i.e.*, $m = n = 2$ in (3). For convenience we rewrite the system as follows:

$$\begin{cases} u_t = u_{xx} - 2u^2 v + 2ku \\ v_t = -v_{xx} + 2uv^2 + 2kv \end{cases} \tag{10}$$

Then the corresponding jet space is $J^\infty(2,2)$ with the coordinates

$$x, t, u_k, v_j, \quad k \geq 0, \quad j \geq 0 \tag{11}$$

and the total derivative operators are given by

$$D_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i} + \sum_{i \geq 0} v_{i+1} \frac{\partial}{\partial v_i}, \tag{12}$$

$$D_t = \frac{\partial}{\partial t} + \sum_{i \geq 0} D_x^i \left(u_2 - 2u^2 v + 2ku \right) \frac{\partial}{\partial u_i} + \sum_{i \geq 0} D_x^i \left(-v_2 + 2uv^2 - 2kv \right) \frac{\partial}{\partial v_i}, \tag{13}$$

where $u_k = \dfrac{\partial^k u}{\partial x^k}, v_j = \dfrac{\partial^j v}{\partial x^j}$.

Set the covering for (1) is given by (6). By the integrability condition (8), we have

$$\begin{aligned}
0 = \left[\tilde{D}_x, \tilde{D}_t \right] &= D_x(T) - D_t(X) + [X,T] = \frac{\partial T^j}{\partial x} \frac{\partial}{\partial w^j} + \sum_{i \geq 0} u_{i+1} \frac{\partial T^j}{\partial u_i} \frac{\partial}{\partial w^j} \\
&+ \sum v_{i+1} \frac{\partial T^j}{\partial v_i} \frac{\partial}{\partial w^j} - \frac{\partial X^j}{\partial t} \frac{\partial}{\partial w^j} - \sum_{i \geq 0} D_x^i \left(u_2 - 2u^2 v + 2ku \right) \frac{\partial X^j}{\partial u_i} \frac{\partial}{\partial w^j} \\
&- \sum_{i \geq 0} D_x^i \left(-v_2 + 2uv^2 - 2kv \right) \frac{\partial X^j}{\partial v_i} \frac{\partial}{\partial w^j} + [X,T]
\end{aligned} \tag{14}$$

Now we consider the WE type coverings for (1) and suppose that both X and T are independent of x, t, *i.e.*,

$$X = X\left(w^1, \cdots, w^l, u, v, u_x, v_x \right), \quad T = T\left(w^1, \cdots, w^l, u, v, u_x, v_x \right). \tag{15}$$

Then the equation (6) becomes

$$\begin{aligned}
0 = u_1 \frac{\partial T^i}{\partial u} \frac{\partial}{\partial w^i} &+ u_2 \frac{\partial T^i}{\partial u_1} \frac{\partial}{\partial w^i} + v_1 \frac{\partial T^i}{\partial v} \frac{\partial}{\partial w^i} + v_2 \frac{\partial T^i}{\partial v} \frac{\partial}{\partial w^i} \\
&- \left(u_2 - 2u^2 v + 2ku \right) \frac{\partial X^i}{\partial u} \frac{\partial}{\partial w^i} - \left(u_3 - 4uu_1 v - 2u^2 v_1 + 2ku_1 \right) \frac{\partial X^i}{\partial u_1} \frac{\partial}{\partial w^i} \\
&- \left(-v_2 + 2uv^2 - 2kv \right) \frac{\partial X^i}{\partial v} \frac{\partial}{\partial w^i} - \left(-v_3 + 2u_1 v^2 + 4uvv_1 - 2kv_1 \right) \frac{\partial X^i}{\partial v_1} \frac{\partial}{\partial w^i} + [X,T]
\end{aligned}$$

Notice that the right hand side of the above equation is a polynomial in u_3, v_3, u_2, v_2. Therefore the coefficients at u_3, v_3, u_2, v_2 must vanish. Consequently we get

$$-\frac{\partial X^i}{\partial u_1} = 0, \quad \frac{\partial X^i}{\partial v_1} = 0 \tag{16}$$

$$\frac{\partial T^i}{\partial u_1} - \frac{\partial X^i}{\partial u} = 0, \quad \frac{\partial T^i}{\partial v_1} + \frac{\partial X^i}{\partial v} = 0 \tag{17}$$

$$\begin{aligned}
u_1 \frac{\partial T^i}{\partial u} \frac{\partial}{\partial w^i} &+ v_1 \frac{\partial T^i}{\partial v} \frac{\partial}{\partial w^i} + \left(2u^2 v - 2ku \right) \frac{\partial X^i}{\partial u} \frac{\partial}{\partial w^i} \\
&- \left(2uv^2 - 2kv \right) \frac{\partial X^i}{\partial v} \frac{\partial}{\partial w^i} + [X,T] = 0
\end{aligned} \tag{18}$$

By (16), X is independent of u_1, v_1, hence $X = X\left(w^1, \cdots, w^l, u, v \right)$. By (17), T can be expressed in the following form:

$$T = u_1 \frac{\partial X}{\partial u} - v_1 \frac{\partial X}{\partial v} + R\left(w^1, \cdots, w^l, u, v \right) \tag{19}$$

Substituting (19) into (18), we get

$$0 = u_1^2 \frac{\partial^2 X}{\partial u^2} + u_1 \frac{\partial R}{\partial u} - v_1^2 \frac{\partial^2 X}{\partial v^2} + v_1 \frac{\partial R}{\partial v} + \left(2u^2 - 2ku\right)\frac{\partial X}{\partial u}$$
$$- \left(2uv^2 - 2kv\right)\frac{\partial X}{\partial v} + u_1\left[X, \frac{\partial X}{\partial u}\right] - v_1\left[X, \frac{\partial X}{\partial v}\right] + [X, R]. \tag{20}$$

Similarly we can regard the left hand side of the above equation as a polynomial in u_1, v_1. Hence the coefficients at u_1, v_1 must vanish. Accordingly we get

$$\frac{\partial^2 X}{\partial u^2} = 0, \quad \frac{\partial^2 X}{\partial v^2} = 0, \tag{21}$$

$$\frac{\partial R}{\partial u} + \left[X, \frac{\partial X}{\partial u}\right] = 0, \quad \frac{\partial R}{\partial v} - \left[X, \frac{\partial X}{\partial u}\right] = 0 \tag{22}$$

$$\left(2u^2 v - 2ku\right)\frac{\partial X}{\partial u} - \left(2uv^2 - 2kv\right)\frac{\partial X}{\partial v} + [X, R] = 0 \tag{23}$$

Since $\dfrac{\partial^2 X}{\partial u^2} = 0, \quad \dfrac{\partial^2 X}{\partial v^2} = 0.$

From (21) X can be written as follows:

$$X = Auv + Bu + Cv + D, \tag{24}$$

where A, B, C, D are only dependent on nonlocal variables $w_1, \cdots, w_l$. Substituting (24) into (22), we get:

$$\frac{\partial R}{\partial u} = [A, C]v^2 + [A, D]v + [B, C]v + [B, D]$$
$$\frac{\partial R}{\partial v} = [B, A]u^2 + [D, A]u + [B, C]u + [D, C] \tag{25}$$

By (25) we have

$$\frac{\partial^2 R}{\partial u \partial v} = 2[A, C]v + [A, D] + [B, C]$$
$$\frac{\partial^2 R}{\partial v \partial u} = 2[B, A]u + [D, A] + [B, C]. \tag{26}$$

Since $\dfrac{\partial^2 R}{\partial u \partial v} = \dfrac{\partial R}{\partial v \partial u},$ we have

$$[A, D] = 0, \quad [A, C] = 0, \quad [B, A] = 0 \tag{27}$$

Putting the above condition into (25), we have

$$\frac{\partial R}{\partial u} = [B, C]v + [B, D], \quad \frac{\partial R}{\partial v} = [B, C]u + [D, C] \tag{28}$$

Hence

$$R = [B, C]uv + [B, D]u + [D, C]v + E, \tag{29}$$

where E is only dependent on nonlocal variables, *i.e.*, $E = E(w_1, \cdots, w_l)$.

Substituting (29) and (24) into (19), we have

$$T = Au_1 v - Av_1 u + Bu_1 - Cv_1 + [B, C]uv + [B, D]u + [D, C]v + E \tag{30}$$

Substituting (30) and (24) into (23), we have

$$2Bu^2v - 2Cuv^2 - 2kBu + 2kCv + [A,E]uv + [B,[B,C]]u^2v$$
$$+ [B,[B,D]]u^2 + [B,[D,C]]uv + [B,E]u + [C,[B,C]]uv^2$$
$$+ [C,[B,D]]uv + [C,[D,C]]v^2 + [C,E]v + [D,[B,C]]uv \qquad (31)$$
$$+ [D,[B,D]]u + [D,[D,C]]v + [D,E] = 0$$

The right hand side of the above equality is a polynomial in u, v. Thus we have

$$2B + [B,[B,C]] = 0, \quad [B,[B,D]] = 0, \quad [C,[D,C]] = 0, \quad [C,[B,C]] - 2C = 0 \quad (32)$$

$$[B,E] - 2kB + [D,[B,D]] = 0, \quad 2kC + [C,E] + [D,[D,C]] = 0, \quad [D,E] = 0 \quad (33)$$

$$[A,E] + [B,[D,C]] + [C,[B,D]] + [D,[B,C]] = [A,E] + 2[D,[B,C]] = 0. \quad (34)$$

Hence we have proved the following statement:

Theorem 3.1. *For the reaction-diffusion system* (1), *any WE prolongation type coverings*

$$D_x(T) - D_t(X) + [X,T] = 0,$$

$$X = X(w^1, \cdots, w^l, u, v, u_x, v_x), \quad T = T(w^1, \cdots, w^l, u, v, u_x, v_x)$$

are given by

$$X = Auv + Bu + Cv + D, \qquad (35)$$

$$T = Au_1v - Av_1u + Bu_1 - Cv_1 + [B,C]uv + [B,D]u + [D,C]v + E$$

where the vector fields A, B, C, C, E are all depend on nonlocal variables $w^1, \cdots, w^l$ *only and satisfy the following brackets*

$$2B + [B,[B,C]] = 0, \quad [B,[B,D]] = 0, \quad [C,[D,C]] = 0, \quad [C,[B,C]] - 2C = 0 \quad (36)$$

$$[B,E] - 2kB + [D,[B,D]] = 0, \quad 2kC + [C,E] + [D,[D,C]] = 0, \quad [D,E] = 0 \quad (37)$$

$$[A,E] + 2[D,[B,C]] = 0, \quad [A,D] = 0, \quad [A,C] = 0, \quad [B,A] = 0. \qquad (38)$$

Next we will discuss about the realizations and classifications of one dimensional WE coverings of (1).

Assume $A \neq 0$, then a suitable nonlocal variable w can be chosen such that $A = \dfrac{\partial}{\partial w}$. Let $B = \beta \dfrac{\partial}{\partial w}, C = \gamma \dfrac{\partial}{\partial w}, D = \delta \dfrac{\partial}{\partial w}, E = \epsilon \dfrac{\partial}{\partial w}$, where $\beta, \gamma, \delta, \epsilon$ are dependent on w only. By Theorem (3.1), we can get

$$B = 0, C = 0, D = \delta \frac{\partial}{\partial w}, E = \epsilon \frac{\partial}{\partial w}$$

where δ, ϵ are constants. Meanwhile, when this happens, all the brackets in Theorem (3.1) are satisfied automatically.

Hence the above covering is equivalent to

$$X = (uv + \delta)\frac{\partial}{\partial w}, \qquad T = (u_1v - uv_1 + \epsilon)\frac{\partial}{\partial w}, \qquad (39)$$

where δ, ϵ are constants.

Secondly let $A = 0$. Then we will consider $B \neq 0$ and $B = 0$ respectively.

1. Assume that $B \neq 0$ then we can choose a suitable variable w in such a way that $B = \dfrac{\partial}{\partial w}$. By (36), we have

$$C = \left(-w^2 + \alpha w + \beta\right)\frac{\partial}{\partial w}, \quad D = \left(pw + q\right)\frac{\partial}{\partial w}, \tag{40}$$

where α, β, p, q satisfying that

$$\alpha^2 + 4\beta = 0, \quad 2q + \alpha p = 0. \tag{41}$$

Substituting (40) into (37) and (38), we have

$$E = r\left(pw + q\right)\frac{\partial}{\partial w}, \tag{42}$$

where $r > 0$ satisfying that

$$\begin{cases} p^2 - pr + 2k = 0 \\ k\alpha + q(r - p) = 0 \\ 2k\beta + \beta rp - \alpha qr - 2q^2 - pq\alpha + p^2\beta = 0 \end{cases} \tag{43}$$

Notice that the latter two constraints of the above formula can be deduced from the first one and (41). Hence, when $A = 0, B = \dfrac{\partial}{\partial w}$, we have:

$$C = \left(-w^2 + \alpha w + \beta\right)\frac{\partial}{\partial w}, \ D = \left(pw + q\right)\frac{\partial}{\partial w}, \ E = r\left(pw + q\right)\frac{\partial}{\partial w} \tag{44}$$

where α, β, p, q, r satisfying

$$\begin{cases} \alpha^2 = -4\beta \\ 2q + \alpha p = 0 \\ p^2 - p\gamma + 2k = 0 \end{cases} \tag{45}$$

Furthermore, the above covering is equivalent to

$$X = \left(u - vw^2 + 2wv + \beta v + pw + q\right)\frac{\partial}{\partial w} \tag{46}$$

$$T = \Big[u_1 - v_1\left(-w^2 + \alpha w + \beta\right) + \left(-2w + \alpha\right)uv$$

$$+ pu + \left(-pw^2 - 2qw + p\beta\right)v + r\left(pw + q\right) \Big]\frac{\partial}{\partial w},$$

where α, β, p, q, r satisfying (45).

2. If $A = 0, B = 0$, then from Theorem (3.1), we know that $C = 0$. Hence the covering is trivial.

In summary, we have proved the following result from the above discussion:

Theorem 3.2. *For the reaction-diffusion system* (1), *any WE prolongation type coverings are locally equivalent to one of the followings:*

$$\tau^1 : \tilde{D}_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1}\frac{\partial}{\partial u_i} + \sum_{i \geq 0} v_{i+1}\frac{\partial}{\partial v_i} + \left(uv + \delta\right)\frac{\partial}{\partial w},$$

$$\tilde{D}_t = \frac{\partial}{\partial t} + \sum_{i \geq 0} D_x^i\left(u_2 - 2u^2 v + 2ku\right)\frac{\partial}{\partial u_i}$$

$$+ \sum_{i \geq 0} D_x^i\left(-v_2 + 2uv^2 - 2kv\right)\frac{\partial}{\partial v_i} + \left(u_1 v - uv_1 + \epsilon\right)\frac{\partial}{\partial w},$$

where δ, ϵ *are constants.*

$$\tau^2 : \tilde{D}_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1}\frac{\partial}{\partial u_i} + \sum_{i \geq 0} v_{i+1}\frac{\partial}{\partial v_i} + \left(u - vw^2 + 2wv + \beta v + pw + q\right)\frac{\partial}{\partial w},$$

$$\tilde{D}_t = \frac{\partial}{\partial t} + \sum_{i \geq 0} D_x^i \left(u_2 - 2u^2 v + 2ku \right) \frac{\partial}{\partial u_i} + \sum_{i \geq 0} D_x^i \left(-v_2 + 2uv^2 - 2kv \right) \frac{\partial}{\partial v_i}$$
$$+ \left[u_1 - v_1 \left(-w^2 + \alpha w + \beta \right) + \left(-2w + \alpha \right) uv + pu \right.$$
$$\left. + \left(-pw^2 - 2qw + p\beta \right) v + r \left(pw + q \right) \right] \frac{\partial}{\partial w},$$

where α, β, p, q, r satisfying (45).

Remark 3.1. *Obviously* τ^1 *is not dependent on nonlocal variable* w *, hence it is Abelian covering for the system* (1). *By Theorem* (2.1), *we can obtain a conservation law for the reaction-diffusion system* (1):

$$\omega = \left(uv + \delta \right) \mathrm{d}x + \left(u_1 v - uv_1 + \epsilon \right) \mathrm{d}t \tag{47}$$

4. Concluding Remarks

We have investigated prolongation structure for the system (1) by using the covering theory. For this prolongation structure theory, the realizations and the classifications of the one-dimensional coverings of the system can be obtained. By comparison with the result by Alfinito *et al.* [1], we find they are actually the same. Here we are from the point view of tangent bundle theory, but WE prolongation theory is based on the cotangent bundle theory. Other than that, we can easily get the corresponding conservation law from the one-dimensional Abelian coverings.

It should be mentioned that a lot of questions remain to be understood. Firstly, it should be important to try to extend the prolongation technique to the study of higher dimension nonlinear field equations. For example, the more general (2 + 1)-dimensional reaction C diffusion equations can be written as

$$\begin{cases} u_t = D_1 \Delta u + b_1 u^2 v + b_2 uv^2 + b_3 u + b_4 v + b_5 \\ v_t = D_2 \Delta v + c_1 u^2 v + c_2 uv^2 + c_3 u + c_4 v + c_5 \end{cases} \tag{48}$$

where $u = u\left(x, y, t \right), v = v\left(x, y, t \right), \Delta$ is the Laplace operator in two-dimensional orthonormal coordinates; D_1 and D_2 are the diffusion constants; b_i and $c_i, i = 1, 2, \cdots, 5$, are the coefficients. How to give the corresponding prolongation structure? Secondly, another important aspect which deserves to be explored is the comparison between the covering theory and covariant theory [2]. Thirdly, how to obtain the realizations and the classifications of higher dimensional coverings of the system? Some of them will be in the forthcoming publication.

Acknowledgements

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