

The Pfaffian Technique: A $(2 + 1)$ -Dimensional Korteweg de Vries Equation

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Abstract

The $(2 + 1)$ -dimensional Korteweg de Vries (KdV) equation, which was first derived by Boiti *et al.*, has been studied by various distinct methods. It is known that this $(2 + 1)$ -dimensional KdV equation has rich solutions, such as multi-soliton solutions and dromion solutions. In the present article, a unified representation of its N -soliton solution is given by means of pfaffian. We'll show that this $(2 + 1)$ -dimensional KdV equation is nothing but the Plücker identity when its τ -function is given by pfaffian.

Keywords

The $(2 + 1)$ -Dimensional Korteweg de Vries Equation, Hirota Bilinear Method, Pfaffian, Plücker Identity

1. Introduction

The solitary wave, so-called because it often occurs as a single entity and is localized, was first observed by J. Scott Russell on the Edinburgh-Glasgow Canal in 1834. It is known that many nonlinear evolution equations have soliton solutions, such as the Korteweg de Vries equation, the Sin-Gordon equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili equation, the Davey-Stewartson equation, etc. In order to study the property of nonlinear evolution equations, methods are developed to derive solitary wave solution or soliton solution to nonlinear evolution equations. Some of the most important methods are the inverse scattering transformation (IST) [1] method, the bilinear method [2]-[7], symmetry reduction method [8], the Bäcklund or Darboux transformation method [9] and so on. Having soliton solutions is one of the basic integrable properties of nonlinear evolution equations.

In this paper, we are interested in the general expression of N -soliton solution to the

(2 + 1)-dimensional KdV equation,

$$\begin{aligned}\frac{\partial u}{\partial t} + 3\frac{\partial(uv)}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= 0, \\ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0,\end{aligned}\tag{1}$$

which was first derived by Boiti *et al.* by using the idea of the weak Lax pair [10]. This system can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [11]. Recently, the dromion solutions and some exact solutions are studied by Lou and Wazwaz respectively [12]-[14]. While as for the uniformed expression of its N -soliton solution is unknown yet.

In this article, we'll study the N -soliton solution to the (2 + 1)-dimensional KdV system (1). A compact form of the N -soliton solution to Equation (1) is obtained by means of pfaffian technique, which is given in section 2. Conclusion and further discussions are given in section 3.

2. N -Soliton Solution to the (2 + 1)-Dimensional KdV Equation

Given a nonlinear evolution equation, if it has 3-soliton solution, then this equation is of great possibility of having N -soliton ($3 \leq N$) solution. Pfaffian technique is one of the methods that can help us to determine whether the evolution equation has multi-soliton solutions or not. In this section, we first review some properties of pfaffian.

2.1. Pfaffian

Pfaffians are antisymmetric functions with respect to its independent variables

$$\text{pf}(a, b) = -\text{pf}(b, a), \quad \text{for any } a \text{ and } b.$$

An n -th order pfaffian $\text{pf}(1, 2, \dots, 2n)$ can be defined inductively by the expansion rule [3]

$$\text{pf}(1, 2, \dots, 2n) = \sum_{j=2}^{2n} (-1)^j \text{pf}(1, j)(2, 3, \dots, \hat{j}, \dots, 2n),$$

where \hat{j} denotes the absence of letter j . For example, when $n = 2$, we have

$$\text{pf}(1, 2, 3, 4) = \text{pf}(1, 2)\text{pf}(3, 4) - \text{pf}(1, 3)\text{pf}(2, 4) + \text{pf}(1, 4)\text{pf}(2, 3).$$

There are various kinds of pfaffian identities. In this article, we just introduce the so-called Plücker relation for pfaffians [3]

$$\begin{aligned}&\text{pf}(\alpha_1, \alpha_2, \alpha_3, 1, 2, \dots, 2n-1)\text{pf}(1, 2, \dots, 2n) \\ &= \text{pf}(\alpha_1, 1, 2, \dots, 2n-1)\text{pf}(\alpha_2, \alpha_3, 1, 2, \dots, 2n) \\ &\quad - \text{pf}(\alpha_2, 1, 2, \dots, 2n-1)\text{pf}(\alpha_1, \alpha_3, 1, 2, \dots, 2n) \\ &\quad + \text{pf}(\alpha_3, 1, 2, \dots, 2n-1)\text{pf}(\alpha_1, \alpha_2, 1, 2, \dots, 2n),\end{aligned}\tag{2}$$

which we are going to use. Hereafter, we let $(1, 2, \dots, 2n)$ denote pfaffian $\text{pf}(1, 2, \dots, 2n)$ for simplicity.

2.2. N -Soliton Solutions

The Hirota form of the $(2 + 1)$ -dimensional KdV system (1) is

$$D_y [D_t + D_x^3] f \cdot f = 0, \quad (3)$$

which is obtained by the dependent variable transformations

$$u = 2 \frac{\partial^2 \ln f(x, y, t)}{\partial x \partial y}, \quad v = 2 \frac{\partial^2 \ln f(x, y, t)}{\partial x^2}. \quad (4)$$

Here the Hirota bilinear operator $D_x^m D_t^n$ is defined by

$$D_x^m D_t^n a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x, t) b(x', t') \Big|_{x'=x, t'=t},$$

with n and m are arbitrary nonnegative integers.

In [14], the 3-soliton solution to the $(2 + 1)$ -dimensional KdV system (3) is obtained

$$f = 1 + \sum_{i=1}^3 \exp(\theta_i) + \sum_{1 \leq i < j \leq 3} c_{ij} \exp(\theta_i + \theta_j) + c_{123} \exp(\theta_1 + \theta_2 + \theta_3) \quad (5)$$

where

$$\theta_i = k_i x + m_i y + q_i t + \theta_i^0, \quad q_i = -k_i^3, \quad k_i \neq 0, m_i \neq 0 \text{ and } \theta_i^0 \text{ are constants;}$$

$$c_{ij} = \frac{(p_i - p_j)(m_i - m_j)}{(p_i + p_j)(m_i + m_j)}, \quad c_{123} = c_{12} c_{23} c_{13},$$

via the perturbation method. It claims that the N -soliton solutions for $4 \leq N$ can also be obtained by using perturbation method, but the explicit expression of the multi-soliton solution is not given.

In this article, we'll study the the multi-soliton solution to Equation (3) using the pfaffian technique [15] [16]. A compact form of the N -soliton solution is given in terms of N -th order pfaffian.

Proposition 1. *If the τ -function f of the $(2 + 1)$ -dimensional KdV system (3) is given by the pfaffian function*

$$f = (d_0, \beta_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N), \quad (6)$$

whose entries, for $j, k = 1, 2, \dots, N$, are defined by

$$\begin{aligned} (d_0, a_j) &= \exp(\eta_j), \quad (d_0, b_j) = -1, \quad (d_0, \beta_0) = 1, \quad (a_j, a_k) = a_{jk} \exp(\eta_j + \eta_k), \\ (b_j, \beta_0) &= 1, \quad (a_j, \beta_0) = 0, \quad (b_j, b_k) = b_{jk}, \quad (a_j, b_k) = \delta_{jk}, \\ \text{with } a_{jk} &= \frac{p_j - p_k}{p_j + p_k}, \quad b_{jk} = \frac{m_j - m_k}{m_j + m_k}, \quad \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \end{aligned} \quad (7)$$

$$\eta_j = p_j x + m_j y - p_j^3 t + \eta_j^0, \quad p_j \neq 0, m_j \neq 0 \text{ and } \eta_j^0 \text{ are constants,}$$

then this particular pfaffian function (6) gives an N -soliton solutions to the $(2 + 1)$ -dimensional KdV system (3).

Proof. In the following, we will prove that the pfaffian function (6) satisfies the $(2 + 1)$ -dimensional KdV Equation (3). By defining "differential operators" d_n ($n = 1, 2, \dots$)

$$(d_n, a_j) = p_j^n \exp(\eta_j), \quad (d_n, b_j) = (d_n, d_m) = (d_n, \beta_0) = (d_n, d_0) = 0, \quad (8)$$

we obtain the following differential formulae

$$\begin{aligned} \frac{\partial f}{\partial x} &= -(d_0, d_1, a_1, \dots, a_N, b_1, \dots, b_N), \quad \frac{\partial^2 f}{\partial x^2} = -(d_0, d_2, a_1, \dots, a_N, b_1, \dots, b_N), \\ \frac{\partial^3 f}{\partial x^3} &= -(d_0, d_3, a_1, \dots, a_N, b_1, \dots, b_N) - (d_0, d_1, d_2, \beta_0, a_1, \dots, a_N, b_1, \dots, b_N), \\ \frac{\partial f}{\partial t} &= (d_0, d_3, a_1, \dots, a_N, b_1, \dots, b_N) - 2(d_0, d_1, d_2, \beta_0, a_1, \dots, a_N, b_1, \dots, b_N). \end{aligned} \quad (9)$$

In order to find the pfaffian expression for the differential functions with derivative of variable y , we need to define another letter β [17]

$$(\beta, b_j) = -m_j, \quad (\beta, a_j) = (\beta, \beta_0) = (\beta, d_n) = 0. \quad (10)$$

Then we have

$$\begin{aligned} \frac{\partial f}{\partial y} &= (d_0, \beta, a_1, \dots, a_N, b_1, \dots, b_N), \\ \frac{\partial^2 f}{\partial x \partial y} &= (d_0, d_1, \beta, \beta_0, a_1, \dots, a_N, b_1, \dots, b_N), \\ \frac{\partial^3 f}{\partial x^2 \partial y} &= (d_0, d_2, \beta, \beta_0, a_1, \dots, a_N, b_1, \dots, b_N), \\ \frac{\partial^4 f}{\partial x^3 \partial y} &= (d_0, d_3, \beta, \beta_0, a_1, \dots, a_N, b_1, \dots, b_N) - (d_0, d_1, d_2, \beta, a_1, \dots, a_N, b_1, \dots, b_N), \\ \frac{\partial^2 f}{\partial y \partial t} &= -(d_0, d_3, \beta, \beta_0, a_1, \dots, a_N, b_1, \dots, b_N) - 2(d_0, d_1, d_2, \beta, a_1, \dots, a_N, b_1, \dots, b_N). \end{aligned} \quad (11)$$

Substituting formulae (9) and (11) into the right hand side of Equation (3), we obtain nothing but the Plücker relation for pfaffians (2)

$$\begin{aligned} D_y [D_t + D_x^3] f \cdot f &= f \left(\frac{\partial^2 f}{\partial y \partial t} + \frac{\partial^4 f}{\partial x^3 \partial y} \right) - 3 \frac{\partial f}{\partial x} \frac{\partial^3 f}{\partial x^2 \partial y} + 3 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} \right) \\ &= 3 \left[(d_1, \dots) (d_2, \beta_3, \beta_0, \dots) - (d_2, \dots) (d_1, \beta_3, \beta_0, \dots) \right. \\ &\quad \left. + (\beta_3, \dots) (d_1, d_2, \beta_0, \dots) - (\beta_0, \dots) (d_1, d_2, \beta_3, \dots) \right] \\ &\equiv 0, \end{aligned}$$

where (\dots) denotes $(d_0, a_1, \dots, a_N, b_1, \dots, b_N)$. Therefore the pfaffian function (6) solves the $(2 + 1)$ -dimensional KdV system (3).

Note that in order to derive the differential formulae of the pfaffian function (6), we have to define another extra letter β besides the “differential operators” d_n . The multi-soliton solution to the nonlinear $(2 + 1)$ -dimensional KdV system (1) can be obtained by substituting pfaffian function (6) into the dependent variable transformation (4) directly.

3. Conclusion

In this article, a compact form of the multi-soliton solution to the $(2 + 1)$ -dimensional

KdV system is given via the pfaffian technique. As one can see, the key point of the proof is to derive suitable expressions of the differential formulae of pfaffian τ -function f . It is worth pointing out that the method used in this article is different as the one for the proof of the BKP equation, which the differential formulae of the pfaffian τ -function depend only on the “differential operators” d_n .

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