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B-Spline Collocation Method for Solving Singularly Perturbed Boundary Value Problems

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Abstract

We use fifth order B-spline functions to construct the numerical method for solving singularly perturbed boundary value problems. We use B-spline collocation method, which leads to a tri-diagonal linear system. The accuracy of the proposed method is demonstrated by test problems. The numerical results are found in good agreement with exact solutions.

Keywords

Fifth Order B-Spline Functions, B-Spline Collocation Method, Singularly Perturbed Boundary Value Problems

1. Introduction

Consider following singularly perturbed boundary value problem

$$Ly(x) = -\varepsilon y'' + p(x)y' + q(x)y = f(x), \quad a < x < b \tag{1}$$

with boundary conditions

$$y(a) = A, \quad y(b) = B, \quad y'(a) = \gamma, \quad y'(b) = \delta \tag{2}$$

where $0 < \varepsilon < 1$, ε is a small positive parameter, $p(x)$ and $q(x)$ are sufficiently smooth real-valued functions. Typically, these problems arise very frequently in fluid dynamics, elasticity, quantum mechanics, chemical reactor theory and many other allied areas. Up to now, different numerical methods have been proposed by various authors [1]-[3] for this singularly perturbed problem arising in transport phenomena in chemistry and biology [4]. It is so attractive to mathematicians due to the fact that the solution exhibits a multi-scale character, *i.e.*, there is a thin layer where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly. So the usual numerical treatment of singular perturbation problems gives rise to major computational difficulties and fails to give accurate solutions.

B-spline functions are useful wavelet basis functions; the stiffness matrix is sparse when it is used as trial functions. B-splines were introduced by Schoenberg in 1946 [5]. Up to now, B-spline approximation method for numerical solutions has been researched by various researchers [6]-[8].

2. Description of the B-Spline Collocation Method

The expression of fifth order B-spline function is as follows:

$$N_6(x)=\begin{cases}(x+3)^5 & [-3,-2)\\ (x+3)^5-6(x+2)^5 & [-2,-1)\\ (x+3)^5-6(x+2)^5+15(x+1)^5 & [-1,0)\\ (3-x))^5-6(2-x)^5+15(1-x)^5 & [0,1)\\ (3-x))^5-6(2-x)^5 & [1,2)\\ (3-x)^5 & [2,3)\\ 0 & \text{others}\end{cases} \tag{3}$$

The fifth order B-spline function $N_6(x)$ is used to calculate in this work and possesses the following characters: piecewise smooth, compact support, Symmetry, rapidly decaying, differentiability, linear combination.

The region $[a,b]$ is partitioned into uniformly sized finite elements of length h by the knots x_j such that $a=x_0<x_1<x_2<\cdots<x_N=b$ with $h=x_j-x_{j-1}$, $x_j=a+jh$, $j=1,2,\cdots,N$. Let $\phi_m(x)$ be fifth order B-spline function with knots at the points x_m, $m=0,1,\cdots,N$. The set of splines $\{\phi_{-2},\phi_{-1},\phi_0,\phi_1,\cdots,\phi_N,\phi_{N+1},\phi_{N+2}\}$ forms a basis for functions defined over $[a,b]$.

In the proposed algorithm, The fifth order B-spline function $N_6(x)$ is used as a single mother wavelet, *i.e.* $\phi(x)=N_6(x)$ and dilation and translation of mother wavelet functions can construct any function of $L^2(R)$.

$$S(x)=\sum_k c_k\phi_k^J(x)=\sum_k c_k 2^{J/2}\phi\left(2^J x-k\right)=\sum_k a_k\phi\left(\frac{x-2^{-J}k}{2^{-J}}\right)=\sum_k a_k N_6\left(\frac{x-x_k}{h}\right) \tag{4}$$

where $h=1/2^J$, $x_k=k/2^J$, $a_k=2^{J/2}c_k$.

So the global approximation $S(x)$ to the function $y(x)$ can be written in terms of the B-spline as follows

$$S(x)=\sum_{i=-2}^{N+2}a_i N_6\left(\frac{x-x_i}{h}\right), \tag{5}$$

where $h=\dfrac{b-a}{n}$, a_i are unknown real coefficients.

Using the fifth order B-spline function and the approximate solution Equation (5), the nodal values $S(x_j)$, $S'(x_j)$ and $S''(x_j)$ at the node x_j are given in terms of element parameters by

$$S(x_j)=\frac{1}{120}\left(a_{j-2}+26a_{j-1}+66a_j+26a_{j+1}+a_{j+2}\right) \quad (6)$$

$$S'(x_j)=\frac{1}{24h}\left(-a_{j-2}-10a_{j-1}+10a_{j+1}+a_{j+2}\right) \quad (7)$$

$$S''(x_j)=\frac{1}{6h^2}\left(a_{j-2}+2a_{j-1}-6a_j+2a_{j+1}+a_{j+2}\right) \quad (8)$$

where the symbols $'$ and $''$ denote first and second differentiation with respect to *x*, respectively.

Substituting Equations (6)-(8) into Equation (1) and Equation (2), we can obtain following linear equations

$$Ba=r \quad (9)$$

where

$$r=\left(120y(a),24hy'(a),-120h^2 f_0/\varepsilon,\cdots,-120h^2 f_N/\varepsilon,\ 24hy'(b),120y(b)\right)^{\mathrm{T}}$$

$$a=\left(a_{-2},a_{-1},a_0,a_1,\cdots,a_N,a_{N+1},a_{N+2}\right)^{\mathrm{T}},\quad f_i=f(a+ih)$$

Note $N_6\left(\dfrac{x_j-x_i}{h}\right)=B_{ij}$

$$B=\begin{bmatrix} 1 & 26 & 66 & 26 & 1 & 0 & \\ -1 & -10 & 0 & 10 & 1 & 0 & \\ LB_{-2,0} & LB_{-1,0} & LB_{0,0} & LB_{1,0} & LB_{2,0} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & LB_{N-2,N} & LB_{N-1,N} & LB_{N,N} & LB_{N+1,N} & LB_{N+2,N} \\ & & -1 & -10 & 0 & 10 & 1 \\ & & 1 & 26 & 66 & 26 & 1 \end{bmatrix}$$

where

$$LB_{j-2,j}=20+5hp_j/\varepsilon-h^2q_j/\varepsilon,\ LB_{j-1,j}=40+50hp_j/\varepsilon-26h^2q_j/\varepsilon$$

$$LB_{j,j}=-120-66h^2q_j/\varepsilon,\qquad LB_{j+1,j}=40-50h\,p_j/\varepsilon-26h^2q_j/\varepsilon$$

$$LB_{j+2,j}=20-5h\,p_j/\varepsilon-h^2q_j/\varepsilon,\quad p_j=p(a+jh),\quad q_j=q(a+jh)$$

It is easily seen that the matrix B is strictly diagonally dominant and hence nonsingular. Since B is nonsingular, we can solve the system $Ba=r$ for $a_{-2},a_{-1},a_0,a_1,\cdots,a_N,a_{N+1},a_{N+2}$. Hence the method of collocation using the fifth order B-spline function $N_6(x)$ as a basis function applied to the singularly perturbed boundary value problem has a unique solution $S(x)$ given by Equation (5).

3. Numerical Results

In the section, we illustrate the numerical techniques discussed in the previous section by the following problems.

Example 1. Consider the convention-dominated equation:

$$-\varepsilon y''+y'+y=1\qquad (0<x<1) \quad (10)$$

with boundary conditions: $y(0)=y(1)=0$,

$$y'(0)=\left[\lambda_1\left(e^{\lambda_2}-1\right)+\lambda_2\left(1-e^{\lambda_1}\right)\right]/\left(e^{\lambda_1}-e^{\lambda_2}\right),$$

$$y'(1)=\left[\lambda_1\left(e^{\lambda_2}-1\right)e^{\lambda_1}+\lambda_2\left(1-e^{\lambda_1}\right)e^{\lambda_2}\right]/\left(e^{\lambda_1}-e^{\lambda_2}\right).$$

The exact solution is given by

$$y(x)=\left(e^{\lambda_2}-1\right)e^{\lambda_1 x}/\left(e^{\lambda_1}-e^{\lambda_2}\right)+\left(1-e^{\lambda_1}\right)e^{\lambda_2 x}/\left(e^{\lambda_1}-e^{\lambda_2}\right)+1 \tag{11}$$

where $\lambda_1=\left(1+\sqrt{1+4\varepsilon}\right)/(2\varepsilon)$, $\lambda_2=\left(1-\sqrt{1+4\varepsilon}\right)/(2\varepsilon)$.

Comparison of the numerical results and point-wise errors is given in Table 1.

It observed that

1) when h decreases (*i.e.* collocation number increases) for fixed ε the point-wise errors decrease;

2) when ε decreases for fixed h the point-wise errors increase;

3) when $\varepsilon=0.0015$, $x\to 1$ the errors are very large.

Example 2. Solve the following non-homogeneous equation:

$$-\varepsilon y''+py'+y=\cos\pi x \quad (0<x<1) \tag{12}$$

with boundary conditions

$$y(0)=y(1)=0,\quad y'(0)=b\pi+A\lambda_1+\lambda_2 B e^{-\lambda_2},\quad y'(1)=-b\pi+A\lambda_1 e^{\lambda_1}+\lambda_2 B.$$

The analytical solution is given by

$$y(x)=a\cos\pi x+b\sin\pi x+A\exp(\lambda_1 x)+B\exp\left[-\lambda_2(1-x)\right] \tag{13}$$

where

$$a=\frac{\varepsilon\pi^2+1}{p^2\pi^2+\left(\varepsilon\pi^2+1\right)^2},\quad b=\frac{p\pi}{p^2\pi^2+\left(\varepsilon\pi^2+1\right)^2},\quad A=-a\frac{1+\exp(-\lambda_2)}{1-\exp(\lambda_1-\lambda_2)},\quad B=a\frac{1+\exp(\lambda_1)}{1-\exp(\lambda_1-\lambda_2)}.$$

And $\lambda_1<0$ and $\lambda_2>0$ are the real solutions of the characteristic equation $-\varepsilon\lambda^2+p\lambda+1=0$.

Approximation solutions for different values of ε and for fixed p are given in Figure 1. It observed that

Table 1. Example 1. Comparison of results and point-wise errors.

X	$\varepsilon=0.1$, $h=1/32$	$\varepsilon=0.1$, $h=1/128$			$\varepsilon=0.01$ $h=1/32$	$\varepsilon=0.01$, $h=1/128$			$\varepsilon=0.0015$, $h=1/1024$
	error	numerical	Exact	error	error	Numerical	Exact	error	error
1/16	0.0036	0.0565	0.0556	0.0009	0.0095	0.0613	0.0600	0.0013	0.0001560
2/16	0.0037	0.1090	0.1082	0.0008	0.0092	0.1176	0.1164	0.0012	0.0001465
4/16	0.0033	0.2053	0.2045	0.0007	0.0082	0.2204	0.2193	0.0011	0.0001294
6/16	0.0030	0.2907	0.2901	0.0006	0.0074	0.3111	0.3102	0.0010	0.0001142
12/16	0.0021	0.0004	0.4582	0.4578	0.0052	0.5248	0.5241	0.0007	0.0000785
14/16	0.0014	0.3984	0.3981	0.0003	0.0025	0.5801	0.5795	0.0006	0.0000693
1	0.0004	0.0462	0.0462	0	0.0199	0.3402	0.3401	0.0001	0.0107

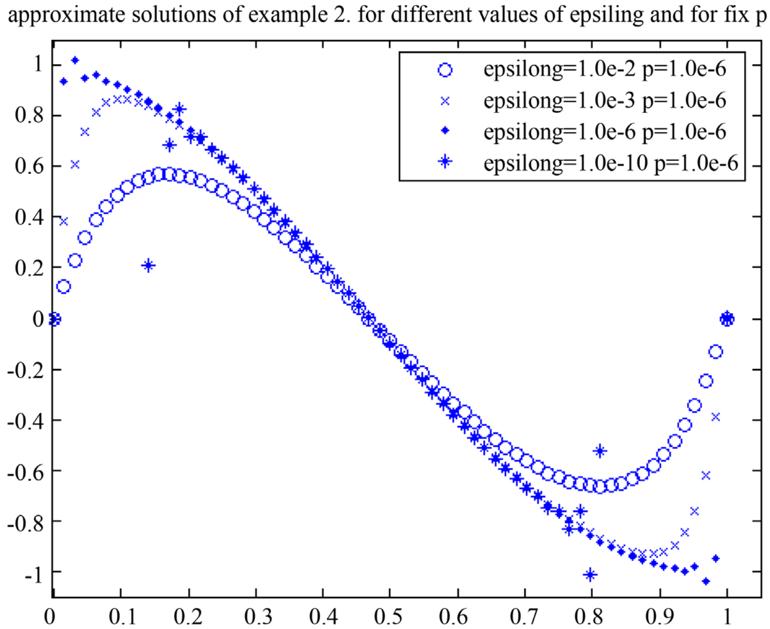

Figure 1. Approximation solutions of example 2 for different values of epsilon g and for fixed p.

1) when $\varepsilon = 10^{-2}$ and 10^{-3}, the approximation solutions are in good agreement with exact solution; 2) when $\varepsilon = 10^{-6}$ and 10^{-10}, $x \to 0$ and $x \to 1$ the errors are very large; 3) when ε decreases for fixed p the width of boundary layer becomes small and wave shape change more and more stiff at $x = 0$ and $x = 1$.

4. Conclusion

The numerical results show clearly the effect of ε on the boundary layer and the B-spline collocation method solving singular boundary value problems is relatively simple to collocate the solution at the mesh points. It is applicable technique and approximates the exact solution very well.

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