

# Forward ( $\Delta$ ) and Backward ( $\nabla$ ) Difference Operators Basic Sets of Polynomials in $\mathbb{C}^n$ and Their Effectiveness in Reinhardt and Hyperelliptic Domains

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## Abstract

We generate, from a given basic set of polynomials in several complex variables  $\{P_m(z)\}_{m \geq 0}$ , new basic sets of polynomials  $\{\tilde{P}_m(z)\}_{m \geq 0}$  and  $\{\tilde{Q}_m(z)\}_{m \geq 0}$  generated by the application of the  $\nabla$  and  $\Delta$  operators to the set  $\{P_m(z)\}_{m \geq 0}$ . All relevant properties relating to the effectiveness in Reinhardt and hyperelliptic domains of these new sets are properly deduced. The case of classical orthogonal polynomials is investigated in details and the results are given in a table. Notations are also provided at the end of a table.

## Keywords

Effectiveness, Cannon Condition, Cannon Sum, Cannon Function, Reinhardt Domain, Hyperelliptic Domain

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## 1. Introduction

Recently, there has been an upsurge of interest in the investigations of the basic sets of polynomials [1]-[27]. The inspiration has been the need to understand the common properties satisfied by these polynomials, crucial to gaining insights into the theory of polynomials. For instance, in numerical analysis, the knowledge of basic sets of polynomials gives information about the region of convergence of the series of these polynomials in a given domain. Namely, for a particular differential equation admitting a polynomial solution, one can deduce the range

of convergence of the polynomials set. This is an advantage in numerical analysis which can be exploited to reduce the computational time. Besides, if the basic set of polynomials satisfies the Cannon condition, then their fast convergence is guaranteed. The problem of derived and integrated sets of basic sets of polynomials in several variables has been recently treated by A. El-Sayed Ahmed and Kishka [1]. In their work, complex variables in complete Reinhardt domains and hyperelliptic regions were considered for effectiveness of the basic set. Also, recently the problem of effectiveness of the difference sets of one and several variables in disc  $D(\mathbb{R})$  and polydisc  $\prod_{i=1}^m (R_i)$  has been treated by A. Anjorin and M.N Hounkonnou [27].

In this paper, we investigate the effectiveness, in Reinhardt and hyperelliptic domains, of the set of polynomials generated by the forward ( $\Delta$ ) and backward ( $\nabla$ ) difference operators on basic sets. These operators are very important as they involve the discrete scheme used in numerical analysis. Furthermore, their composition operators form the most of second order difference equations of Mathematical Physics, the solutions of which are orthogonal polynomials [25] [26].

Let us first examine here some basic definitions and properties of basic sets, useful in the sequel.

**Definition 1.1** Let  $z = (z_1, z_2, \dots, z_n)$  be an element of the space of several complex variables  $\mathbb{C}^n$ . The hyperelliptic region of radii  $r_s > 0, s \in I = \{1, 2, \dots, n\}$ , is denoted by  $E_{[r]}$  and its closure by  $\bar{E}_{[r]}$  where

$$E_{[r]} = \{w \in \mathbb{C} : |w| < 1\}, \bar{E}_{[r]} = \{w \in \mathbb{C} : |w| \leq 1\}$$

And

$$w = \{w_1, w_2, \dots, w_n\}, w_s = \frac{z_s}{r_s}; s \in I.$$

**Definition 1.2** An open complete Reinhardt domain of radii  $\rho_s > 0, s \in I$  is denoted by  $\Gamma_{(\rho)}$  and its closure by  $\bar{\Gamma}_{(\rho)}$ , where

$$\Gamma_{(\rho)} = \Gamma_{(\rho_1, \rho_2, \dots, \rho_n)} = \{z \in \mathbb{C}^n : |z_s| < \rho_s : s \in I\},$$

$$\bar{\Gamma}_{(\rho)} = \bar{\Gamma}_{(\rho_1, \rho_2, \dots, \rho_n)} = \{z \in \mathbb{C}^n : |z_s| \leq \rho_s : s \in I\}.$$

The unspecified domains  $D(\bar{\Gamma}_{(\rho)})$  and  $D(\bar{E}_{(r)})$  are considered for both the Reinhardt and hyperelliptic domains. These domains are of radii  $\rho_s^* > \rho_s, r_s^* > r_s, s \in I$ . Making a contraction of this domain, we get the domain  $D((\rho)^+) = D((\rho_1^+, \rho_2^+, \dots, \rho_n^+))$  where  $\rho_s^+$  stands for the right-limits of  $\rho_s^*, s \in I$ :

$$D(\bar{\Gamma}_{(\rho)}) \equiv D((\rho)^+) = D((\rho_1^+, \rho_2^+, \dots, \rho_n^+)) = \{z \in \mathbb{C}^n : |z_s| \leq \rho_s^* : s \in I\}.$$

Thus, the function  $F(z)$  of the complex variables  $z_s, s \in I$ , which is regular in  $\bar{E}_{(r)}$  can be represented by the power series

$$F(z) = \sum_{m=0}^{\infty} a_m z^m = \sum_{(m_1, m_2, \dots, m_n)} a_{(m_1, m_2, \dots, m_n)} (z_1^{m_1}, z_2^{m_2}, \dots, z_n^{m_n}), \tag{1}$$

where  $m = (m_1, m_2, \dots, m_n)$  represents the mutli indicies of non-negative integers for the function  $F(z)$ . We have [1]

$$M(F(z), [r]) = M(F(z) : (r_1, r_2, \dots, r_n)) = \max_{\bar{E}_{(r)}} |F(z)| \tag{2}$$

where  $r = (r_1, r_2, \dots, r_n)$  is the radius of the considered domain. Then for hyperelliptic domains  $\bar{E}_{(r)}$  [1]

$$\sigma_m = \inf_{|t|=1} \frac{1}{t^m} \frac{\{\langle m \rangle\}^{(m)/2}}{\prod_{s=1}^n m_s^{m_s/2}},$$

$t$  being the radius of convergence in the domain,  $\langle m \rangle = m_1 + m_2 + \dots + m_n$  assuming  $1 \leq \sigma_m \leq (\sqrt{n})^{(m)}$  and  $m_s^{m_s/2} = 1$ , whenever  $m_s = 0; s \in I$ . Since  $\omega_s = \frac{z_s}{r_s}; s \in I$ , we have (1)

$$\overline{\lim}_{\langle m \rangle \rightarrow \infty} \left( \frac{|a_m|}{\sigma_m \prod_{s=1}^n (r_s)^{\langle m \rangle - m_s}} \right)^{\frac{1}{\langle m \rangle}} = \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left\{ \frac{|a_m|^{\frac{1}{\langle m \rangle}}}{\sigma_m^{\frac{1}{\langle m \rangle}} \left( \prod_{s=1}^n (r_s)^{\frac{\langle m \rangle - m_s}{\langle m \rangle}} \right)} \right\} \leq \frac{1}{\prod_{s=1}^n r_s}$$

where also, using the above function  $F(z)$  of the complex variables  $z_s, s \in I$ , which is regular in  $\overline{\Gamma}(\rho)$  and can be represented by the power series above (1), then we obtain

$$|a_m| \leq \frac{M(F(z), (\rho'_m))}{\rho'_m}, m_s \geq 0; s \in I,$$

$$\rho'_m \in (\rho_{m_1}, \rho_{m_2}, \dots, \rho_{m_n}) \text{ and}$$

$$M(F(z), (\rho')) = \max_{\overline{\Gamma}(\rho')} |F(z)|. \tag{3}$$

Hence, we have for the series  $F(z)$

$$\sum_{m=0}^{\infty} a_m z^m = \sum_{m=0}^{\infty} a_m \left( \sum_{k=0}^m \pi_{m,k} P_k(z) \right) \equiv \sum_{m=0}^{\infty} a_m \left( \sum_{k=0}^m \pi_{m,k} z^k \right),$$

$k = (k_1, k_2, \dots, k_n), m = (m_1, m_2, \dots, m_n), \pi_{m,k} = \binom{m}{k}, \max_{\overline{\Gamma}(\rho')} |F(z)| = M(F(z), (\rho'))$ . Since  $\rho'_s$  can be taken arbitrary near to  $\rho_s, s \in I$ , we conclude that

$$\overline{\lim}_{\langle m \rangle \rightarrow \infty} \left\{ |a_m| \prod_{s=1}^n \rho_s^{(m_s - \langle m \rangle)} \right\}^{\frac{1}{\langle m \rangle}} \leq \frac{1}{\prod_{s=1}^n \rho_s},$$

With  $\overline{\lim} = \text{lmsup}$  and  $\langle m \rangle \gg m_s$ .

**Definition 1.3** A set of polynomials  $\{P_m(z)\}_{m \geq 0} = \{P_0(z), P_1(z), \dots\}_{m \geq 0}$  is said to be basic when every polynomial in the complex variables  $z_s; s \in I$  can be uniquely expressed as a finite linear combination of the elements of the basic set  $\{P_m(z)\}_{m \geq 0}$ .

Thus, according to [4], the set  $\{P_m(z)\}_{m \geq 0}$  will be basic if and only if there exists a unique row-finite-matrix  $\overline{P}$  such that  $\overline{P}P = P\overline{P} = \mathbb{1}$ , where  $P = (P_{m,h})_{m \geq 0}$  is a matrix of coefficients of the set  $\{P_m(z)\}_{m \geq 0}$ ;  $h = (h_1, h_2, \dots, h_n)$  are multi indices of nonnegative integers,  $\overline{P}$  is the matrix of operators deduced from the associated set of the set  $\{P_m(z)\}_{m \geq 0}$  and  $\mathbb{1}$  is the infinite unit matrix of the basic set  $\{P_m(z)\}_{m \geq 0}$ , the inverse of which is  $\{\overline{P}_m(z)\}_{m \geq 0}$ . We have

$$P_m(z) = \sum_{h=0}^m P_{m,h} z^h z^m = \sum_{h=0}^m \overline{P}_{m,h} P_h(z) = \sum_{h=0}^m P_{m,h} \overline{P}_m(z) \overline{P}_m(z) = \sum_{h=0}^m \overline{P}_{m,h} z^h. \tag{4}$$

Thus, for the function  $F(z)$  given in (1), we get  $F(z) = \sum_{m=0}^{\infty} \pi_{m,h} P_m(z)$  where

$$\pi_{m,h} = \sum_{h=0}^m \overline{P}_{m,h} a_h = \sum_{h=0}^m \overline{P}_{m,h} \frac{f^{(h)}(0)}{h_s!} \neq \pi_{m,k} = \binom{m}{k}, h! = h(h-1)(h-2) \dots 3 \cdot 2 \cdot 1. \text{ The series } \sum_{m=0}^{\infty} \pi_{m,h} P_h(z)$$

is an associated basic series of  $F(z)$ . Let  $N_m = N_{m_1}, N_{m_2}, \dots, N_{m_n}$  be the number of non zero coefficients  $\overline{P}_{m,h}$  in the representation (4).

**Definition 1.4** A basic set satisfying the condition

$$\overline{\lim}_{\langle m \rangle \rightarrow \infty} N_m^{\frac{1}{\langle m \rangle}} = 1 \tag{5}$$

Is called a Cannon basic set. If

$$\overline{\lim}_{\langle m \rangle \rightarrow \infty} N_m^{\frac{1}{\langle m \rangle}} = a > 1,$$

Then the set is called a general basic set.

Now, let  $D_m = D_{m_1, m_2, m_3, \dots, m_n}$  be the degree of polynomials of the highest degree in the representation (4). That is to say  $D_h = D_{h_1, h_2, h_3, \dots, h_n}$  is the degree of the polynomial  $P_h(z)$ ; the  $D_n \leq D_m \forall n_s \leq m_s : s \in I$  and since the element of basic set are linearly independent [6], then  $N_m \leq 1 + 2 + \dots + (D_m + 1) < \lambda D_m$ , where  $\lambda$  is a constant. Therefore the condition (5) for a basic set to be a Cannon set implies the following condition [6]

$$\overline{\lim}_{\langle m \rangle \rightarrow \infty} D_m^{\langle m \rangle} = 1. \tag{6}$$

For any function  $F(z)$  of several complex variables there is formally an associated basic series  $\sum_{h=0}^{\infty} P_h(z)$ . When the associated basic series converges uniformly to  $F(z)$  in some domain, in other words as in classical terminology of Whittaker (see [5]) the basic set of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which they are represented. To study the convergence property of such basic sets of polynomials in complete Reinhardt domains and in hyperelliptic regions, we consider the following notations for Cannon sum

$$\mu(P_m(z), (\rho)) = \prod_{s=1}^n \rho_s^{\langle m \rangle} \sum_{h=0}^m |\bar{P}_{m,h}| M(P_h(z), (\rho)) \tag{7}$$

For Reinhardt domains [24],

$$\sigma_m \prod_{s=1}^n (r_s)^{\langle m \rangle} \sum_{h=0}^m |\bar{P}_{m,h}| M(P_h(z), (r)) = \Omega(P_m(z), (r)) \tag{8}$$

For hyperelliptic regions [1].

## 2. Basic Sets of Polynomials in $\mathbb{C}^n$ Generated by $\nabla$ and $\Delta$ Operators

Now, we define the forward difference operator  $\Delta$  acting on the monomial  $z^m$  such that

$$J(\Delta) = \Delta^n z^m$$

$$\Delta^n \equiv (E + (-\mathbb{1}))^n = \sum_{k=0}^n \binom{n}{k} E^{n-k} (-\mathbb{1})^k \text{ by binomial expansion.}$$

where  $E$  is the shift operator and  $\mathbb{1}$ -the identity operator. Then

$$\Delta^n P_m(z) = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k E^{n-k} P_m(z) = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k P_m(z + n - k).$$

So, considering the monomial  $z^m$

$$\Delta^n z^m = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k (z + n - k)^m = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k (z + \alpha)^m, \quad \alpha = n - k.$$

$$(z + \alpha)^m = \sum_{j=0}^m \binom{m}{j} z^{m-j} \alpha^j.$$

Hence

$$\Delta^n z^m = \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} (-\mathbb{1})^k \alpha^j z^{m-j}.$$

Since  $\alpha = n - k$ ,

$$\alpha^j = (n - k)^j = \sum_{t=0}^j \binom{j}{t} n^{j-t} (-k)^t = \sum_{t=0}^j \binom{j}{t} n^{j-t} (-1)^t (k)^t.$$

Hence

$$\Delta^n z^m = \sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^{m-j} \binom{n}{k} \binom{m}{j} \binom{j}{t} \times \zeta (-\mathbb{1})^{t+k} n^{j-t} (k)^t z^s,$$

where  $\zeta = \binom{m-j}{s}$  and  $z^{m-j} = \sum_{s=0}^{m-j} \zeta z^s$  by definition. Similarly, we define the backward difference operator  $\nabla$  acting on the monomial  $z^m$  such that

$$J(\nabla) = \nabla^n z^m, \nabla = \mathbb{1} - E^{-1}. \tag{9}$$

Equivalently, in terms of lag operator  $L$  defined as  $LF(z) = f(z-1)$ , we get  $\nabla = \mathbb{1} - L$ . Remark that the advantage which comes from defining polynomials in the lag operator stems from the fact that they are isomorphic to the set of ordinary algebraic polynomials. Thus, we can rely upon what we know about ordinary polynomials to treat problems concerning lag-operator polynomials. So,

$$\nabla^n = (\mathbb{1} - E^{-1})^n = \sum_{k=0}^n \binom{n}{k} (-E^{-1})^k = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k E^{-k}. \tag{10}$$

The Cannon functions for the basic sets of polynomials in complete Reinhardt domain and in hyperelliptical regions [1], are defined as follows, respectively:

$$\begin{aligned} \mu(P(z), (\rho)) &= \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left\{ \mu(P_m(z), (\rho)) \right\}^{\frac{1}{\langle m \rangle}} \\ \Omega(P(z), (r)) &= \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left\{ \Omega(P_m(z), (r)) \right\}^{\frac{1}{\langle m \rangle}}. \end{aligned}$$

Concerning the effectiveness of the basic set  $\{P_m(z)\}_{m \geq 0}$  in complete Reinhardt domain we have the following results:

**Theorem 2.1** A necessary and sufficient condition [7] for a Cannon set  $\{P_m(z)\}_{m \geq 0}$  to be:

1. effective in  $\bar{\Gamma}_{(\rho)}$  is that  $\mu(P(z), (\rho)) = \prod_{s=1}^n \rho_s$ ;
2. effective in  $D(\bar{\Gamma}_{(\rho)})$  is that  $\mu(P(z), (\rho^+)) = \prod_{s=1}^n \rho_s$ .

**Theorem 2.2** The necessary and sufficient condition for the Cannon basic set  $\{P_m(z)\}_{m \geq 0}$  of polynomials of several complex variables to be effective [1] in the closed hyperelliptic  $\bar{E}_{(r)}$  is that  $\Omega(P(z), (r)) = \prod_{s=1}^n r_s$  where  $r = (r_1, r_2, \dots, r_n)$ .

The Cannon basic set  $\{P_m(z)\}_{m \geq 0}$  of polynomials of several complex variables will be effective in  $D(\bar{E}_{(r)})$  if and only if  $\Omega(P(z), (r^+)) = \prod_{s=1}^n r_s$ . See also [1]. We also get for a given polynomial set  $\{Q_m(z)\}_{m \geq 0}$ :

$$\nabla^n Q_m(z) = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k E^{-k} Q_m(z) \nabla^n Q_m(z) = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k Q_m(z-k).$$

So, considering the monomial  $z^m$ ,

$$\nabla^n z^m = \sum_{k=0}^n \binom{n}{k} (-\mathbb{1})^k (z-k)^m = \sum_{s=0}^{m-j} \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \times \zeta (-\mathbb{1})^{j+k} k^j z^s.$$

Let's prove the following statement:

**Theorem 2.3** The set of polynomials  $\{\tilde{P}_m(z)\}_{m \geq 0}$  and  $\{\tilde{Q}_m(z)\}_{m \geq 0}$

$$\begin{aligned} \{P_m(J(\Delta)(z))\}_{m \geq 0} &= \{\tilde{P}_m(z)\}_{m \geq 0} \\ \{Q_m(J(\nabla)(z))\}_{m \geq 0} &= \{\tilde{Q}_m(z)\}_{m \geq 0} \end{aligned}$$

Are basic.

**Proof:** To prove the first part of this theorem, it is sufficient to show that the initial sets of polynomials

$\{P_m(z)\}_{m \geq 0}$  and  $\{Q_m(z)\}_{m \geq 0}$ , from which  $\{\tilde{P}_m(z)\}_{m \geq 0}$  and  $\{\tilde{Q}_m(z)\}_{m \geq 0}$  are generated, are linearly independent. Suppose there exists a linear relation of the form

$$\sum_{i=0}^m c_i P_{m,i}(z) \equiv 0, c_i \neq 0 \tag{11}$$

For at least one  $i, i \in I$ . Then

$$(P_m(J(\Delta))) \left( \sum_{i=0}^m c_i P_{m,i}(z) \right) \equiv 0.$$

Hence, it follows that  $\sum_{i=0}^m c_i P_{m,i}(z) \equiv 0$ . This means that  $\{P_m(z)\}_{m \geq 0}$  would not be linearly independent. Then the set would not be basic. Consequently (11) is impossible. Since  $1, z^1, z^2, \dots, z^n$  are polynomials, each of them can be represented in the form  $z^{(m)} = \sum_{\langle m \rangle, \langle k \rangle} \pi_{\langle m \rangle, \langle k \rangle} P_k(z)$ . Hence, we write

$$\begin{aligned} m_1 = 0, \quad 1 &= \sum_k \pi_{0,k} P_k(z) \\ m_2 = 1, \quad z &= \sum_k \pi_{1,k} P_k(z) \\ m_3 = 2, \quad z &= \sum_k \pi_{2,k} P_k(z) \\ &\vdots \\ m_n = n, \quad z^n &= \sum_k \pi_{n,k} P_k(z) \end{aligned}$$

with  $z = (z_1, z_2, \dots, z_n), k = (k_1, k_2, \dots, k_n)$ .

In general, given any polynomial  $P_m(z) = \sum_{i=0}^m c_i z^i$  and using

$$\begin{aligned} P_m(z) &= \sum_{i=0}^m c_i \left( \sum_{j=0}^i \pi_{i,j} P_j(z) \right) \\ &= \sum_{i=0}^m c_i (\pi_{i,0} P_0(z) + \dots) \\ &= \sum_{i=0}^m c_i \pi_{i,0} P_0(z) + c_i \pi_{i,1} P_1(z) + \dots \\ &= \sum_{m_n=m_1}^m \tilde{c}_{m_n} P_{m_n}(z); \quad m = (m_1, m_2, \dots, m_n); z = (z_1, z_2, \dots, z_n). \end{aligned}$$

Hence the representation is unique. So, the set  $\{\tilde{P}_m(z)\}_{m \geq 0}$  is a basic set. Changing  $\Delta$  to  $\nabla$  leads to the same conclusion. We obtain the following result.

**Theorem 2.4** *The Cannon set  $\{\tilde{P}_m(z)\}_{m \geq 0}$  of polynomials in several complex variables  $z_s; s \in I$  is Effective*

*in the closed complete Reinhardt domain  $\Gamma_{(\rho)}$  and in the closed Reinhardt region  $D(\bar{\Gamma}_{(\rho)})$ .*

**Proof:** In a complete Reinhardt domain for the forward difference operator  $\Delta$ , the Cannon sum of the monomial  $z^m$  is given by

$$\mu(\tilde{P}_m(z), (\rho)) = \frac{\prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \sum_h |\bar{P}_{m,h}| M(\tilde{P}_{m,h}(z), (\rho_s))}{\sum_{s=0}^{m-j} \sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta (-1)^{t+k} n^{j-t} k^t}.$$

Then

$$M(\tilde{P}_{m,h}(z), (\rho)) = \max_{\Gamma_{(\rho)}} |\tilde{P}(z)| \leq N_m M(P_{m,h}(z), (\rho))$$

where  $N_m$  is a constant. Therefore,

$$\mu(\tilde{P}_{m,h}(z), (\rho)) \leq \frac{N_m \left( \prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \sum_h |\bar{P}_{m,h}| M(P_{m,h}(z), (\rho)) \right)}{\sum_{s=0}^{m-j} \sum_{t=0}^j \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta (-1)^{t+k} n^{j-t} k^t}$$

which implies that

$$\mu(\tilde{P}_m(z), (\rho)) \leq \frac{N_m \left( \prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \mu(P_m(z), (\rho)) \right)}{\sum_s \sum_{t=0}^{m-j} \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta (-1)^{t+k} n^{j-t} k^t}.$$

Then the Cannon function

$$\mu(\tilde{P}(z), (\rho)) \leq \overline{\lim}_{\langle m \rangle \rightarrow \infty} \mu(P_m(z), (\rho))^{\frac{1}{\langle m \rangle}}$$

$$\mu(\tilde{P}(z), (\rho)) \leq \prod_{s=1}^n \rho_s; s \in I.$$

But  $\mu(\tilde{P}(z), (\rho)) \geq 0$ . Hence

$$\mu(\tilde{P}(z), (\rho)) = \overline{\lim}_{\langle m \rangle \rightarrow \infty} \mu(P_m(z), (\rho))^{\frac{1}{\langle m \rangle}}$$

$$\mu(\tilde{P}(z), (\rho)) = \prod_{s=1}^n \rho_s; s \in I.$$

Similarly, for the backward difference operator  $\nabla$ , the Cannon sum

$$\mu(\tilde{Q}_m(z), (\rho)) = \frac{\prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \sum_j |\bar{Q}_{m,j}| M(Q_{m,j}(z), (\rho))}{\sum_s \sum_{k=0}^{m-j} \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \zeta (-1)^{k+j} k^j}$$

$$\mu(\tilde{Q}_m(z), (\rho)) = \frac{\prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \sum_j |\bar{Q}_{m,j}| M(\tilde{Q}_{m,j}(z), (\rho))}{\sum_s \sum_{k=0}^{m-j} \sum_{j=0}^m (-1)^j \binom{n}{k} \binom{m}{j} \zeta (-1)^k k^j}.$$

Then

$$\mu(\tilde{Q}_m(z), (\rho)) \leq \frac{\prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} (K_m \mu(Q_m(z), (\rho)))}{\sum_{s=0}^{m-j} \sum_{k=0}^n \sum_{j=0}^m (-1)^{j+k} \binom{n}{k} \binom{m}{j} \zeta k^j}.$$

where  $K_m = |\nabla^m| = \text{constant}$  as  $\nabla$  is bounded for the Reinhardt domain is complete. Thus,

$$\begin{aligned} \mu(\tilde{Q}(z), (\rho)) &= \overline{\lim}_{\langle m \rangle \rightarrow \infty} \mu(\tilde{Q}_m(z), (\rho))^{\frac{1}{\langle m \rangle}} \\ &\leq \overline{\lim}_{\langle m \rangle \rightarrow \infty} \mu(Q_m(z), (\rho))^{\frac{1}{\langle m \rangle}} \leq \prod_{s=1}^n \rho_s. \end{aligned}$$

But

$$\mu(\tilde{Q}(z), [\rho]) \geq \prod_{s=1}^n \rho_s.$$

Hence, we deduce that  $\mu(Q_m(z), (\rho)) = \prod_{s=1}^n \rho_s$ .

**Theorem 2.5** *If the Cannon basic set  $\{P_m(z)\}_{\geq 0}$  (resp.  $\{Q_m(z)\}_{\geq 0}$ ) of polynomials of the several complex variables  $z_s, s \in I$  for which the condition (5) is satisfied, is effective in  $\bar{E}_{(r)}$ , then the  $(\Delta)$  and  $(\nabla)$ -set  $\{\tilde{P}_m(z)\}_{m \geq 0}$  (resp.  $\{\tilde{Q}_m(z)\}_{m \geq 0}$ ) of polynomials associated with the set  $\{P_m(z)\}_{m \geq 0}$  (resp.  $\{Q_m(z)\}_{m \geq 0}$ ) will be effective in  $\bar{E}_{(r)}$ .*

The Cannon sum  $\tilde{\Omega}(\tilde{P}(z), (r))$  of the forward difference operator  $\Delta$  of the set  $\{\tilde{P}_m(z)\}_{\geq 0}$  in  $\bar{E}_{(r)}$  will have the form

$$\tilde{\Omega}(\tilde{P}(z), (r)) = \frac{\sigma_m \prod_{s=1}^n (r_s)^{\langle m \rangle - m_s} \sum_{h=0}^m |\bar{P}_{m,h}| M(\tilde{P}_{m,h}(z), (r))}{\sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^* (-1)^{k+t} n^{j-k} k^t}$$

where  $\zeta^* = \sum_{s=0}^{m-j} \zeta$  and

$$\begin{aligned} M(\tilde{P}_{m,h}(z), (r)) &= \max_{E_{(r)}} |\tilde{P}_h(z)| \leq \sum |\tilde{P}_{m,h}| \frac{\prod_{s=1}^n r_s^{h_s}}{\sigma^h} \\ &\leq \left| \sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j (-1)^{k+t} \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^* n^{j-t} k^t \right| \times \frac{\sigma^h M(P_{m,h}(z), (r)) \prod_{s=1}^n r_s^{h_s}}{\prod_{s=1}^n r_s^{h_s} \sigma^h} \\ &\leq K_1 N_m D_m^2 M(P_{m,h}(z), (r)) \leq K_1 D_m^{m+2} M(P_{m,h}(z), (r)) \end{aligned}$$

where  $K_1$  is a constant. Then

$$\begin{aligned} \tilde{\Omega}(\tilde{P}_m[z], [r]) &\leq \frac{K_1 \sigma_m D_m^{n+2} \prod_{s=1}^n [r_s]^{\langle m \rangle - m_s} \sum_{h=0}^m |\bar{P}_{m,h}|}{\sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^* (-1)^{k+t} n^{j-t} k^t} \\ &= K_2 \prod_{s=1}^n [r_s]^{\langle m \rangle - m_s} \sum_{h=0}^m |\bar{P}_{m,h}| \end{aligned}$$

where

$$K_2 = \frac{K_1 \sigma_m D_m^{n+2}}{\sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^* (-1)^{t+k} n^{j-t}}$$

So, by similar argument as in the case of Reinhardt domain we obtain

$$\begin{aligned} \tilde{\Omega}(\tilde{P}(z), (r)) &= \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left( \tilde{\Omega}(\tilde{P}_m(z), (r)) \right)^{\frac{1}{\langle m \rangle}} \\ &= \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left( K_2 \prod_{s=1}^n (r_s)^{\langle m \rangle - m_s} \alpha \right)^{\frac{1}{\langle m \rangle}} = \prod_{s=1}^n r_s, \end{aligned}$$

where  $\alpha = \sum_h |\bar{P}_{m,h}|$ , since the Cannon function is such that [1]  $\tilde{\Omega}(\tilde{P}(z), (r)) \geq \prod_{s=1}^n r_s$ . Similarly, for the backward difference operator

$$\tilde{\Omega}(\tilde{Q}_m(z), (r)) = \frac{\sigma_m \prod_{s=1}^n (r_s)^{\langle m \rangle - m_s} \sum_{h=0}^m |\bar{Q}_{m,h}| M(\tilde{Q}_{m,h}(z), (r))}{\sum_{s=0}^{m-j} \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \zeta^* (-1)^{j+k} k^j}$$



Such that the Cannon function writes as

$$\tilde{\Omega}(\tilde{Q}(z), (r)) \leq \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left( \tilde{\Omega}(\tilde{Q}_m(z), (r)) \right)^{\frac{1}{\langle m \rangle}} = \prod_{s=1}^n r_s.$$

But

$$\tilde{\Omega}(\tilde{Q}(z), (r)) \geq \prod_{s=1}^n r_s$$

Since the Cannon function is non-negative. Hence

$$\tilde{\Omega}(\tilde{Q}(z), (r)) = \prod_{s=1}^n r_s.$$

### 3. Examples

Let us illustrate the effectiveness in Reinhardt and hyperelliptic domains, taking some examples. First, suppose that the set of polynomials  $\{P_m(z)\}_{m \geq 0}$  is given by

$$\begin{aligned} P_0(z) &= 1 \\ P_m(z) &= z^m + 2^{m^2} \quad \text{for } m \geq 1. \end{aligned}$$

Then

$$\begin{aligned} z^m &= P_m(z) - 2^{m^2} \\ M_m(\rho) &= \sup_{|z| < \rho_s} |P_m(z)| = \rho_s^m + 2^{m^2}, \quad s \in I \\ M_0(P_0(z), (\rho)) &= \sup_{|z| \leq \rho} |P_0(z)| = 2^{m^2}. \end{aligned}$$

Hence

$$\mu(P_m(z), (\rho)) = \rho_s^m + 2 \cdot 2^{m^2} = \sum_{h=0}^m |P_{m,h}| M(P_m(z), (\rho))$$

which implies

$$\begin{aligned} \mu(P(z), (\rho)) &= \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left\{ \mu(P_m(z), (\rho)) \right\}^{\frac{1}{\langle m \rangle}} \\ &= \overline{\lim}_{\langle m \rangle \rightarrow \infty} \left\{ \rho_s^m + 2 \cdot 2^{m^2} \right\}^{\frac{1}{\langle m \rangle}} \end{aligned}$$

for  $\rho_s < 2$ ;  $\mu(P(z), (\rho)) = \infty$ .

Now consider the new polynomial from the polynomial defined above:

$$\begin{aligned} \Delta^n z^m &= \sum_{k=0}^n \binom{n}{k} (-1)^k (z + n - k)^m \\ &= \sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^s (-1)^{k+t} n^{j-t} (k)^t z^s. \end{aligned}$$

Hence by Theorem 2.4,

$$\mu(\tilde{P}_m(z), (\rho)) = \frac{\prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \sum_{h=0}^n |\tilde{P}_{m,h}| M(\tilde{P}_{m,h}(z), (\rho))}{\sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^s (-1)^{k+t} n^{j-t} k^t}.$$

where

$$M(\tilde{P}_{m,h}(z), (\rho)) = \max_{\Gamma(\rho)} |\tilde{P}(z)| \leq N_m M(P_{m,h}(z), (\rho))$$

where  $N_m$  is a constant. Hence,

$$\mu(\tilde{P}_{m,h}(z), (\rho)) \leq \frac{N_m \left( \prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \sum_h |\bar{P}_{m,h}| M(P_{m,h}(z), (\rho)) \right)}{\sum_{t=0}^j \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^* (-1)^{k+t} n^{j-t} k^t}$$

The Cannon function

$$\mu(\tilde{P}(z), (\rho)) \leq \lim_{\langle m \rangle \rightarrow \infty} \frac{\left\{ N_m \left( \prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \sum_{h=0}^n |\tilde{P}_{m,h}| M(P_{m,h}(z), (\rho)) \right) \right\}^{\frac{1}{\langle m \rangle}}}{\left\{ \sum_{t=0}^j \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^* (-1)^{k+t} n^{j-t} k^t \right\}^{\frac{1}{\langle m \rangle}}}$$

which implies

$$\mu(\tilde{P}(z), (\rho)) = \left( K_1 \mu(\tilde{P}_m(z), (\rho)) \right)^{\frac{1}{\langle m \rangle}} = \infty \quad \text{as } m \rightarrow \infty$$

where

$$K_1 = \frac{N_m^{\frac{1}{\langle m \rangle}} \prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s}}{\sum_{t=0}^j \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \binom{j}{t} \zeta^* (-1)^{k+t} n^{j-t} k^t}$$

and

$$\mu(\tilde{P}_m(z), (\rho)) = \sum_{h=0}^m |\tilde{P}_{m,h}| M(P_{m,h}(z), (\rho)).$$

Hence

$$\mu(P(z), (\rho)) = \infty. \quad \text{Then } \mu(\tilde{P}(z), (\rho)) = \infty.$$

Similarly, for the operator  $\nabla$ , we have

$$\mu(\tilde{Q}(z), (\rho)) \leq \frac{N_m^{\frac{1}{\langle m \rangle}} \left( \prod_{s=1}^n (\rho_s)^{\langle m \rangle - m_s} \right)^{\frac{1}{\langle m \rangle}} \left( \sum_{h=0}^m |\tilde{Q}_{m,h}| M(Q_m(z), (\rho)) \right)^{\frac{1}{\langle m \rangle}}}{\left( \sum_{s=0}^{m-j} \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \zeta (-1)^{j+k} k^j \right)^{\frac{1}{\langle m \rangle}}}$$

Since

$$\sum |\tilde{Q}_{m,h}|^{\frac{1}{\langle m \rangle}} \left( M(Q_{m,h}(z), (\rho)) \right)^{\frac{1}{\langle m \rangle}} = \mu((Q_m(z), (\rho)))^{\frac{1}{\langle m \rangle}} = \infty.$$

Then

$$\mu(\tilde{Q}(z), (\rho)) = \infty; \quad m \rightarrow \infty$$

**Table 1.** Region of effectiveness: (1) Disc  $\lambda(R) = \prod_{j=1}^n R_j$ ; (2) Hyperelliptic  $\lambda(R) = \prod_{s=1}^n r_s$ ; Reinhardt domain  $\lambda(R) = \prod_{j=1}^n \rho_s$ .

Polynomials ( $P_m$ )	$\Delta^n P_m [z]$	$\nabla^n P_m [z]$
Monomials		
$Z^m$	$\sum_{k=0}^n \sum_{j=0}^m \sum_{t=0}^j \sum_{s=0}^{m-j} \zeta^t (-1)^{k+t} k^t \times {}^n C_k^m C_j^i C_i(n)^{j-t} Z^s$	$\sum_{k=0}^n \sum_{j=0}^m \sum_{s=0}^{m-j} C_k^m C_j^i \zeta^k k^i (-1)^{k+j} Z^s$
Chebyshev (first kind)		
$P_m(Z)$ $= \sum_{\mu=0}^{[m/2]} C_{2\mu}^m Z^{m-2\mu} (Z^2 - 1)^\mu$	$\sum_{\mu=0}^{[m/2]} \sum_{k=0}^{\mu} \sum_{s=0}^{\mu-2\mu} \eta_1 \times {}^m C_{2\mu}^{m-2\mu} C_j^n C_k^\mu C_s$ $\times (-1)^{s+k} (n-k)^{s+t} Z^q$	$\sum_{\mu=0}^{[m/2]} \sum_{k=0}^{\mu} \sum_{j=0}^{m-2\mu} \eta_2$ $\times {}^n C_k^m C_{2\mu}^{m-2\mu} C_j \times (-1)^{q+s+j} Z^q$
Chebyshev (second kind)		
$\sum_{\mu=0}^{[m/2]} (-1)^\mu {}^{m-\mu} C_\mu (2Z)^{m-2\mu}$	$\sum_{\mu=0}^{[m/2]} \sum_{k=0}^{\mu} \sum_{q=0}^{m-2\mu} \times {}^n C_{m-2\mu}^{m-\mu} C_q^{m-\mu} C_\mu$ $\times \eta_3 2^{m-2\mu} (-1)^k (n-k)^t Z^s$	$\sum_{\mu=0}^{[m/2]} \sum_{k=0}^{\mu} \sum_{s=0}^{m-2\mu} {}^n C_k^{m-\mu} C_\mu (-1)^k$ $\times {}^{m-2\mu} C_s \eta_4 (-1)^{s+k} 2^{m-2\mu} Z^s$
Hermite		
$\sum_{k=0}^{[m/2]} (-1)^k k! (2z)^{m-2k} (m-2k)!$	$\sum_{j=0}^n \sum_{k=0}^{[m/2]} \sum_{t=0}^{m-2k} \eta_5$ $\times {}^n C_j^{m-2k} C_i (-1)^j 2^{m-2k} k! (m-2k)! \times (n-j)^t Z^s$	$\sum_{s=0}^{m-2k-t} \sum_{t=0}^n \sum_{r=0}^{m-2k} \sum_{k=0}^{[m/2]} {}^n C_j (-1)^{j+t} {}^{m-2k} C_r 2^{m-2k} k! (m-2k)!$ $\times j^t Z^s$

Implication: The new sets are nowhere effective since the parents sets are nowhere effective. By changing  $\prod_{s=1}^n \rho_s^{\langle m \rangle - m_s}$  in Reinhart domain to  $\sigma_m \prod_{s=1}^n [r_s]^{\langle m \rangle - m_s}$ , where  $\sigma_m = \inf_{|t|=1} \frac{1}{t^m} \frac{\{\langle m \rangle\}^{\langle m \rangle / 2}}{\prod_{s=1}^n m_s^{m_s / 2}}$ , we obtain the same condition of effectiveness as in Reinhart domain for both operators  $\Delta$  and  $\nabla$  in the hyperelliptic domain.

The following notations are relevant to the table below.

$$\eta_1 = \sum_{t=0}^{2(\mu-s)m-2s-t-j} \sum_{q=0}^t \binom{2\mu-2s}{t} \binom{m-2s-t-j}{q} \tag{12}$$

$$\eta_2 = \sum_{q=0}^{\mu} \sum_{s=0}^{2(\mu-q)m-2q-j} \sum_{v=0}^q (-1)^{q+j+v}, \eta_3 = \sum_{s=0}^{m-2\mu-q} \binom{m-2\mu-q}{s} \tag{13}$$

$$\eta_4 = \sum_{q=0}^{m-2\mu-s} \binom{m-2\mu-s}{q} \tag{14}$$

$$\eta_5 = \sum_{s=0}^{m-2k-t-s} \binom{m-2k-t-s}{s} \tag{15}$$

Finally, for the classical orthogonal polynomials, the explicit results of computation are given in a **Table 1** below.

Thus, in this paper, we have provided new sets of polynomials in C, generated by  $\nabla$  and  $\Delta$  operators, which satisfy all properties of basic sets related to their effectiveness in specified regions such as in hyperelliptic and Reinhardt domains. Namely, the new basic sets are effective in complete Reinhardt domain as well as in closed Reinhardt domain. Furthermore, we have proved that if the Cannon basic set  $\{P_m(z)\}_{m \geq 0}$  is effective in hyperelliptic domain, then the new set  $\{\tilde{P}_m(z)\}_{m \geq 0}$  is also effective in the hiperelliptic domain.

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## Appendix

### Key Notations

- 1)  $\mu(\tilde{P}_{m,h}[z],[\rho])$  = Cannon sum of the new  $\Delta$ -set in Reinhardt domain.
- 2)  $\mu(\tilde{Q}_{m,h}[z],[\rho])$  = Cannon sum of the new  $\nabla$ -set in Reinhardt domain.
- 3)  $\mu(\tilde{P}_{m,h}[z],[r])$  = Cannon sum of the new  $\Delta$ -set in Hyperelliptic domain.
- 4)  $\mu(\tilde{Q}_{m,h}[z],[r])$  = Cannon sum of the new  $\nabla$ -set in Hyperelliptic domain.
- 5)  $\mu(\tilde{P}[z],[\rho])$  = Cannon function of the new  $\Delta$ -set in Reinhardt domain.
- 6)  $\mu(\tilde{Q}[z],[\rho])$  = Cannon function of the new  $\nabla$ -set in Reinhardt domain.
- 7)  $\mu(\tilde{P}_{m,h}[z],[r])$  = Cannon sum of the new  $\Delta$ -set in Hyperelliptic domain.
- 8)  $\mu(\tilde{Q}_{m,h}[z],[r])$  = Cannon sum of the new  $\nabla$ -set in Hyperelliptic domain.
- 9)  $M(F(z),[r']) = \max_{\Gamma[r']}|F(z)|$ .
- 10)  $M(F(z),[\rho']) = \max_{\Gamma[\rho']}|F(z)|$ .
- 11)  $M(\tilde{P}_{m,h}[z],[r]) = \max_{\Gamma[r]}|\tilde{P}[z]| \leq N_m M(P_{m,h}[z],[r])$

$$M(\tilde{P}_{m,h}[z],[\rho]) = \max_{\Gamma[\rho]}|\tilde{P}[z]| \leq N_m M(P_{m,h}[z],[\rho])$$

$$M(\tilde{Q}_{m,h}[z],[r]) = \max_{\Gamma[r]}|\tilde{Q}[z]| \leq N_m M(Q_{m,h}[z],[r])$$

$$M(\tilde{Q}_{m,h}[z],[\rho]) = \max_{\Gamma[\rho]}|\tilde{Q}[z]| \leq N_m M(Q_{m,h}[z],[\rho])$$

where  $N_m$  is a constant.  $Q_{m,h} = Q\left(\begin{matrix} m \\ h \end{matrix}\right)$  is a coefficient corresponding to polynomials set  $\{Q_m(z)\}_{m \geq 0}$ ,

$P_{m,h} = P\left(\begin{matrix} m \\ h \end{matrix}\right)$  is a coefficient corresponding to polynomial set  $\{P_m(z)\}_{m \geq 0}$ . We should note that  $Q_{m,h} \neq P_{m,h}$

or  $Q_{m,h} = P_{m,h}$ .



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