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3D Radiative Transfer Equation Coupled with Heat Conduction Equation with Realistic Boundary Conditions Applied on Complex Geometries

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Abstract

This paper presents the solution of coupled radiative transfer equation with heat conduction equation in complex three-dimensional geometries. Due to very different time scales for both physics, the radiative problem is considered steady-state but solved at each time iteration of the transient conduction problem. The discrete ordinate method along with the decentered streamline-upwind Petrov-Galerkin method is developed. Since specular reflection is considered on borders, a very accurate algorithm has been developed for calculation of partition ratio coefficients of incident solid angles to the several reflected solid angles. The developed algorithms are tested on a paraboloid-shaped geometry used for example on concentrated solar power technologies.

Keywords

Radiative Transfer Equation, Heat Conduction Equation, Finite Element Methods, SUPG, DOM, Specular Reflection, Complex Geometry

1. Introduction

The study of the thermal and radiative heat transfer in semitransparent media plays an important role for industrial applications such as thermal insulation [1], photo-thermal therapy [2], glass forming [3] [4], porous media [5] and many others [6]. The steady thermal equation is commonly used to give a global and sometimes sufficient solution [7] [8] but, in some applications [2] [3], the knowledge of the evolution of the thermal heat transfer is necessary.

The coupling takes into account of the steady-state radiative transfer equation (RTE), as well as the transient heat conduction equation (HCE). Such a transient coupling is well derived in [9] [10]. The RTE is an integro-differential equation that contains an advection term and also an angular integral term corresponding to a gain by scattering. Deterministic and statistical methods are both popular in the radiative transfer community to solve the RTE. For the determinist methods, the most well-known angular discretization methods are the dis-

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crete ordinate methods [9] [11] [12] and the P_N methods [2] [10] [13]. The RTE being a hyperbolic equation, the finite volume methods (FVM) are widely used for such kind of equation, for the spatial discretization [14] [15]. To add more, Finite Element Methods (FEM) are useful for complex geometries. In FEM, the classical Galerkin FEM in its original version, is not suited for the RTE, due to first order differential [16]. In such case, the Streamline-Upwind Petrov-Galerkin (SUPG) stabilizes the solution by adding artificial diffusion [11] [17]. Other methods have been developed in the past [9] [10] [18]-[23]. For the statistical methods, the Monte Carlo [23] [24] and Ray Tracing [25] model the transport of photons using samples and randoms. The statistical methods are easy to handle for simple geometries and consume low memory. Moreover, to calculate an exchange between two surfaces/volumes, the methods are fast. However, when the geometries are complex, when the solution is to be found in a whole enclosure, and when the radiative properties are heterogeneous, then the statistical methods are difficult to handle and the CPU time needed to obtain an accurate solution may become extremely high.

Due to the second-order diffusion operator, the Galerkin finite element methods are efficient to solve the HCE as long as the nonlinear partfrom the blackbody radiance is properly dealt with. Lattice Boltzmann [7] and meshless methods [8] may also be good alternative to finite elements for this rather simple physics. Recently, Monte Carlo methods have been developed specifically for coupled conduction, convection and radiative transfers [26].

The paper is organized as follow. In Section 2, the physical models are presented: in one hand the radiative transfer equation along with mixed diffuse/specular boundary conditions, and, in the other hand, the transient heat conduction equation along with its specific boundary conditions. The Section 3 deals with the numerical methods which are used to solve the coupling RTE-HCE. A discrete ordinate method for angular discretization, combined with SUPG, a decentered finite element scheme for space discretization, allow the solution of the RTE. For the HCE, Euler implicit scheme combined with a Galerkin finite element method are used to solve the energy problem. Section 4 finally deals with numerical results. A three-dimensional paraboloid is considered. Numerical results are given for several refractive index coefficients yielding to model on some cases highly reflecting materials as well as, on other cases, non-reflecting boundary reflections. According to cases, the temperature evolution inside the medium of concern greatly changes.

2. Mathematical Models

Two models are presented: the radiative transfer equation and the heat transfer equation, both with their respective boundary conditions. The speed of the light being much higher than the conduction time constant, the steady solution of the radiative transfer equation is considered for all given conduction time steps.

The radiative transfer equation is written as follow:

$$\boldsymbol{s}.\nabla I\left(\boldsymbol{x},\boldsymbol{s}\right)+\beta I\left(x,s\right)=\sigma_s\int_{4\pi}\Phi\left(\boldsymbol{s},\boldsymbol{s}'\right)I\left(\boldsymbol{x},\boldsymbol{s}'\right)\mathrm{d}\boldsymbol{s}'+\kappa I_b\left(T\left(t,\boldsymbol{x}\right)\right) \tag{1}$$

where I is the radiative intensity for a monochromatic wavelength in $\mathrm{Wm^{-2}\mu m^{-1}sr^{-1}}$, solution of the radiative transfer equation which is to be solved for all direction $\boldsymbol{s}$ in the unit sphere and for all $\boldsymbol{x}$ into the open bounded domain D. σ_s is the scattering coefficient, κ is the absorption coefficient and $\beta=\sigma_s+\kappa$ is the so-called extinction coefficient in $\mathrm{m^{-1}}$, Φ is the scattering phase function and $I_b\left(T\right)$ is the given Planck function defined for a given wavelength in vacuum λ_0 [9]:

$$I_b\left(T\left(t,\boldsymbol{x}\right)\right)=C_1n^3\lambda_0^{-5}\left(e^{\frac{C_2}{\lambda_0 T}}-1\right)^{-1}\text{ in }\mathrm{Wm^{-2}\mu m^{-1}} \tag{2}$$

where $C_1=1.191\times10^{-16}\,\mathrm{Wm^2}$, $C_2=1.4388\times10^{-2}\,\mathrm{Km}$. Moreover, the behavior of the radiative intensity on borders is important to be well taken into account in order to simulate a physics close to the reality. For a smooth media, specular reflection is considered:

$$I\left(\boldsymbol{x},\boldsymbol{s}\right)=\tilde{I}\left(\boldsymbol{x},\boldsymbol{s}\right)+\rho(\boldsymbol{s}.\boldsymbol{n})I\left(\boldsymbol{x},[\boldsymbol{s}-2\left(\boldsymbol{s}.\boldsymbol{n}\right)\boldsymbol{n}]\right)\text{ on }\partial D\text{ and }\forall\boldsymbol{s}\text{ such as }\boldsymbol{s}.\boldsymbol{n}<0 \tag{3}$$

The first term in the right-hand-side of the equality sign is the Dirichlet contribution which may explain external sources for example. The other term is the gain by specular reflection from the incident direction $\boldsymbol{s}-2\left(\boldsymbol{s}.\boldsymbol{n}\right)\boldsymbol{n}$. $\rho\left(\boldsymbol{s}.\boldsymbol{n}\right)\in[0,1]$ is the reflection coefficient based on the Fresnel formulation and the Snell-Descartes law. The reflectivity coefficient depends of the scalar product $\boldsymbol{s}.\boldsymbol{n} = \cos\theta_i$ and the index of refraction$\tilde{n}$.

The blackbody emissivity $I_b(T)$ depends explicitly of the temperature $T(t,\boldsymbol{x})$ supposed unknown. The temperature is the solution of the following unsteady-conduction equation:

$$\frac{\partial T}{\partial t}(t,\boldsymbol{x}) = D_T \Delta \mathrm{T}(\mathrm{t},\mathbf{x}) - \nabla.\mathrm{q}_\mathrm{r}(t,\boldsymbol{x}) \tag{4}$$

where $D_T = \dfrac{k}{\rho c_p}$ is the thermal diffusivity coefficient, k the thermal conductivity $\left(\mathrm{WK}^{-1}\mathrm{m}^{-1}\right)$, ρ is the density $\left(\mathrm{kgm}^{-3}\right)$ and c_p is the specific heat capacity $\left(\mathrm{J\,kg}^{-1}\mathrm{K}^{-1}\right)$. The divergence of the radiative flux $\nabla.q_r(t,\boldsymbol{x}) = \kappa I_b - \kappa\int_{4\pi} I(\boldsymbol{x},\boldsymbol{s})\mathrm{d}s$ depends on the radiative intensity and the temperature. The temperature at $t=0$ is supposed known $T(0,\boldsymbol{x}) = T_0(\boldsymbol{x})$. Also, Robin boundary conditions areapplied on borders to simulate convective transfers with an external fluid at temperature T_{ext}:

$$\frac{\partial T}{\partial \boldsymbol{n}}(t,\boldsymbol{x}) = h\left(T_{ext}(\boldsymbol{x}) - T(t,\boldsymbol{x})\right) \text{ on } \partial D$$

where h is the exchange coefficient.

3. Mathematical Approximations

In the general case, the RTE and the HCE cannot be solved analytically. Some numerical tools need to be developed to get an approximation of the continuous solution. The discretization of the RTE and the HCE are respectively presented.

To cut off the integral problem into the radiative transfer equation, the unit sphere is discretized into N_d solid angles with a main direction $\boldsymbol{s_m}$. The radiative transfer equation becomes a system of N_d equations with N_d unknowns, noted $I_m(\boldsymbol{x})$ each being continuous in space. As the number of directionsis limited, the specular condition is also discretized accordingly:

$$\boldsymbol{s_m}.\nabla I_m(\boldsymbol{x}) + \beta I_m(\boldsymbol{x}) = \sigma_s \sum\nolimits_{j=1}^{N_d} \omega_j \Phi_{m,j} I_j(\boldsymbol{x}) + \kappa I_b(T), \forall m = 1,\ldots,N_d \tag{7}$$

$$I_m(\boldsymbol{x}) = \tilde{I}_m(\boldsymbol{x}) + \rho(\boldsymbol{s_m}.\boldsymbol{n}) \sum\nolimits_{\boldsymbol{s_j}.\boldsymbol{n}>0} \delta_{m,j}(\boldsymbol{n}) I_j(\boldsymbol{x}) \tag{8}$$

where ω_j corresponds to the weight associated to the direction $\boldsymbol{s_j}$, and $\delta_{m,j}(\boldsymbol{n})$ is the partition ratio coefficient representing the proportion of the radiative intensity I_j which is reflected towards the direction $\boldsymbol{s_m}$, taking into account of the weight $\rho(\boldsymbol{s_m}.\boldsymbol{n})$ according to Fresnel law.

The Galerkin finite element method being as well known unstable for the radiative transfer equation due to the advection term $\boldsymbol{s}.\nabla I_m$, the streamline-upwind Petrov-Galerkin method uses an additional term to the test function v to throw off the scheme and to get stability. To obtain the weak formulation, the m^{th} equation of the global system is multiplied by the test function $v + \gamma \boldsymbol{s_m}.\nabla v$, it is then integrated on the full domain D, and the Green theorem is finally used to express the boundary conditions. To add stability, $\gamma = 0.3 h_D$, with h_D depend of the mesh refinement of D [17]. The variational formulation reads, with $\tilde{\beta}_m = \kappa + \sigma_s\left(1 - \omega_m \Phi_{m,m}\right)$:

$$\begin{aligned}
&\int_D \left[I_m - \gamma\left(\boldsymbol{s_m}.\nabla I_m\right)\right][\tilde{\beta}_m v - \boldsymbol{s_m}.\nabla v]\mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{s_m}.\boldsymbol{n}>0} \left(1 + \tilde{\beta}_m \gamma\right) I_m v \boldsymbol{s_m}.\boldsymbol{n}\mathrm{d}\Gamma \\
&- \sum_{j \neq m} \omega_j \Phi_{m,j} \int_D \sigma_s I_j \left(v + \gamma \boldsymbol{s_m}.\nabla v\right)\mathrm{d}\boldsymbol{x} + \rho\left(\boldsymbol{s_m},\boldsymbol{n}\right) \int_{\boldsymbol{s_i}.\boldsymbol{n}>0 \cap \boldsymbol{s_m}.\boldsymbol{n}<0} \delta_{mj}(\boldsymbol{n}) I_j v \boldsymbol{n}.\boldsymbol{s_m}\mathrm{d}\Gamma \\
&= \int_{\boldsymbol{s_m}.\boldsymbol{n}>0} \tilde{I}_m v \boldsymbol{s_m}.\boldsymbol{n}\mathrm{d}\Gamma + \int_D I_b\left(v + \gamma \boldsymbol{s_m}.\nabla v\right)\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{9}$$

To cut off the temporal derivative, the first order implicit Euler scheme is used. Moreover, at a given time step, the divergence of the flux q_r is calculated at the previous time step to remove the nonlinearity due to the blackbody term $I_b(T)$. The weak formulation of the conduction problem reads:

$$\int_D \frac{T^{N+1}}{\delta t} + D_T \nabla T^{N+1}.\nabla v\mathrm{d}\boldsymbol{x} + \int_{\partial D} \frac{Dh}{k} T^{N+1} v\mathrm{d}\Gamma = \int_D \frac{T^N}{\delta t} v - \nabla.q_r^N v\mathrm{d}\boldsymbol{x} + \int_{\partial D} \frac{Dh}{k} T_{\mathrm{ext}} v\mathrm{d}\Gamma \tag{11}$$

4. Numerical Solution

The set of the varatiational Formulations (9)-(11) gives us steady-state solutions of radiative intensities along with the transient solution of the temperature in the whole domain. The geometry of concern is a paraboloid with a height of 4/3m and a diameter of 4m. The equation of the paraboloid surface is given by $z = 0.333(x^2 + y^2)$. The physical properties are the following. The absorption coefficient is $\kappa = 0.4\text{m}^{-1}$, the isotropic scattering coefficient is $\sigma_s = 0.1\text{m}^{-1}$. Next, the thermal conductivity is $k = 1\text{Wm}^{-1}\text{K}^{-1}$, the density is $\rho = 2\text{kgm}^{-3}$, the heat capacity is $c_p = 5\text{J kg}^{-1}\text{K}^{-1}$, and the convective exchange coefficient is $h = 5\text{m}^{-2}$. At $t = 0\text{s}$, the temperature is $T_0(\boldsymbol{x}) = 300\text{K}$. A collimated beam is entering to the medium on the full plan surface such as $I(\boldsymbol{x}, \boldsymbol{s}_0) = 10^{-3}\text{Wm}^{-2}\mu\text{m}^{-1}\text{sr}^{-1}$. Solutions are presented below for three values of refractive index. The first case, with n = 1, considers the border is transparent, there is no reflection. Another case, with n = 1.8, considers that the reflected part is very important. The last case stands in between, with n = 1.4.

Figure 1 presents the evolutions of the radiative intensity and of temperature at t = 0.5 s along the longitudinal axis, and **Figure 2** presents the same data in cross-sections. It can be observed that the maximum radiative intensity increases with the refraction index. Hence worth, the temperature inside the medium also greatly in-

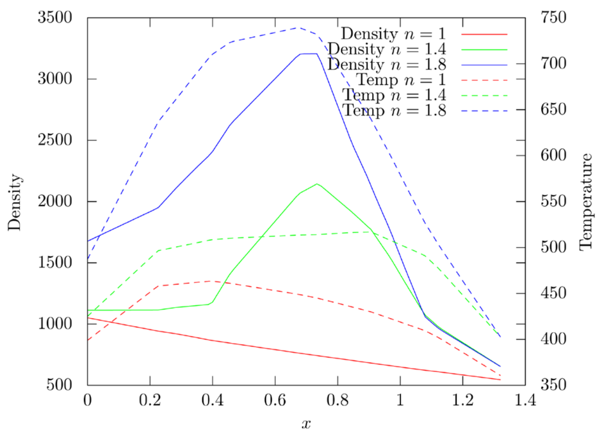

Figure 1. Evolution of the radiative intensity and of temperature à t = 0.5 s along the longitudinal axis.

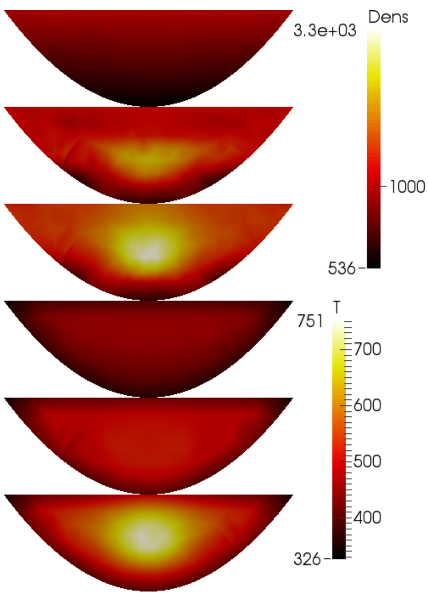

Figure 2. Top: radiative intensity; bottom: temperature at t = 0.5 s. For each, the first is for n = 1, the second is for n = 1.4, the third is for n = 1.8.

creases with the refraction index. As an example an increase of the index factor from 1 to 1.4 increases the maximum temperature difference from 164 to 220 K. In the same manner, an increase of the index factor from 1.4 to 1.8 increases the maximum temperature difference from 220 to 551 K. This confirms that the design of materials for such systems is highly important.

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