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Moment Identities for Skorohod Integrals on Guichardet-Fock Spaces

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Abstract

In this paper, we define expectation of $f \in F$, *i.e.* $E(f) = f(\varnothing)$, accordingto Wiener-Ito-Segal isomorphic relation between Guichardet-Fock space *F* and Wienerspace *W*. Meanwhile, we prove a moment identity for the Skorohod integrals aboutvacuum state.

Keywords

Moment Identities, Skorohod Integral, Guichardet-Fock Spaces

1. Introduction

The quantum stochastic calculus [1] [2] developed by Hudson and Parthasarathy is essentially a noncommutative extension of classical Ito stochastic calculus. In this theory, annihilation, creation, and number operator processes in boson Fock space play the role of "quantum noises", [3] which are in continuous time. In 2002, Attal [4] discussed and extended quantum stochastic calculus by means of the Skorohod integral of anticipation processes and the related gradient operator on Guichardet-Fock spaces. Usually, Fock spaces as the models of the Particle Systems are widely used in quantumphysics. Meanwhile, vacuum states described by empty set on Guichardet-Fockspaces play very important role at quantum physics.

Recently Privault [5] [6] developed a Malliavin-type theory of stochastic calculus on Wiener spaces and showed its several interesting applications. In his article, Privault surveyed the moment identities for Skorohod integral on Wiener spaces. It is well known that Guichardet-Fock space F and Wiener space W are Wiener-Ito-Segal isomorphic. Motivated by the above, we would like to study the momentidentities for Skorohod integraon Guichardet-Fock spaces.

This paper is organized as follows. Section 2, we fix some necessarynotations and recall main notions and facts about Skorohod integralin Guichardet-Fock spaces. Section 3 states our main results.

2. Notations

In this section, we fix some necessary notations and recall mainnotions in Guichardet-Fock spaces. For detail formulation of Skorohod integrals, we refer reader to [4].

Let R_+ be the set of all nonnegative real numbers and Γ the finite power set of R_+, namely

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$$\Gamma := \{\sigma \mid \sigma \subset R_+, \#\sigma < \infty\},$$

where $\#\sigma$ denotes the cardinality of σ as a set. Particularly, let $\varnothing \in \Gamma^{(0)}$ be an atom of measure 1. We denote by $L^2(\Gamma)$ the usual space of square integral real-valued functions on Γ.

Fixing a complex separable Hilbert space η, Guichardet-Fock space tensor product $\eta \otimes L^2(\Gamma)$, which we identify with the space of square-integrable functions $L^2(\Gamma;\eta)$, and is denoted by *F*.

For a Hilbert space-valued map $x:\Gamma \times R_+ \to \eta$, let

$$\delta(x):\sigma \mapsto \sum_{s\in\sigma} x_s(\sigma \setminus s)$$

denotes the Skorohod integral operator. For a vector space-valued map $f:\Gamma \to V$, let ∇f and Df be the maps $\Gamma \times R_+ \to V$ given by

$$\nabla f(\omega,s) = f(\omega \bigcup s),\ Df(\omega,s) = \mathbf{1}_{\{\omega<s\}} f(\omega \bigcup s)$$

respectively denote the stochastic gradient operator of *f* and the adapted gradient operator of *f*. Moreover, we write $\mathrm{Dom}\nabla$ for the domain of the stochastic gradient as an unbounded Hilbert apace operator:

$$\mathrm{Dom}\nabla := \{ f \in F : \nabla f \in L^2(\Gamma \times R_+;\eta)\}.$$

Definition 2.1 For the map $x:\Gamma \times R_+ \to \eta$, the value of Skorohod integral $\delta(x)$ at empty set is called the expectation of $\delta(x)$ on Guichardet-Fock space and is denoted by $E(\delta(x))$ *i.e.* $E(\delta(x)) = \delta(x)(\varnothing)$.

Lemma 2.1 Let *x* be a map $\Gamma \times R_+ \to \eta$, if *x* is square integrable and the function $(\omega,s,t) \to \langle x_s(\omega \bigcup t), x_t(\omega \bigcup s)\rangle$ is integrable, then $x \in \mathrm{Dom}\delta$ and

$$\| \delta(x)\|^2 = \int \| x\|^2 ds + \int\int\int \langle x_s(\omega \bigcup t), x_t(\omega \bigcup s)\rangle d\omega dt ds, \tag{2.1}$$

we denote

$$\begin{aligned}\mathrm{trace}(Dx)^2 &= \langle \nabla x, \nabla^* x\rangle \\ &= \int_0^\infty \int_0^\infty \langle \nabla_t x_s, \nabla_s x_t\rangle dt ds \\ &= \int_0^\infty \int_0^\infty \langle x_s(\omega \cup t), x_t(\omega \cup s)\rangle dt ds.\end{aligned}$$

Lemma 2.2 Let $f \in F$ and let $x:\Gamma \times R_+ \to \eta$ be Skorohod integrable, if the map

$$(\omega,s) \mapsto \langle x_s(\omega), f(\omega \cup s)\rangle$$

is integrable, then

$$\langle \delta(x), f\rangle = \int\int \langle x_\omega, \nabla_s f(\omega)\rangle d\omega ds. \tag{2.2}$$

Lemma 2.3 Let $x:\Gamma \times R_+ \to \eta$ be measurable. For $a.a.t$, we have

$$D_t\delta(x) = \delta_0^t(D_t x_\cdot) + P_t x_t, \tag{2.3}$$

where $P_t x_t = \mathbf{1}_{\Gamma_t} x_t$, $\Gamma_t := \{\omega \in \Gamma : \omega \subset [0,t[\}$.

Proof In view of the identity

$$\mathbf{1}_{\{\sigma<t\}}\delta(x)(\sigma \cup t) = \sum_{s\in\sigma} \mathbf{1}_{\{\sigma<t\}} \mathbf{1}_{[0,t[}(s) x_s((\sigma/s)\cup t) + \mathbf{1}_{\{\sigma<t\}} x_t(\sigma),$$

we have

$$D_t\delta(x)(\sigma) = \delta(\mathbf{1}_{[0,t[}(\cdot) D_t x_\cdot)(\sigma) + P_t x_t(\sigma).$$

3. Moment Identities for Skorohod Integrals

Theorem 3.1 For any $n \ge 1$ and $x \in F$, we have

$$E(\delta(x)^{n+1}) = \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\delta(x)^{n-k} (\langle (\nabla x)^{k-1} x, x\rangle + \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (\nabla x)^{k-i} x, \nabla \mathrm{trace}(\nabla x)^i\rangle)], \tag{3.1}$$

where

$$\mathrm{trace}(\nabla x)^{k+1} = \int_0^\infty \cdots \int_0^\infty \langle \nabla^*_{t_{k-1}} x_{t_k}, \nabla_{t_{k-2}} x_{t_{k-1}} \cdots \nabla_{t_0} x_{t_1} \nabla_{t_k} x_{t_0} \rangle dt_0 \cdots dt_k .$$

For $n = 1$ the above identity coincides with (2.1).
We will need the following lemma.
Lemma 3.1 Let $n \geq 1$ and $x \in F$. Then for all $1 \leq k \leq n$ we have

$$E(\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla \delta(x) \rangle)(\varnothing) - (n-k)(\delta(x)^{n-k-1} \langle (\nabla x)^k x, \nabla \delta(x) \rangle)$$
$$= E[\delta(x)^{n-k} (\langle (\nabla x)^{k-1} x, x \rangle + \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (\nabla x)^{k-i} x, \nabla \mathrm{trace}(\nabla x)^i \rangle)].$$

Proof Using relation (2.2), (2.3), we obtain

$$\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla \delta(x) \rangle = \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, x + \delta(\nabla^* x) \rangle$$
$$= \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, x \rangle) + (\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \delta(\nabla^* x) \rangle$$
$$= \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, x \rangle) + (\langle (\nabla^* x), \nabla(\delta(x)^{n-k} (\nabla x)^{k-1} x) \rangle$$
$$= \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, x \rangle) + (\delta(x)^{n-k} \langle \nabla^* x, \nabla((\nabla x)^{k-1} x) \rangle + \langle \nabla^* x, ((\nabla x)^{k-1} x) \otimes \nabla(\delta(x)^{n-k}) \rangle$$
$$= \delta(x)^{n-k} (\langle (\nabla x)^{k-1} x, x \rangle + \langle \nabla^* x, \nabla((\nabla x)^{k-1} x) \rangle) + (n-k) \delta(x)^{n-k-1} \langle \nabla^* x, ((\nabla x)^{k-1} x) \otimes \nabla \delta(x) \rangle$$
$$= \delta(x)^{n-k} (\langle (\nabla x)^{k-1} x, x \rangle + \langle \nabla^* x, \nabla((\nabla x)^{k-1} x) \rangle) + (n-k) \delta(x)^{n-k-1} \langle (\nabla x)^k x, \nabla \delta(x) \rangle,$$

and

$$\langle \nabla^* x, \nabla((\nabla x)^{k-1} x) \rangle = \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_{k-1}} x_{t_k}, \nabla_{t_k} (\nabla_{t_{k-2}} x_{t_{k-1}} \cdots \nabla_{t_0} x_{t_1} x_{t_0}) \rangle dt_0 \cdots dt_k$$
$$= \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_{k-1}} x_{t_k}, \nabla_{t_{k-2}} x_{t_{k-1}} \cdots \nabla_{t_0} x_{t_1} \nabla_{t_k} x_{t_0} \rangle dt_0 \cdots dt_k$$
$$= \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_{k-1}} x_{t_k}, \nabla_{t_{k-2}} x_{t_{k-1}} \cdots \nabla_{t_0} x_{t_1} \nabla_{t_k} x_{t_0} \rangle dt_0 \cdots dt_k$$
$$= \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_{k-1}} x_{t_k}, \nabla_{t_k} x_{t_{k+1}} \cdots \nabla_{t_{i+1}} x_{t_{i+2}} (\nabla_{t_i} \nabla_{tk} x_{t_{i+1}}) \nabla_{t_{i-1}} x_{t_i} \cdots \nabla_{t_0} x_{t_1} x_{t_0} \rangle dt_0 \cdots dt_k$$
$$= \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \int_0^\infty \cdots \int_0^\infty \langle \nabla_{t_i} \langle \nabla^\dagger_{t_{k-1}} x_{t_k}, \nabla_{t_k} x_{t_{k+1}} \cdots \nabla_{t_{i+1}} x_{t_{i+2}} \nabla_{t_k} x_{t_{i+1}} \rangle, \nabla_{t_{i-1}} x_{t_i} \cdots \nabla_{t_0} x_{t_1} x_{t_0} \rangle dt_0 \cdots dt_k$$
$$= \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \langle (\nabla x)^i x, \nabla \mathrm{trace}(\nabla x)^{k-i} \rangle,$$

Proof of Theorem 3.1, We decompose

$$E(\delta(x)^{n+1}) = E(\langle x, \nabla(\delta(x))^n \rangle) = E(n(\delta(x))^{n-1} \langle x, \nabla \delta(x) \rangle)$$
$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\delta(x)^{n-k} (\langle (\nabla x)^{k-1} x, \nabla \delta(x) \rangle - (n-k)(\delta(x))^{n-k-1} \langle (\nabla x)^{k-1} x, \nabla \delta(x) \rangle],$$

then we apply lemma 3.1, which yields (3.1).

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References

[1] Hudson, R.L. and Parthasarathy, K.R. (1984) Quantum Ito's Formula and Stochastic Evolutions. *Communications in*

Mathematical Physics, **3**, 301-323. http://dx.doi.org/10.1007/BF01258530

[2] Meyer, P.A. (1993) Quantum Probability for Probabilists. Lecture Notes in Mathematics, Spring-Verlag, Berlin. http://dx.doi.org/10.1007/978-3-662-21558-6

[3] Wang, C.S., Lu, Y.C. and Chai, H.F. (2011) An Alternative Approach to Privault's Discrete-Time Chaotic Calculus. *Journal of Mathematical Analysis and Applications*, **2**, 643-654. http://dx.doi.org/10.1016/j.jmaa.2010.08.021

[4] Attal, S. and Lindsay, J.M. (2004) Quantum Stochastic Calculus with Maximal Operator Domains. *The Annals of Probability*, **32**, 488-529. http://dx.doi.org/10.1214/aop/1078415843

[5] Privault, N. (2009) Moment Identities for Skorohod Integrals on the Wiener Space and Applications. *Electronic Communications in Probability*, **14**, 116-121. http://dx.doi.org/10.1214/ECP.v14-1450

[6] Privault, N. (2010) Random Hermite Polynomials and Girsanov Identities on the Wiener Space. *Infinite Dimensional Analysis*, **13**, 663-675. http://dx.doi.org/10.1142/S0219025710004218