

Existence of Positive Periodic Solutions for a Time-Delay Biological Model

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Abstract

Based on the classic Lotka-Volterra cooperation model, we establish a time-delay model of which a species cannot survive independently. By continuation theorem, we discuss existence of positive periodic solutions of the model.

Keywords

Biological Model, Existence, Periodic Solution, Time Delay, Independent Survival

1. Introduction

The plants can survive independently and insect pollination can improve the growth rate of plants in [1] and [2]. According to this phenomenon, based on the classical Lotka-Volterra model, we establish a model of two populations of Lotka-Volterra which cannot survive independently, finally he analyzes the stability of the model.

There is still less research work of the model which cannot exist independently. The existing researches basically are the autonomous models (see [3] and [4]). In this paper, we establish a Lotka-Volterra model with time delay which a species cannot survive independently. The main aim is to discuss existence of periodic positive solution for the model.

Suppose that there are two plant populations (A and B) living in their natural environment, which are free from other interference factor. Let $y_1(t)$ and $y_2(t)$ are the population density of plant A and plant B, a_1, a_2, b_1, b_2 are continuous functions with periodic $\omega > 0$, and $\int_0^\omega r_i(t)dt > 0 (i = 1, 2)$. The constants a_1, a_2, b_1, b_2 are stimulations of living environment. By the thought of [1]-[4], we could have got the following Lotka-Volterra model with time delay which a species cannot survive independently.

$$\begin{cases} \frac{dy_1(t)}{dt} = y_1(t)[-r_1(t) - a_1(t)y_1(t - \tau(t)) + b_1(t)y_2(t - \sigma(t))], \\ \frac{dy_2(t)}{dt} = y_2(t)[r_2(t) + a_2(t)y_1(t - \tau(t)) - b_2(t)y_2(t - \sigma(t))]. \end{cases} \quad (1)$$

The main aim of the paper is to discuss existence of periodic positive solution for the model.

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2. Lemma 1 and Lemma 2

Assume X and Z are normed vector space, $L: DomL \subset X \rightarrow Z$ and $N: X \rightarrow Z$ are linear mappings. If L is Fredholm mapping which Zero is index, and there are continuous projection $P: X \rightarrow X$ and $Q: Z \rightarrow Z$, such that $Im P = KerL$ and $Im KerQ = Im(I - Q)$, we can get that $L|_{DomL \cap KerP}: (I - P)X \rightarrow Im L$ is reversible. If Inverse mapping $K_p(I - Q)N: \bar{\nabla} \rightarrow X$ is tight, we call N is tight on $\bar{\nabla}$.

Lemma 1 (Continuation theorem) [5] Let L be the mapping of Fredholm with zero index, collection N is tight on collection $\bar{\nabla}$. Suppose the following: for any $\lambda \in (0,1)$, the solution $x \notin \partial \bar{\nabla}$ of equation $Lx = \lambda Nx$; for any $x \in \partial \bar{\nabla} \cap KerL, QNx \neq 0$ and $deg\{JQN, \bar{\nabla} \cap KerL, 0\} \neq 0$. Then, there is at least a solution for $Lx = Nx$ on $DomL \cap \bar{\nabla}$.

Lemma 2 $R_+^2 = \{(x, y) | x > 0, y > 0\}$ is positive invariant set of model (1).

Proof. By formula (2), we have

$$\begin{cases} y_1(s) = y_1(0) \exp\left\{\int_0^s [-r_1(t) - a_1(t)y_1(t - \tau(t)) + b_1(t)y_2(t - \sigma(t))]dt\right\}, \\ y_2(s) = y_2(0) \exp\left\{\int_0^s [-r_2(t) - a_2(t)y_1(t - \tau(t)) + b_1(t)y_2(t - \sigma(t))]dt\right\}. \end{cases} \tag{2}$$

Since formula (2) is always true for $t \geq 0$, lemma 2 is proved.

For the convenience of discuss, we give following notations.

$$\bar{g} = \frac{1}{\omega} \int_0^\omega g(t)dt, \quad g^M = \max_{t \in [0, \omega]} g(t), \quad g^m = \min_{t \in [0, \omega]} g(t)$$

$$a_1^m = \min_{0 \leq t \leq \omega} \{a_1(t)\}, \quad a_2^M = \max_{0 \leq t \leq \omega} \{a_2(t)\}, \quad b_1^M = \max_{0 \leq t \leq \omega} \{b_1(t)\}, \quad b_2^m = \min_{0 \leq t \leq \omega} \{b_2(t)\}.$$

3. Existence of Periodic Solutions

In order to apply Continuation theorem to system (3), we define

$$X = Z = \{x(t) = (x_1(t), x_2(t))^T \in C(R, R^2), x(t + \omega) = x(t)\},$$

and

$$\|x\| = \|(x_1(t), x_2(t))^T\| = \max_{t \in [0, \omega]} |x_1(t)| + \max_{t \in [0, \omega]} |x_2(t)|, \quad \|x\| = \|(x_1(t), x_2(t))^T\| = \max_{t \in [0, \omega]} |x_1(t)| + \max_{t \in [0, \omega]} |x_2(t)|,$$

then X, Z is Banach space under the norm $\|\bullet\|$ (see [6] and [7]).

Let $y_1(t) = \exp\{x_1(t)\}, y_2(t) = \exp\{x_2(t)\}$, the Equation (1) can be turns into

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -r_1(t) - a_1(t) \exp\{x_1(t - \tau(t))\} + b_1(t) \exp\{x_2(t - \sigma(t))\}, \\ \frac{dx_2(t)}{dt} &= r_2(t) + a_1(t) \exp\{x_1(t - \tau(t))\} - b_1(t) \exp\{x_2(t - \sigma(t))\}. \end{aligned} \tag{3}$$

Since $u(t) = (x_1(t), x_2(t))$ is periodic, we know that

$$\Phi_1(u, t) = -r_1(t) - a_1(t) \exp\{x_1(t - \tau(t))\} + b_1(t) \exp\{x_2(t - \sigma(t))\}$$

and

$$\Phi_2(u, t) = r_2(t) + a_1(t) \exp\{x_1(t - \tau(t))\} - b_1(t) \exp\{x_2(t - \sigma(t))\}$$

are continuous function with the periodicity ω .

$$\text{Let } Nx = \begin{pmatrix} \Phi_1(u, t) \\ \Phi_2(u, t) \end{pmatrix}, \quad Lx = \dot{x} = \frac{dx(t)}{dt}, \quad Px = \frac{1}{\omega} \int_0^\omega x(t)dt, \quad \text{for } x \in X, \quad Qz = \frac{1}{\omega} \int_0^\omega z(t)dt, \quad \text{for } z \in Z, z \in Z,$$

then $KerL = R^2, Im L = \{z \in Z : \int_0^\omega z(t)dt = 0\}$ is closed set in set Z ,

$\dim KerL = 2 = \text{codim } Im L$, and P, QP and Q is the continuous projection, such that $Im P = KerL, KerQ = Im L = Im(I - Q)$. Thus there is the inverse mapping of $L, Kp: Im L \rightarrow KerP \cap DomL$ and

$$Kp(z) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s)dsdt.$$

So that we get

$$QNx = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega \Phi_1(u, s) ds \\ \frac{1}{\omega} \int_0^\omega \Phi_2(u, s) ds \end{pmatrix},$$

$$Kp(I - Q)Nx = \begin{pmatrix} \int_0^t \Phi_1(u, s) ds \\ \int_0^t \Phi_2(u, s) ds \end{pmatrix} - \begin{pmatrix} \frac{1}{\omega} \int_0^\omega \int_0^t \Phi_1(u, s) ds dt \\ \frac{1}{\omega} \int_0^\omega \int_0^t \Phi_2(u, s) ds dt \end{pmatrix} - \begin{pmatrix} (\frac{t}{\omega} - 0.5) \int_0^\omega \Phi_1(u, s) ds \\ (\frac{t}{\omega} - 0.5) \int_0^\omega \Phi_2(u, s) ds \end{pmatrix}.$$

It is obvious that QN and $Kp(I - Q)N$ is continuous.

We assume that $\bar{\nabla}$ is bounded open set. It is obvious that $QN(\bar{\nabla})$ is bounded. We have that $\overline{Kp(I - Q)N(\bar{\nabla})}$ is compact set by Arzela-Ascoli theorem, so we get N is L -tight on $\bar{\nabla}$.

The corresponding operator equation is $Lx = \lambda Nx$ with $\lambda \in (0, 1)$, we have the following formula

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \lambda \{-r_1(t) - a_1(t) \exp\{x_1(t - \tau(t))\} + b_1(t) \exp\{x_2(t - \sigma(t))\}\}, \\ \frac{dx_2(t)}{dt} &= \lambda \{r_2(t) + a_2(t) \exp\{x_1(t - \tau(t))\} - b_2(t) \exp\{x_2(t - \sigma(t))\}\}. \end{aligned} \tag{4}$$

We assume that $x = x(t) \in X$ is the solution for system (4) with $\lambda \in (0, 1)$, by integral we get the following formula (5)

$$\begin{aligned} \int_0^\omega -r_1(t) - a_1(t) \exp\{x_1(t - \tau(t))\} + b_1(t) \exp\{x_2(t - \sigma(t))\} dt &= 0, \\ \int_0^\omega r_2(t) + a_2(t) \exp\{x_1(t - \tau(t))\} - b_2(t) \exp\{x_2(t - \sigma(t))\} dt &= 0. \end{aligned} \tag{5}$$

To move term from one side of an algebraic equation to the other side, reversing its sign to maintain equality, we get the following

$$\int_0^\omega -r_1(t) + b_1(t) \exp\{x_2(t - \sigma(t))\} dt = \int_0^\omega a_1(t) \exp\{x_1(t - \tau(t))\} dt. \tag{6}$$

$$\int_0^\omega r_2(t) + a_2(t) \exp\{x_1(t - \tau(t))\} dt = \int_0^\omega b_2(t) \exp\{x_2(t - \sigma(t))\} dt. \tag{7}$$

From formula (5), formula (6) and formula (7), we have

$$\begin{aligned} \int_0^\omega \left| \dot{x}_1(t) \right| dt &= \lambda \int_0^\omega -r_1(t) - a_1(t) \exp\{x_1(t - \tau(t))\} + b_1(t) \exp\{x_2(t - \sigma(t))\} dt \\ &\leq \int_0^\omega -r_1(t) + b_1(t) \exp\{x_2(t - \sigma(t))\} dt + \int_0^\omega a_1(t) \exp\{x_1(t - \tau(t))\} dt \\ &\leq 2 \int_0^\omega a_1(t) \exp\{x_1(t - \tau(t))\} dt. \end{aligned} \tag{8}$$

and

$$\begin{aligned} \int_0^\omega \left| \dot{x}_2(t) \right| dt &= \lambda \int_0^\omega r_2(t) + a_2(t) \exp\{x_1(t - \tau(t))\} - b_2(t) \exp\{x_2(t - \sigma(t))\} dt \\ &\leq \int_0^\omega r_2(t) + a_2(t) \exp\{x_1(t - \tau(t))\} dt + \int_0^\omega b_2(t) \exp\{x_2(t - \sigma(t))\} dt \\ &\leq 2 \int_0^\omega b_2(t) \exp\{x_2(t - \sigma(t))\} dt. \end{aligned} \tag{9}$$

From formula (8) and formula (9) we can get

$$b_1^m \int_0^\omega \exp\{x_1(t - \tau(t))\} dt \leq -\bar{r}_1 \omega + b_1^M \int_0^\omega \exp\{x_2(t - \sigma(t))\} dt. \tag{10}$$

$$b_2^m \int_0^\omega \exp\{x_2(t - \sigma(t))\} dt \leq \bar{r}_2 \omega + a_2^M \int_0^\omega \exp\{x_1(t - \tau(t))\} dt. \tag{11}$$

From formula (10) and formula (11) we get

$$\begin{aligned} b_2^m \int_0^\omega \exp\{x_2(t - \sigma(t))\} dt &\leq \bar{r}_2 \omega + a_2^M \int_0^\omega \exp\{x_1(t - \tau(t))\} dt \\ &\leq \bar{r}_2 \omega + \frac{a_2^M}{a_1^m} (-\bar{r}_1 \omega + b_1^M \int_0^\omega \exp\{x_2(t - \sigma(t))\} dt). \end{aligned}$$

So that

$$\int_0^\omega \exp\{x_2(t - \sigma(t))\} dt \leq \frac{(\bar{r}_2 a_1^m - \bar{r}_1 a_2^M) \omega}{a_1^m b_2^m + a_2^M b_1^M} \leq \frac{(\bar{r}_2 a_1^m + \bar{r}_1 a_2^M) \omega}{a_1^m b_2^m + a_2^M b_1^M}.$$

Similarly, we have

$$\int_0^\omega \exp\{x_1(t - \tau(t))\} dt \leq \frac{(\bar{r}_1 b_2^m - \bar{r}_1 b_2^M) \omega}{a_1^m b_2^m - a_2^M b_1^M} \leq \frac{(\bar{r}_1 b_2^m + \bar{r}_1 b_2^M) \omega}{a_1^m b_2^m - a_2^M b_1^M}.$$

Using formula (7) we get

$$b_2^M \int_0^\omega \exp\{x_2(t - \sigma(t))\} dt \geq \bar{r}_2 \omega.$$

Thus

$$\frac{r_2 \omega}{b_1^M} \leq \int_0^\omega \exp\{x_2(t - \sigma(t))\} dt \leq \frac{(\bar{r}_2 a_1^m + \bar{r}_1 a_2^M) \omega}{a_1^m b_2^m + a_2^M b_1^M}. \tag{12}$$

Similarly, we have

$$-\frac{r_1 \omega}{a_1^M} \leq \int_0^\omega \exp\{x_1(t - \tau(t))\} dt \leq \frac{(\bar{r}_1 b_2^m + \bar{r}_1 b_2^M) \omega}{a_1^m b_2^m - a_2^M b_1^M}. \tag{13}$$

From formula (8) and formula (13) we get

$$\int_0^\omega \left| \dot{x}_1(t) \right| dt \leq \frac{2a_1^M (\bar{r}_1 b_2^m + \bar{r}_1 b_2^M) \omega}{a_1^m b_2^m - a_2^M b_1^M} \stackrel{def}{=} W_1.$$

From formula (9) and formula (12) we get

$$\int_0^\omega \left| \dot{x}_2(t) \right| dt \leq \frac{2b_2^M (\bar{r}_2 a_1^m + \bar{r}_1 a_2^M) \omega}{a_1^m b_2^m + a_2^M b_1^M} \stackrel{def}{=} W_2.$$

Since $(x_1(t), x_2(t))^T \in X$, it exists $\theta_i, \varphi_i \in [0, \omega]$, such that

$$x_i(\theta_i) = \min_{t \in [0, \omega]} x_i(t), x_i(\varphi_i) = \max_{t \in [0, \omega]} x_i(t), i = 1, 2. \tag{14}$$

By formula (12), formula (13) and formula (14), we get

$$\begin{aligned} x_1(\theta_1) &\leq \ln \left[\frac{(\bar{r}_1 b_2^m + \bar{r}_1 b_2^M) \omega}{a_1^m b_2^m - a_2^M b_1^M} \right] \stackrel{def}{=} V_{11}, \quad x_1(\varphi_1) \geq \ln \left(-\frac{r_1 \omega}{a_1^M} \right) \stackrel{def}{=} V_{12} \\ x_2(\theta_2) &\leq \ln \left[\frac{(\bar{r}_2 a_1^m + \bar{r}_1 a_2^M) \omega}{a_1^m b_2^m + a_2^M b_1^M} \right] \stackrel{def}{=} V_{21}, \quad x_2(\varphi_2) \geq \ln \left(\frac{r_2 \omega}{b_2^M} \right) \stackrel{def}{=} V_{22}. \end{aligned}$$

So that

$$x_i(t) \leq x_i(\theta_i) + \int_0^\omega \left| \dot{x}_i(t) \right| dt \leq V_{i1} + W_i, \quad x_i(t) \geq x_i(\varphi_i) - \int_0^\omega \left| \dot{x}_i(t) \right| dt \leq V_{i2} - W_i.$$

It is obvious that $V_{ij}, W_i (i, j = 1, 2)$ has nothing with choose of λ . Thus the following formula (15) has a unique positive solution $(z_1^*, z_2^*) \in \text{int } R_+^2$.

$$\begin{cases} -\bar{r}_1 - \bar{a}_1 z_1 + \bar{b}_1 z_2 = 0, \\ \bar{r}_2 + \bar{a}_2 z_1 - \bar{b}_2 z_2 = 0. \end{cases} \tag{15}$$

Let $V = \max\{|V_{i1} + C_i| + V_1, |V_{i2} + C_i| + V_1, i = 1, 2\}$, where $V_1 > 0$ is sufficiently large and $\|(\ln z_1^*, \ln z_2^*)^T\| = |\ln z_1^*| + |\ln z_2^*| < V_1$. Then, $\nabla = \{x(t) = (x_1(t), x_2(t))^T \in X : \|x\| < V\}$, then ∇ satisfying the first condition for Lemma 1. When $x \in \partial\nabla \cap \text{Ker}L = \partial\nabla \cap R^2$,

$$QNx = \begin{pmatrix} -\bar{r}_1 - \bar{a}_1 z_1 + \bar{b}_1 z_2 = 0 \\ \bar{r}_2 + \bar{a}_2 z_1 - \bar{b}_2 z_2 = 0 \end{pmatrix} \neq 0.$$

So that $\text{deg}\{JQN, \nabla \cap \text{Ker}L, 0\} \neq 0$, ∇ satisfying all the conditions for Lemma 1. We have that there is at least a solution with $Lx = Nx$ on $\text{Dom}L \cap \bar{\nabla}$ from Lemma 1.

Let $y_1^*(t) = \exp\{x_1^*(t)\}$, $y_2^*(t) = \exp\{x_2^*(t)\}$, when $a_1^m b_2^m > a_1^M b_2^M$, $(y_1^*(t), y_2^*(t))^T$ is positive periodic solution for system (1) which the length of ω . Hence, there is at least a positive periodic solution for system (1).

Theorem If $a_1^m b_2^m > a_1^M b_2^M$, then $(y_1^*(t), y_2^*(t))^T$ is a positive periodic solution for system (1). In other word, there is at least one positive periodic solution for system (1).

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