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# The Riesz Decomposition of Set-Valued Superpramart

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## Abstract

**The paper proves the convergence theorem of set-valued Superpramart in the sense of weak convergence under the $X^*$ separable condition. Using support function and results about real-valued Superpramart, we give the Riesz decomposition of set-valued Superpramart.**

## Keywords

**Set-Valued, Superpramart, Weak Convergence, Riesz Decomposition**

## 1. Introduction

Reference [1] gives Riesz decomposition of set-valued supermartingale in real space and promotes the results to reflexive Banach spaces (reference [2]). Reference [2] gives the counter-example that not all of the set-valued martingale has Riesz decomposition in a two-dimensional plane case. The fundamental reason is the defects of algebraic operation on hyperspace. Therefore, the research can pursue the unstrict sense of Riesz decomposition instead of studying various sense of Riesz decomposition. Reference [2] shows the other Riesz decomposition of set-valued supermartingale in real space. Reference [3] gives Riesz decomposition of set-valued supermartingale in the general Banach space under the $X^*$ separable condition. References [4] and [5] research Riesz decomposition of set-valued submartingale in the general Banach space. Reference [6] studies Riesz decomposition in weak set-valued Amart. Reference [7]-[9] gives every sense of Riesz decomposition of set-valued Pramart in the general Banach space under the $X^*$ separable condition. Reference [10] studies the problems of Riesz decomposition of set-valued Pramart. All of the above studies have given the necessary and sufficient conditions for Riesz decomposition. The research of every sense of Riesz decomposition of set-valued Superpramart is still rare.

The paper firstly demonstrates convergence theorem that set-valued Superpramart is in the sense of weak convergence under the $X^*$ separable condition. On this basis, using support function and results about real-valued Superpramart, we give a class of Riesz decomposition of set-valued Superpramart.

## 2. Method

Assume $(X,\|\cdot\|)$ as a separable Banach space, $D_1$ is X-fan subset of the columns that can be condensed. $X^*$ is the dual space $X$. $X^*$ is separable. $D^* = \{x_k^* \in X^*, k \geq 1\}$ is $X^*$-fan subset of the columns that can be condensed, remember

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$$P_{f(c)}(X)=\{A\subset X: AisNon-empty\ closed\ (convex)\ subset\}$$

Any $A,B\in P_f(X)$, define

$$\|A\|=\sup\{\|x\|: x\in A\}$$

$$S(x^*,A)=\sup\{<x^*,x>: x\in A\}, x^*\in X^*$$

$$d(x,A)=\inf\{|x-y|: y\in A\}, x\in X$$

$$h^+(A,B)=\sup\{d(x,B), x\in A\}\quad h^-(A,B)=h^+(B,A)$$

$$h\ (A,B)=\max\{\ h^-(A,B), h^+(B,A)\}$$

$\{A_n, n\ge 1, A\}\subset P_f(X)$ If for any $x^*\in X^*, \lim_n S(x^*,A_n)=S(x^*,A\ )$, we call $A_n$ weak convergence in *A*, denote as

$$A_n\xrightarrow{w}A\text{, or }(\mathrm{w})\lim_{\mathrm{n}}\mathrm{A}_n=A.$$

$$w-\overline{\lim_n}A_n=\{x\in X:\exists\ x_k\in A_{n_k}, k\ge 1, k\to+\infty,\quad x_k\xrightarrow{w}x\}$$

$$s-\underline{\lim_n}A_n=\{x\in X:\lim_n d(x,A_n)=0\}$$

If $s-\underline{\lim_n}A_n=w-\overline{\lim_n}A_n=A$, then we call $\{A_n, n\ge 1\}$ Kuratowski-Mosco significance of convergence in A, denote as (K-M) $\lim_n A_n=A$, or $A_n\xrightarrow{K-M}A$.

Assume (Ω,$\mathcal{G}$,P) is a complete probability space. $\{\mathcal{G}_n, n\ge 1\}$ is the $\mathcal{G}$'s rise $\sigma$, and $\mathcal{G}=\vee\mathcal{G}_n$, $\overline{T}(T)$ indicates stop (bounded stopping), $T(\sigma)=\{\tau\in T;\tau\ge\sigma\}$. $\sigma\in T$, said the set value is mapped $\mathrm{F}:\Omega\to\mathrm{P_f}(\mathrm{X})$ to a random set (or on G measurable). If for any open set *G*, it has $F^-(G)=\{\omega: F(\omega)\cap G\ne\Phi\}\in\mathcal{G}$. We define $\{F_n, \mathcal{G}_n, \mathrm{n}\ge 1\}$ as adapted random set columns. If $\forall n\ge 1$, $F_n$ can be measured by $\mathcal{G}_n$. If $\int_\Omega\|F\|dP<\infty$, F is bounded integrable. $L^1_{fc}[\Omega,X]$ represents that the value of $P_{fc}(X)$ is all the integrable bounded random set.

In order to write simply, often eliminating the almost certainly established under the meaning of the equations, inequalities and tag contain relations sense "a.s.", $\{F_n, \mathcal{G}_n, n\ge 1\}$. $\{x_n, \mathcal{G}_n, n\ge 1\}$ is often referred as $\{F_n, n\ge 1\}$, $\{x_n, n\ge 1\}$.

**Definition 1** Supposing $\{x_n, n\ge 1\}$ is a real-valued integrable adapted column

1) If $\forall\varepsilon>0$, $\lim_{\sigma\in T}\sup_{\tau\in T(\sigma)}P\{\omega:[x_\sigma-E(x_\tau\mid\mathcal{G}_\sigma)]>\varepsilon\}=0$, call $\{x_n, n\ge 1\}$ as Subpramart.

2) If $\forall\varepsilon>0$, $\lim_{\sigma\in T}\sup_{\tau\in T(\sigma)}P\{\omega:[x_\sigma-E(x_\tau\mid\mathcal{G}_\sigma)]^->\varepsilon\}=0$, call $\{x_n, n\ge 1\}$ as Superpramart.

**Definition 2** Supposing $\{F_n, n\ge 1\}$ is a $L^1_{fc}[\Omega,X]$ valued adapted random set column

If $\forall\varepsilon>0$, $\lim_{\sigma\in T}\sup_{\tau\in T(\sigma)}P\{\omega: h^-(F_\sigma, E(F_\tau\mid\mathcal{G}_\sigma))>\varepsilon\}=0$, call $\{F_n, n\ge 1\}$ as set-valued Superpramart.

**Definition 3** Supposing $A,B\in P_{fc}(X)$, we call A and B are homothetic, if $C\in P_{fc}(X)$ exists, then it has $A=B+C$.

**Definition 4** We call set-valued Superpramart $\{F_n, n\ge 1\}$ has Riesz decomposition, if set-valued martingale $\{G_n, n\ge 1\}\subset L^1_{fc}[\Omega,X]$ and set-valued Superpramart $\{Z_n, n\ge 1\}\subset L^1_{fc}[\Omega,X]$ exist, $Z\xrightarrow{K-M}\{0\}, (n\to\infty)$ lets

$$F_n=G_n+Z_n, n\ge 1.$$

**Lemma 1 [11]** If $\{F_n, n\ge 1\}$ is set-valued Superpramart, then

1) $\forall x^*\in X^*$, $\{S(x^*,F_n), n\ge 1\}$ is real-valued Superpramart.

2) $\forall x\in X$, $\{d(x,F_n), n\ge 1\}$ is real-valued Subpramart.

**Lemma 2** Supposing $\{x_n, n\ge 1\}$ is real-valued Superpramart and $\underline{\lim_\tau}E|x_\tau|<\infty$, then $\lim_n x_n$ exist and is integrable.

Proof: $\{-x_n, n \geq 1\}$ is the real-valued Subpramart, and reference [12] theorem 5 (corollary 1) is known.

**Lemma 3 [7]** Supposing $\{A_n, n \geq 1\} \subset P_{fc}(X)$, if

1) $\sup \| A_n \| < \infty$;

2) $\forall x^* \in D^*, \lim_n S(x^*, A_n)$ are limited existing, if it has $A \in P_{fc}(X)$, let $A_n \xrightarrow{w} A$.

**Lemma 4** Supposing $\{F_n, n \geq 1\} \subset L^1_{fc}(X)$ is set-valued Superpramart and $\sup_{\tau \in T} E \| F_\tau \| < \infty$, then it has random set $F \in P_{fc}(X)$, let $F_n \xrightarrow{w} F$.

Proof: $\forall x^* \in X^*$, we know $\{S(x^*, F_n), n \geq 1\}$ is a real-valued Superpramart from reference [11] theorem 3.1, and because $\varliminf_T E \left| \sigma\left(x^*, F_\tau\right) \right| \leq \| x^* \| \varliminf_T E \| F_\tau \| < \infty$, we know $\lim_n S(x^*, F_n)$ exists and is limited from reference [12] theorem 5, and through the list of $D^*$, we know $\lim_n S(x^*, F_n)$ exists and is limited in little-known set of N, $\omega \in \Omega \setminus N$, $x^* \in D^*$, by the maximum inequality and Lemma 3, the existence of F lets $\lim_n S(x^*, F_n) = S\left(x^*, F\right), \omega \in \Omega \setminus N, x^* \in X^*$, then by reference [2] corollary 2.1.1 and theorem 2.1.19, we know F is a random set, the conclusion is proved.

**Lemma 5** Supposing $\{x_n, n \geq 1\}$ is a real-valued Superpramart, and $\varliminf_n E\|F_n\| < \infty$, then it has an unique factorization $x_n = y_n + z_n, n \geq 1$, where $\{y_n, n \geq 1\}$ is a real-valued martingale, $\{z_n, n \geq 1\}$ is a real-valued Superpramart, and $| z_n | \to 0, (n \to \infty)$.

Proof: From Lemma 2, we know $\lim_n x_n = x \in L^1$, noting $y_n = E(x | \mathcal{G}_n)$, it's easy to find taht $\{y_n, n \geq 1\}$ is a real-valued martingale, making $z_n = x_n - y_n, n \geq 1$, it's easy to know $\{z_n, n \geq 1\}$ is a real-valued Superpramart, and $\lim_n z_n = \lim_n x_n - \lim_n E(x | \mathrm{B}_n) = 0$, it also has $| z_n | \to 0, (n \to \infty)$.

The uniqueness is proved by the following: Supposing $x_n = y_n^{(i)} + z_n^{(i)}, n \geq 1, i = 1, 2$, so $y_n^{(1)} - y_n^{(2)} = z_n^{(2)} - z_n^{(1)}, n \geq 1$, because $\left\{y_n^{(i)}, n \geq 1\right\}$ ($i$ = 1, 2) is a real-valued martingale, $\{z_n^{(2)} - z_n^{(1)}, n \geq 1\}$ is a real-valued martingale from the above equation, then

$| y_n^{(1)} - y_n^{(2)} | = \varliminf_k | E(y_{n+k}^{(1)} - y_{n+k}^{(2)} | \mathcal{G}_n) | \leq \varliminf_k E(| z_{n+k}^{(2)} - z_{n+k}^{(1)} \| \mathcal{G}_n) = 0$, (from reference [12] theorem 7), so

$y_n^{(1)} = y_n^{(2)}, z_n^{(2)} = z_n^{(1)}, n \geq 1$, the uniqueness is proved.

**Lemma 6** Supposing $\{F_n, n \geq 1\} \subset L^1_{fc}(X)$ is a set-valued Superpramart, and $\sup_{\tau \in T} E \|F_\tau\| < \infty$, if $s - \varliminf F_n = F$, then $F_n \xrightarrow{w} F$.

Proof: From Lemma 4, we know the random set $G \in P_{fc}(X)$ exists, then it has $F_n \xrightarrow{w} G$, and

$$d\left(x, F_n\right) = \sup_i [< x_i^*, x > - S(x_i^*, F_n)] \quad x^*_i \in D^*$$

noting $y_n^i = < x_i^*, x > - S(x_i^*, F_n)$, from reference [11] theorem 3.1, we know $\left\{y_n^i, n \geq 1\right\}, i \geq 1$ is a real-valued consistent Subpramart, then from reference [2] Lemma 4.4.2, we know the little-known set $N$ exists,

$\lim_n d(x, F_n) = \sup_i (\lim_n y_n^i) = \sup_i [< x_i^*, x > - S(x_i^*, G)] = d(x, G), \forall x \in D_1$, $\omega \in \Omega \setminus N$, From inequality

$| d(x, A) - d(y, A) | \leq \| x - y \|$ and the usual density method, we known $\lim_n d(x, F_n) = d(x, G), \forall x \in X$, $\omega \in \Omega \setminus N$, it indicates $s - \varliminf F_n = G$, then $F_n \xrightarrow{w} F$.

**Lemma 7** Supposing $\{F_n, n \geq 1\} \subset L^1_{fc}(X)$ is a set-valued Superpramart, and $\sup_{\tau \in T} E \|F_\tau\| < \infty$, the followings are equivalent:

1) $\{F_n, n \geq 1\}$ can be the Riesz decomposition.

2) $\forall n \geq 1$, $F_n$ and $E(F \mid \mathcal{G}_n)$(n ≥ 1) are homothetic, where $F_n \xrightarrow{w} F$.

Proof: We prove $F \in L^1_{fc}[\Omega, X]$ firstly, because $\|F_n\| = \sup_i S\left(x_i^*, F_n\right)$, it's easy to know

$\varliminf_n \|F_n\| \geq \sup_i \left[\lim_n S(x_i^*, F_n)\right] = \sup_i S(x_i^*, F) = \|F\|$, by the lemma Fatou, we know

$\int_\Omega \|F\| dP \leq \int_\Omega \varliminf_n \| F_n \| dp \leq \varliminf_n \int_\Omega \|F_n\| dP \leq \varliminf_n E \|F_n\| < \infty$, then $F \in L^1_{fc}[\Omega, X]$.

1) $\Longrightarrow$ 2) because $F_n = G_n + Z_n$, $Z_n \xrightarrow{K-M} \{0\}, (n \to \infty)$,

Then, $S(x^*, F_n) = S(x^*, G_n) + S(x^*, Z_n)$.

From Lemma 1, Lemma 6 and reference [2] lemma 4.1.3, it's easy to know the above equation is the Riesz decomposition of real-valued Superpramart $\{S(x^*,F_n),n\geq 1\}$, and from Lemma 5 and its proof process, we notice $\lim_n S(x^*,F_n)=S\left(x^*,F\right).,x^*\in X^*$ and know $S(x^*,G_n)=E(S(x^*,F)\,|\,\mathcal{G}_n)=S(x^*,E(F|\ G_n))$, by the separability of $X^*$ and the continuity of $X^*$ support function, we know from reference [2] corollary 1.4.1 that: $G_n=E(F\,|\,\mathcal{G}_n)$, namely, $F_n$ and $E(F \mid \mathcal{G}_n)$ are homothetic.

2)$\Longrightarrow$1) Noting $G_n = E(F \mid \mathcal{G}_n)$, it's easy to know $\{G_n, n\geq 1\}$ is the value martingale of $L^1_{fc}[\Omega,X]$, and $\sup_n E\|G_n\|\leq E\|F\|<\infty$, making $F_n=G_n+Z_n$, the following is the proof that $\{Z_n,n\geq 1\}$ is the value Superpramart of $L^1_{fc}[\Omega,X]$, because

$$S(x^*,F_n) = S(x^*,G_n) + S(x^*,Z_n)$$

$$S(x^*,Z_n) = S(x^*,F_n) - S(x^*,G_n)$$

It's easy to prove $\|Z_n\|\leq\|F_n\|+\|G_n\|$, so $E\|Z_n\|\leq E\|G_n\|+E\|F_n\|<\infty$.

$E(F_\tau|\mathcal{G}_\sigma) = G_\sigma + E(Z_\tau|\mathcal{G}_\sigma)$, $\sigma\in T$, $\tau\in T(\sigma)$, from reference [2] lemma 5.3.6, we know

$$\begin{aligned}&H^+(\,E(Z_\tau\,|\,\mathcal{G}_\sigma),Z_\sigma)\\&=\sup_i[-S(x_i^*,Z_\sigma)+S(x_i^*,E(Z_\tau\,|\,\mathcal{G}_\sigma))],\ x_i^*\in D^*\\&=\sup_i[-S(x_i^*,F_\sigma)+S(x_i^*,G_\sigma)+S(x_i^*,E(F_\tau\,|\,\mathcal{G}_\sigma))-S(x_i^*,G_\sigma)],\ x_i^*\in D^*\\&=\sup_i[-S(x_i^*,F_\sigma)+S(x_i^*,E(F_\tau\,|\,\mathcal{G}_\sigma)]\\&=h^+(\,E(F_\tau\,|\,\mathcal{G}_\sigma),F_\sigma)\end{aligned}$$

Then, we know $\{Z_n,n\geq 1\}$ is set-valued Superpramart the proof is set below $Z_n\xrightarrow{K-M}\{0\},(n\to\infty)$, because

$S(x_i^*,Z_n) = S(x_i^*,F_n) - S(x_i^*,G_n)$, $x^*_i\in D^*$

$=S(x_i^*,F_n) - S(x_i^*,E(F|\mathcal{G}_n))$

$=S(x_i^*,F_n) - E(S(x_i^*,F)|\mathcal{G}_n)$, and from the list of $D^*$, we know the little-known set $N_1$, and $\lim_n S(x^*,Z_n)=0$, $\omega\in\Omega\setminus N_1$, $x^*\in D^*$, using Lemma 3, we know $\lim_n S(x^*,Z_n)=S\left(x^*,\{0\}\right),\omega\in\Omega\setminus N_1,x^*\in X^*$,

$$d\left(x,Z_n\right)=\sup_i[<x_i^*,x>-S(x_i^*,Z_n)]\quad x^*_i\in D^*$$

Noting $y_n^i=<x_i^*,x>-S(x_i^*,Z_n)$, from reference [11] lemma 3.2, we know $\{y_n^i,n\geq 1\},i\geq 1$ is a real-valued consistent Subpramart, and from reference [2] lemma 4.4.2, we know the little-known set $N_2$ exists, $\lim_n d(x,Z_n)=\sup_i(\lim_n y_n^i)=\sup_i[<x_i^*,x>-S(x_i^*,\{0\})]=d(x,\{0\}),\ \ \forall x\in D_1$, $\omega\in\Omega\setminus N_2$, from inequality $|\,d(x,A)-d(y,A)\,|\leq\|\,x-y\,\|$ and the usual density method, we know $\lim_n d(x,F_n)=d(x,\{0\}),\ \forall x\in X$, $\omega\in\Omega\setminus N_2$, then from reference [2] lemma 4.5.4, $Z_n\xrightarrow{K-M}\{0\},(n\to\infty)$.

## 3. Conclusion

The paper proves the convergence theorem of Superpramart in the sense of weak convergence. And on the basis of this certificate, through the support function and the results of real-valued Superpramart, we give the one of Riesz decomposition forms of set-valued Superpramart. It provides new ideas for the research of Riesz decomposition.

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