Journal of Applied Mathematics and Physics, 2016, 4, 1168-1178
Published Online June 2016 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2016.46121

The Impact of the Earth's Movement through the Space on Measuring the Velocity of Light

Miloš Čojanović

Independent Researcher, Montreal, Canada
Email: cojmilmo@gmail.com

Received 16 April 2016; accepted 26 June 2016; published 29 June 2016

Abstract

Goal of this experiment is basically measuring the velocity of light. As usual we will measure two-way velocity of light (from A to B and back). In contrast to the similar experiments we will not assume that speeds of light from A to B and from B to A are equal. To achieve this we will take into account Earth's movement through the space, rotation around its axis and apply "least squares method for cosine function", which will be explained in Section 9. Assuming that direction East-West is already known, one clock, a source of light and a mirror, is all equipment we need for this experiment.

Keywords

Speed of Light, One Way Speed of Light, Least Squares Method for Cosine Function

1. Introduction

Observe the planet Earth. The Earth orbits the Sun. For this motion we will join the vector $\mathbf{v}_1$. Sun orbits the center of the Milky Way. For this motion we will join the vector $\mathbf{v}_2$. In relation to the center of the Milky Way, we can join to the Earth movement sum of vectors

$$\mathbf{v}_1 + \mathbf{v}_2.$$

It is also known that our Galaxy is moving relative to other galaxies (or to a point in the space outside the Milky Way Galaxy). Similarly, to this motion we could join the vector $\mathbf{v}_3$.

Denote by $\mathbf{v}$ the sum of all these vectors

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \cdots. \quad (1)$$

At the end of the sum three points are left, because eventually there may be some other movements.

In the period of 24 h vectors $\mathbf{v}_2$, $\mathbf{v}_3$ can be taken as constants, while the vector $\mathbf{v}_1$ by making a certain error could also be taken as constant.

Thus for the Earth's motion through the space within 24 h, we can join the constant vector **v**.

The speed and direction Earth orbits the Sun are known, and let v_0 represent its avarage speed.

Suppose that some approximate values for vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are known as well. On the basis of these values, let suppose that we have inequality

$$|\mathbf{v}| = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| \geq v_0. \tag{2}$$

2. Planning an Experiment

Suppose that an arbitrary point **A** is given. Earth rotation axis will be taken as the **z** coordinate, and as the plane **xy** we will take the plane passing through point **A** and perpendicular to the **z** axis. In this case it is natural to take section of the plane **xy** and **z** axis as the center of the coordinate system. In addition to point **A** let the points **B** and **D** are given. Line **AB** lies in the plane **xy** and parallel to the direction of the Earth's rotation. Distance **AB** will be marked with L. For the **x** axis, at some initial time $\mathbf{t}_0$, we will take the line in the plane **xy**, parallel to **AB**. The projection of the vector **v** in the plane **xy** denote by $\mathbf{v}_{xy}$. Due to the Earth's rotation the direction of **AB** will be changed, so that it will be changed the angle, marked by Φ, between the **x** axis (which remained fixed) and the line **AB**. Let at point **A** we have a clock and some source of light. Suppose that speed of light in the direction **AB** is given by equation

$$\mathbf{c}_{AB} = \mathbf{c} - |\mathbf{v}_{xy}| * \cos(\Phi). \tag{1}$$

Point **D** will be chosen so the line **AD** is parralel to direction South-North. Distance **AD** is marked by L_1. Angle between line **AD** and **z** axis we will denote by φ. Angle φ actually represents Latitude of point **A** on the Earth's surface, thus it remains unchanged during the experiment.

The projection of the vector **v** on **z** axis denote by $\mathbf{v}_z$ (actually $\mathbf{v}_2 + \mathbf{v}_3$, because $\mathbf{v}_1$ is perpendicular on **z** axis). Assume that the speed of signal in the direction **AD** is given by equation

$$\mathbf{c}_{AD} = \mathbf{c} - |\mathbf{v}_z| * \cos(\varphi) \tag{2}$$

where **c** represents "velocity of light in vacuum for a body at rest". Our aim is to find the constant **c**, vectors $\mathbf{v}_{xy}$ and $\mathbf{v}_z$.

3. Conducting an Experiment

In some moment T_0 we will send signal from point **A** to point **B**. The angle between the axis **x** and $\mathbf{v}_{xy}$ is marked by Θ.

Once the signal arrived at point **B** it will be reflected back to point **A**.

Difference between the time when the signal was being sent from point **A**, and the time when the signal reached to the point **A** is denoted by $\mathbf{t}_0$.

At the same time we will send signal from point **A** to **D** and return back to point **A**. Difference between the time when signal was being sent and reached to point **A** we will denote by $\boldsymbol{\tau}_0$.

The same procedure will be within 24 h repeated N (N > 4) times, whereas the time between the two sets of consecutive procedure to be same and equal to 24 h/N.

In that way we will get the series $\{\mathbf{t}_i\}$ and $\{\boldsymbol{\tau}_i\}$

$$\{t_i\}, \{\tau_i\} \quad i \in \{0, 1, \cdots, N-1\}. \tag{1}$$

To the each t_i we can join an angle α_i between **x** axis and line **AB**.

In that way we get the series

$$\{\alpha_i\} \quad \text{where} \quad \alpha_i = i * 2\Pi/N - \Theta, \; i \in \{0, 1, \cdots, N-1\}. \tag{2}$$

By assumption (3.1) the speed of the signal $\mathbf{c}_i$ in the direction **AB** is equal to

$$\mathbf{c}_i(AB) = \mathbf{c} - |\mathbf{v}_{xy}| * \cos(i * 2\Pi/N - \Theta) \tag{3}$$

and in opposite direction **BA**

$$\mathbf{c}_i(BA) = \mathbf{c} + |\mathbf{v}_{xy}| * \cos(i * 2\Pi/N - \Theta). \tag{4}$$

It follows that

$$t_i = \frac{L}{\mathbf{c} - |\mathbf{v}_{xy}| * \cos(i * 2\Pi/N - \Theta)} + \frac{L}{\mathbf{c} + |\mathbf{v}_{xy}| * \cos(i * 2\Pi/N - \Theta)} \Rightarrow \tag{5}$$

$$t_i = \frac{2 * L * \mathbf{c}}{\mathbf{c}^2 - |\mathbf{v}_{xy}|^2 * \cos^2(i * 2\Pi/N - \Theta)}. \tag{6}$$

If we swap the roles of the points **A** and **B**, we would get the same formula as in (6). Therefore it is completely irrelevant whether direction of the vector $\mathbf{v}_{xy}$ is equal to direction **AB** or **BA**.

We assume that

$$\mathbf{c}^2 - |\mathbf{v}_{xy}|^2 * \cos^2(i * 2\Pi/N - \Theta) > 0 \Leftrightarrow t_i > 0$$

for $i \in \{0,1,\cdots,N-1\}, \Theta \in [-\Pi/2, \Pi/2]$.

It would be in principle our experiment.

4. Computing the Values of c, $|\mathbf{v}_{xy}|$ and Θ

In this section we will deal only with the measurements in direction East-West.

Let t_i is given by (3.6) and

$$c_i = \frac{2 * L}{t_i}, \ i \in \{0,1,\cdots,N\} \tag{1}$$

denote the average speed c_i (from point **A** to point **B** and back to **A**).

It follows that c_i can be written as

$$c_i = \mathbf{c} - \frac{|\mathbf{v}_{xy}|^2 * \cos^2(i * 2\Pi/N - \Theta)}{\mathbf{c}} + e_i \Rightarrow \tag{2}$$

where e_i represents some experimental error. Replacing

$$\cos^2(i * 2\Pi/N - \Theta) = (\cos(2 * i * 2\Pi/N - 2\Theta) + 1)/2$$

we get

$$c_i = \left(\mathbf{c} - \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}}\right) - \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}} * \cos(2 * (i * 2\Pi/N - \Theta)) + e_i \tag{3}$$

in short form

$$c_i = \mathbf{B} - \mathbf{A} * \cos(2 * (i * 2\Pi/N - \Theta)) + e_i, \quad i \in \{0,1,\cdots,N-1\} \tag{4}$$

$$\mathbf{B} = \mathbf{c} - \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}} \tag{5}$$

$$\mathbf{A} = \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}}, \text{ where } \mathbf{A} \ge \mathbf{0}. \tag{6}$$

The coefficients **A**, **B** and **Θ** will be chosen so the sum of squares

$$S_1(\mathbf{B}, \mathbf{A}, \Theta) = \sum e_i^2 = \sum (c_i - \mathbf{B} + \mathbf{A} * \cos(2 * (i * 2\Pi/N - \Theta)))^2 \tag{7}$$

has a minimum value.

To acheive our goal we are going to apply **Theorem 1** for k = 2.

For the sake of simplicity we've only considered cases when

$$\sum a_i * \cos(2*\alpha_i) \neq 0 \text{ and } \mathbf{A}_0 \neq 0.$$

Thus we have

$$\mathbf{B}_0 = \mathbf{c}_m = \left(\sum_{i=0}^{N-1} c_i\right) \Big/ N \tag{8}$$

$$\mathbf{tg}(2*\Theta_0) = \frac{\sum_{i=0}^{N-1} a_i * \sin(2*\alpha_i)}{\sum_{i=0}^{N-1} a_i * \cos(2*\alpha_i)} \tag{9}$$

$$\mathbf{A_0} = -\frac{2*\sum_{i=0}^{N-1} a_i * \cos(2*\alpha_i - 2*\Theta_0)}{N} \tag{10}$$

$$a_i = c_i - \mathbf{c}_m,\ \alpha_i = i*2*\Pi/N,\ \ i \in \{0,1,\cdots,N-1\}.$$

We'll make a small digression. From **Lemma 1** it follows

$$\begin{aligned}\sum a_i * \cos(k*\alpha_i) &= \sum (c_i - \mathbf{c_m}) * \cos(k*\alpha_i)\\ &= \sum c_i * \cos(k*\alpha_i) - \sum \mathbf{c}_m * \cos(k*\alpha_i)\\ &= \sum c_i * \cos(k*\alpha_i)\end{aligned}$$

In the similiar way we can get

$$\sum a_i * \sin(k*\alpha_i) = \sum c_i * \sin(k*\alpha_i).$$

Generally we have $\mathrm{tg}(x) = \mathrm{tg}(x - \Pi) \Rightarrow \mathrm{tg}(2*\Theta) = \mathrm{tg}(2*\Theta - \Pi)$. From (9) $\Rightarrow$

$$\Theta_1 = \frac{1}{2} * \mathbf{Atan}\left(\frac{\sum a_i * \sin(2*\alpha_i)}{\sum a_i * \cos(2*\alpha_i)}\right) \tag{11}$$

Function Atan () takes values at interval (−Π/2, Π/2).

$$\Theta_2 = \Theta_1 - \Pi/2$$

If we consider $\mathbf{A}_0$ as function of $\Theta \Rightarrow \mathbf{A_0}(\Theta_2) = \mathbf{A_0}(\Theta_1 - \Pi/2) = -\mathbf{A_0}(\Theta_1)$.

From (6) it folows that between the values Θ_1 and Θ_2 we have to choose that one for which $\mathbf{A}_0 > 0$.

From (5) and (6) we can derive values for **c** and $|\mathbf{v}_{xy}|$.

$$\mathbf{c} = \mathbf{B}_0 + \mathbf{A}_0 = \mathbf{c}_m + \mathbf{A}_0 \tag{12}$$

$$|\mathbf{v}_{xy}| = \pm\sqrt{2*\mathbf{A}_0 * \mathbf{c}} \tag{13}$$

We don't know exact direction of vector $\mathbf{v}_{xy}$, thus positive and negative value are assigned to $|\mathbf{v}_{xy}|$.

5. Comparison between Two Methods

In this section we will make comparison between "the least squares method" and "the least squares method for cosine function".

Let consider $\{c_i\}$ given by (4.1) as the series of mutually independent measurements.

Let $\mathbf{c}_m$ represents the mean value of serial $\{c_i\}$.

$$\mathbf{c}_m = \left(\sum c_i\right) / N \tag{1}$$

If we apply Least squares method, Variance V_1 is given by

$$V_1 = \sum (c_i - \mathbf{c}_m)^2 \tag{2}$$

and standard deviation σ_1 by

$$\sigma_1 = \sqrt{V_1/N}. \tag{3}$$

Suppose that to the each c_i we joined the time when measurement took place, or rather the angle between the direction of **AB** and vector $\mathbf{v}_{xy}$. Expected value $E_2(\alpha_i)$ for "The Least squares method for cosine function" is given by

$$E_2(\alpha_i) = y_i = \mathbf{B}_0 - \mathbf{A}_0 \cos\left(k*(\alpha_i - \Theta_0)\right) \tag{4}$$

where

$$\alpha_i = i*2\Pi/N,\ i \in \{0,1,\cdots,N-1\}. \tag{5}$$

Denote a_i by

$$a_i = c_i - \mathbf{B}_0 = c_i - \mathbf{c}_m.$$

Let us find Variance V_2 for this method

$$\begin{aligned} V_2 &= \sum (c_i - y_i)^2 = \sum \left(c_i - \mathbf{c}_m + \mathbf{A}_0 * \cos\left(k*(\alpha_i - \Theta_0)\right)\right)^2 \\ &= \sum \left(a_i + \mathbf{A}_0 * \cos\left(k*(\alpha_i - \Theta_0)\right)\right)^2 \\ &= \sum a_i^2 + 2*\mathbf{A}_0 * \sum a_i * \cos\left(k*(\alpha_i - \Theta_0)\right) + \mathbf{A}_0^2 * \sum \frac{1+\cos\left(2*k(\alpha_i - \Theta_0)\right)}{2} \\ &= \sum a_i^2 + 2*\mathbf{A}_0 * \sum a_i * \cos\left(k*(\alpha_i - \Theta_0)\right) + \frac{\mathrm{N}*\mathbf{A}_0^2}{2} \\ &= |\text{from (10.5)}| = \sum a_i^2 - \frac{\mathrm{N}*\mathbf{A}_0^2}{2} = \sum (\mathrm{c_i} - \mathbf{c}_\mathrm{m})^2 - \frac{\mathrm{N}*\mathbf{A}_0^2}{2} \end{aligned} \tag{6}$$

$$V_2 = V_1 - \frac{\mathrm{N}*\mathbf{A}_0^2}{2} \geq 0 \Rightarrow V_1 \geq V_2. \tag{7}$$

Standard deviation σ_2 for this method is given by

$$\sigma_2 = \sqrt{V_2/N}. \tag{8}$$

From (7) $\Rightarrow \sigma_1 \geq \sigma_2$ From (7) $\Rightarrow V_2 \geq 0 \Rightarrow \sum (c_i - \mathbf{c}_m)^2 \geq \frac{N*\mathbf{A}_0^2}{2} \Rightarrow \sqrt{2}*\sigma_1 \geq |\mathbf{A}_0|$.

If standard deviation σ_2 is bigger then some expected value it means either our measurement are not accurate enough or our method (curve) doesn't suit to our data.

6. Analysys of South-North Measurements

In this chapter we will deal with the series $\{\tau_i\}$ given by (3.1).

Just to remind that $\boldsymbol{\tau}_i$ represents time it takes for signal to travel from **A** to **D** and back to **A** in direction South-North.

$$\tau_i = \frac{L_1}{\mathbf{c} - |\mathbf{v}_z| * \cos(\varphi)} + \frac{L_1}{\mathbf{c} + |\mathbf{v}_z| * \cos(\varphi)} \Rightarrow \tag{1}$$

$$\tau_i = \frac{2*L_1*\mathbf{c}}{\mathbf{c}^2 - |\mathbf{v}_z|^2 * \cos^2(\varphi)} \tag{2}$$

Let

$$\gamma_1 = \frac{2 * L_1}{\tau_i} \quad i \in \{0,1,\cdots,N-1\} \tag{3}$$

denote the average speed γ_i. In that way we get the series $\{\gamma_i\}$

$$\gamma_i = \mathbf{c} - \frac{|\mathbf{v_z}|^2 * \cos^2(\varphi)}{\mathbf{c}} + e_i \tag{4}$$

where e_i represents some experimental error.

Since angle φ kept constant value during the experiment we could apply Least squares method to the series given by (4).

Let denote γ_m by

$$\gamma_m = \left(\sum \gamma_i\right) / N \tag{5}$$

mean value of the series $\{\gamma_i\}$.

We can calculate Variance V_1

$$V_1 = \sum (\gamma_i - \gamma_m)^2 \tag{6}$$

and standard deviation σ_1

$$\sigma_1 = \sqrt{\frac{V_1}{N}}. \tag{7}$$

If standard deviation σ_1 is bigger then some expected value we should declare the experiment failed.

Combining equations (4) and (5) we get

$$|\mathbf{v}_z| = \pm \sqrt{\frac{\mathbf{c}^2 - \mathbf{c} * \gamma_m}{\cos(\varphi)}} \quad (\mathbf{c} \geq \gamma_m, \cos\varphi \neq 0). \tag{8}$$

We don't know exact direction of vector $\mathbf{v}_z$, thus positive and negative value were assigned to $|\mathbf{v}_z|$.

7. Conclusions

From (5.13) and (7.8) it follows that length of vector **v** is given by

$$|\mathbf{v}| = \sqrt{\mathbf{v}_{xy}^2 + \mathbf{v}_z^2} \tag{1}$$

while vector **v** is given by

$$\mathbf{v} = \pm \mathbf{v}_{xy} \pm \mathbf{v}_z. \tag{2}$$

Recall (from 2.1) that vector **v** can be written also as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3. \tag{3}$$

Suppose that during one year the same experiments have been repeated 2∗K times. In that way we will get the series

$$\{|\mathbf{v}(\mathrm{i})|\}_{i=1}^{2K} \tag{4}$$

where $|\mathbf{v}(\mathrm{i})|$ represents length of vector given by Equation (2) or (3) at i-*th* try.

Let $\mathbf{v}_1(\mathrm{i}+\mathrm{K})$ and $\mathbf{v}_1(\mathrm{i})$ denote velocity at which Earth orbits the Sun at $(\mathrm{i}+\mathrm{K})$-th and i-th try.

Suppose also that origins of vectors $\mathbf{v}_\mathrm{i}(\mathrm{i}+\mathrm{K})$ and $\mathbf{v}_\mathrm{i}(\mathrm{i})\, \mathrm{i} \in \{1,2,\cdots,\mathrm{K}\}$ lay on the diameter of Earth orbit around the Sun, so they are parallel but in oposite directions.

Mean value v_m of the serial (3) is given by

$$\mathrm{v}_\mathrm{m} = \left(\sum |\mathbf{v}(\mathrm{i})|\right) / (2K). \tag{5}$$

Depending on v_m we will consider following cases:

1) $\frac{v_m}{v_0} \to 0$

In other words v_m is significantly less than v_0 what is in contradiction to our hypotesis (2.2).

In this case we have to reject hypothesis given by (3.1) and declare that velocity of light is not effected by Earth's movement through the space.

This results is consistent with some other experiments, for example with Michelson-Morley experiment.

2) $v_m > v_0$

During the experiments in period of one year $\mathbf{v}_1$ is changing, while $\mathbf{v}_2 + \mathbf{v}_3$ is keeping the constant value.

Recall that vector $\mathbf{v}_1$ is perpendicular to $\mathbf{z}$ axis.

Denote vector $\mathbf{u}$ by

$$\mathbf{u} = \mathbf{v}_2 + \mathbf{v}_3 \tag{6}$$

$$\left(\text{let } \mathrm{proj}_{xy}(\mathbf{a}) \text{ represents orthogonal projection of vector } \mathbf{a} \text{ on plane } \mathbf{xy}\right) \tag{7}$$

$$\mathbf{v}_{xy} = proj_{xy}(\mathbf{v}) = proj_{xy}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = proj_{xy}(\mathbf{v}_1) + proj_{xy}(\mathbf{u}) = \mathbf{v}_1 + \mathbf{u}_{xy} \Rightarrow \tag{8}$$

$$\left|\mathbf{v}_{xy}(i)\right|^2 = \left|\mathbf{v}_1(i)\right|^2 + \left|\mathbf{u}_{xy}\right|^2 + 2 * \mathbf{v}_1(i) * \mathbf{u}_{xy} \tag{9}$$

$$\left|\mathbf{v}_{xy}(i+K)\right|^2 = \left|\mathbf{v}_1(i+K)\right|^2 + \left|\mathbf{u}_{xy}\right|^2 + 2 * \mathbf{v}_1(i+K) * \mathbf{u}_{xy}. \tag{10}$$

If we replace $\left|\mathbf{v}_1(i)\right|$ and $\left|\mathbf{v}_1(i+K)\right|$ by v_0

$$\left|\mathbf{v}_1(i)\right| \approx v_0$$

$$\left|\mathbf{v}_1(i+K)\right| \approx v_0$$

(v_0 represents average speed Earth orbits the Sun).

From (9) and (10) we can get approximate value for $\left|\mathbf{u}_{xy}(i)\right|$

$$\left|\mathbf{u}_{xy}(i)\right| \approx \sqrt{\frac{\left|\mathbf{v}_{xy}(i)\right|^2 + \left|\mathbf{v}_{xy}(i+K)\right|^2 - 2 * v_0^2}{2}} \quad i \in \{1, 2, \cdots, K\}. \tag{11}$$

We can form serial

$$\left\{\left|\mathbf{u}_{xy}(\mathrm{i})\right|\right\}_{i=1}^{K}. \tag{12}$$

Mean value u_{xy} of the serial (12) is given by

$$u_{xy} = \left(\sum_{i=1}^{K} \left|\mathbf{u}_{xy}(\mathrm{i})\right|\right) \Big/ K. \tag{13}$$

Let find standard deviation σ_1 for serial (13).

If σ_1 is bigger then some expected value we have to decline our hypothesis (2.1) and declare the experiment failed.

$$\mathbf{v}_z = proj_z(\mathbf{v}) = proj_z(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = proj_z(\mathbf{v}_1) + proj_z(\mathbf{u}) = \mathbf{u}_z \tag{14}$$

$$\left\{\left|\mathbf{u}_z(\mathrm{i})\right|\right\}_{i=1}^{2K} \tag{15}$$

where $\left|\mathbf{u}_z(\mathrm{i})\right| = \left|\mathbf{v}_z(i)\right|$ at i-th try.

For serial (15) mean value u_z is given by

$$u_z = \left(\sum_{i=1}^{2K} \left|\mathbf{u}_z(\mathrm{i})\right|\right) \Big/ (2K).$$

Let standard deviation for serial (15) is marked by σ_2.

If σ_2 is bigger then some expected value we have to decline our hypothesis (2.1) and declare the experiment failed.

Otherwise hypothesis given by (3.1) holds and we can conclude that velocity of light depends on Earth's movement through space. In other words velocity of light depends on the direction in which has been measured, what would be in contradiction with Michelson-Morley experiment [1].

The speed that Solar system moves in the space in this case is given by equation

$$u = \sqrt{u_{xy}^2 + u_z^2}. \tag{16}$$

Note that while performing the experiment we committed some mistakes.

It was not taken into account the speed of Earth's rotation. This problem can be solved by conducting an experiment at place closer to the Earth's poles, and thus the speed of Earth's rotation taken as small as we want. On other hand this would be counter-productive to our conditions for South-North measurement. Ideally, E-W experiment should be performed on the North/South Pole and S-N experiment at some place on equator.

In addition, within 24 h the Earth changes its direction and the speed at which it revolves around the Sun. We can't solve this problem but we can assume that this speed is relatively small comparing to total speed at which Earth moves through the space.

8. Lemma 1

If N, k are natural numbers ($1 < N$, $0 < k < N$) and Θ an arbitrary angle then

$$\sum_{j=0}^{N-1} \sin\left(\left(j * k/N\right) * 2\Pi - \Theta\right) = 0 \tag{1}$$

$$\sum_{j=0}^{N-1} \cos\left(\left(j * k/N\right) * 2\Pi - \Theta\right) = 0 \tag{2}$$

Proof.

$$\sum_{j=0}^{N-1} \cos\left(\left(j * k/N\right) * 2\Pi - \Theta\right) + \mathbf{i} * \sin\left(\left(j * k/N\right) * 2\Pi - \Theta\right) = e^{-\Theta * \mathbf{i}} * \sum_{j=0}^{N-1} e^{(j * k/N) * 2\Pi * \mathbf{i}} = e^{-\Theta * \mathbf{i}} * \frac{M}{N}$$

where

$$M = e^{((N*k/N)*2\Pi)*\mathbf{i}} - 1 = e^{(k*2\Pi)*\mathbf{i}} = 1 - 1 = 0$$

$$N = e^{((k/N)*2\Pi)*\mathbf{i}} - 1 \neq 0, \quad 0 < \left(k/N\right) * 2\Pi < 2\Pi$$

Q.E.D.

9. Theorem 1. Least Squares Method for Cosine Function

Suppose we are given the series $\{c_i\}$, $c_i > 0$, $i \in \{0, 1, \cdots, N-1\}$ and there are at least two p, q thus c_p <> c_q

Let take arbitrary coefficients **B**, **A**, **Θ** and form equations

$$c_i = \mathbf{B} - \mathbf{A} * \cos\left(k * \left(i * 2\Pi/N - \Theta\right)\right) + e_i, \quad 0 < k < N/2 \tag{1}$$

$$\sum e_i^2 = \sum \left(c_i - \mathbf{B} + \mathbf{A} * \cos\left(k * \left(\alpha_i - \Theta\right)\right)\right)^2$$

Define function g(B, A, Θ) by

$$g\left(\mathbf{B}, \mathbf{A}, \Theta\right) = \sum e_i^2 = \sum \left(c_i - \mathbf{B} + \mathbf{A} * \cos\left(k * \left(\alpha_i - \Theta\right)\right)\right)^2 \tag{2}$$

We will prove that in case $\mathbf{A_0} \neq 0$, function g() has a minimum value at point $\left(\mathbf{B}_0, \mathbf{A}_0, \Theta_0\right)$

$$\mathbf{B}_0 = \mathbf{c}_m = \left(\sum_{i=0}^{N-1} c_i\right) \Big/ N \tag{3}$$

$$\mathbf{tg}\left(k*\Theta_{\mathbf{0}}\right)=\frac{\sum_{i=0}^{N-1}a_i*\sin\left(k*\alpha_i\right)}{\sum_{i=0}^{N-1}a_i*\cos\left(k*\alpha_i\right)} \tag{4}$$

$$\mathbf{A_0}=-\frac{2*\sum_{i=0}^{N-1}a_i*\cos\left(k*\alpha_i-k*\Theta_0\right)}{N} \tag{5}$$

where $a_i=c_i-\mathbf{c}_{\mathrm{m}}$, $\alpha_i=i*2\Pi/N$, $i\in\{0,1,\cdots,N-1\}$.

Proof.

Let **B**, **A** and **Θ** have arbitrary values

$$\begin{aligned} g\left(\mathbf{B},\mathbf{A},\Theta\right) &= \sum\left(c_i-\mathbf{B}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2 \\ &= \sum\left(\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)+\left(c_i-\mathbf{c}_{\mathrm{m}}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)\right)^2 \\ &= \mathrm{N}*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)^2+2*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)*\sum\left(c_i-\mathbf{c}_{\mathrm{m}}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right) \\ &\quad +\sum\left(c_i-\mathbf{c}_{\mathrm{m}}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2 \\ &= \mathrm{N}*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)^2+2*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)*\sum\left(c_i-\mathbf{c}_{\mathrm{m}}\right)+2*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)*\mathbf{A}*\sum\cos\left(k*\left(\alpha_i-\Theta\right)\right)+\mathrm{g}\left(\mathbf{c}_{\mathrm{m}},\mathbf{A},\Theta\right) \\ &= \mathrm{N}*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)^2+\mathrm{g}\left(\mathbf{c}_{\mathrm{m}},\mathbf{A},\Theta\right) \end{aligned}$$

thus we get

$$\mathrm{g}\left(\mathbf{B},\mathbf{A},\Theta\right)\geq\mathrm{g}\left(\mathbf{c}_{\mathrm{m}},\mathbf{A},\Theta\right) \tag{6}$$

In that way we can reduce function g() from function of three variables to fuction of two variables A and **Θ**, keeping coefficent **B** fixed and equal to $\mathbf{c}_{\mathrm{m}}$.

Now we can write the function g() in the form

$$\begin{aligned} g\left(\mathbf{A},\Theta\right) &= \sum\left(c_i-\mathbf{c_m}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2=\sum\left(a_i+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2 \\ &= \sum a_i^2+2*\mathbf{A}*\sum a_i*\cos\left(k*\left(\alpha_i-\Theta\right)\right)+\mathbf{A}^2*\sum\cos^2\left(k*\left(\alpha_i-\Theta\right)\right) \\ &\Rightarrow \left|\cos^2\left(k*\left(\alpha_i-\Theta\right)\right)=\left(\cos\left(2k*\left(\alpha_i-\Theta\right)\right)+1\right)/2\right| \end{aligned}$$

$$\mathrm{g}\left(\mathbf{A},\Theta\right)=\frac{\mathrm{N}*\mathbf{A}^2}{2}+2*\mathbf{A}*\sum a_i*\cos\left(k*\left(\alpha_i-\Theta\right)\right)+\sum a_i^2. \tag{7}$$

In order to find minimum for function g(), first we have to find partial derivates with respect to A and Θ and critical point (A_0, Θ_0)

$$\frac{\partial\mathrm{g}\left(\mathbf{A}_0,\Theta_0\right)}{\partial\mathbf{A}}=0,\quad \frac{\partial\mathrm{g}\left(\mathbf{A}_0,\Theta_0\right)}{\partial\Theta}=0. \tag{8}$$

Let us find the first partial derivatives

$$\begin{aligned} \frac{\partial g}{\partial\Theta} &= 2*\mathrm{k}*\mathbf{A}*\sum a_i*\sin\left(k*\alpha_i-k*\Theta\right) \\ &= 2*\mathrm{k}*\mathbf{A}*\left(\cos\left(\mathrm{k}*\Theta\right)*\sum \mathrm{a_i}*\sin\left(\mathrm{k}*\alpha_i\right)-\sin\left(\mathrm{k}*\Theta\right)*\sum \mathrm{a_i}*\cos\left(\mathrm{k}*\alpha_i\right)\right) \end{aligned} \tag{9}$$

$$\frac{\partial \mathrm{g}}{\partial\Theta}=0\Rightarrow 2*\mathrm{k}*\mathbf{A}*\sum \mathrm{a_i}*\sin\left(\mathrm{k}*\alpha_i-\mathrm{k}*\Theta\right)=0\Rightarrow. \tag{10}$$

1) $\mathbf{A}=0$

In this case we would have

$$g(\mathbf{B},\mathbf{A},\Theta) = g(\mathbf{B}) = \sum e_i^2 = \sum (c_i - \mathbf{B})^2$$

It's easy to prove that g() has minimum at

$$\mathbf{B}_0 = \mathbf{c}_{\mathrm{m}} = \left(\sum_{i=0}^{N-1} c_i\right) \Big/ N.$$

2) $\mathbf{A} \neq 0 \Rightarrow \sum \mathrm{a_i} * \sin(\mathrm{k} * \alpha_i - \mathrm{k} * \Theta) = 0$

$$\Rightarrow \cos(\mathrm{k} * \Theta) * \sum \mathrm{a_i} * \sin(\mathrm{k} * \alpha_i) - \sin(\mathrm{k} * \Theta) * \sum \mathrm{a_i} * \cos(\mathrm{k} * \alpha_i) = 0$$

$$\begin{aligned} \frac{\partial \mathrm{g}}{\partial \mathbf{A}} &= \mathrm{N} * \mathbf{A} + 2 * \sum a_i * \cos(\mathrm{k} * \alpha_{\mathrm{i}} - \mathrm{k} * \Theta) \\ &= \mathrm{N} * \mathbf{A} + 2 * \left(\cos(\mathrm{k} * \Theta) * \sum \mathrm{a_i} * \cos(\mathrm{k} * \alpha_i) + \sin(\mathrm{k} * \Theta) * \sum \mathrm{a_i} * \sin(\mathrm{k} * \alpha_i)\right) \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{\partial \mathrm{g}}{\partial \mathbf{A}} = 0 \Rightarrow \mathbf{A} &= -\frac{2 * \sum a_i * \cos(\mathrm{k} * (\alpha_{\mathrm{i}} - \Theta))}{\mathrm{N}} \\ &= -\frac{2 * \left(\cos(\mathrm{k} * \Theta) * \sum a_i * \cos(\mathrm{k} * \alpha_{\mathrm{i}}) + \sin(\mathrm{k} * \Theta) * \sum a_i * \sin(\mathrm{k} * \alpha_{\mathrm{i}})\right)}{\mathrm{N}} \end{aligned} \quad (12)$$

Let us look at the Equations (10) and (12)

For $\mathrm{A} \neq 0$ we will consider three cases:

1) $\sum \mathrm{a_i} * \cos(\mathrm{k} * \alpha_i) = 0, \ \sum \mathrm{a_i} * \sin(\mathrm{k} * \alpha_i) = 0$

From (12) it follows A = 0. We will reject this posibility because $\mathrm{A} \neq 0$.

2) $\sum \mathrm{a_i} * \cos(\mathrm{k} * \alpha_i) = 0, \ \sum \mathrm{a_i} * \sin(\mathrm{k} * \alpha_i) \neq 0$

From (10) it follows $\cos(\mathrm{k} * \Theta_0) = 0 \Rightarrow \Theta_0 = \pm \Pi / (2 * \mathrm{k})$.

3) $\sum \mathrm{a_i} * \cos(\mathrm{k} * \alpha_i) \neq 0$

From (10) $\Rightarrow$

$$\mathbf{tg}(\mathrm{k} * \Theta_0) = \frac{\sin(\mathrm{k} * \Theta_0)}{\cos(\mathrm{k} * \Theta_0)} = \frac{\sum \mathrm{a}_i * \sin(k * \alpha_i)}{\sum \mathrm{a}_i * \cos(k * \alpha_i)}$$

From (12) $\Rightarrow$

$$\mathbf{A}_0 = -\frac{2 * \sum a_i * \cos(\mathrm{k} * (\alpha_{\mathrm{i}} - \Theta_0))}{\mathrm{N}} \quad (\text{for both cases})$$

Now we have to find the second order partial derivatives of g() with respect to A and Θ.

$$\frac{\partial^2 \mathrm{g}}{\partial^2 \mathrm{A}} = \mathrm{N} \Rightarrow \frac{\partial^2 \mathrm{g}(\mathrm{A}_0, \Theta_0)}{\partial^2 \mathrm{A}} = \mathrm{N} > 0 \quad (13)$$

$$\frac{\partial^2 \mathrm{g}}{\partial^2 \Theta} = -2 * \mathrm{k}^2 * \mathrm{A} * \sum \mathrm{a}_i * \cos(k * (\alpha_i - \Theta)) \Rightarrow \quad (14)$$

$$\frac{\partial^2 \mathrm{g}(\mathrm{A}_0, \Theta_0)}{\partial^2 \Theta} = -2 * \mathrm{k}^2 * \mathrm{A}_0 * \sum \mathrm{a}_i * \cos(k * (\alpha_i - \Theta_0)) \Rightarrow$$

$$\left| \text{from (12)} \Rightarrow \sum a_i * \cos(\mathrm{k} * (\alpha_{\mathrm{i}} - \Theta_0)) = -\frac{\mathrm{N} * \mathrm{A}_0}{2} \right| \Rightarrow$$

$$\frac{\partial^2 \mathrm{g}(\mathrm{A}_0, \Theta_0)}{\partial^2 \Theta} = -2 * \mathrm{k}^2 * \mathrm{A}_0 * \sum \mathrm{a}_i * \cos(k * (\alpha_i - \Theta_0)) \Rightarrow$$

$$\frac{\partial^2 \mathrm{g}(\mathrm{A}_0, \Theta_0)}{\partial^2 \Theta} = \mathrm{N} * \mathrm{k}^2 * \mathrm{A}_0^2$$

$$\frac{\partial^2 g(A,\Theta)}{\partial A \partial \Theta} = \frac{\partial^2 g(A,\Theta)}{\partial \Theta \partial A} = 2 * k * \sum a_i * \sin\left(k * (\alpha_i - \Theta)\right) \Rightarrow \tag{15}$$

$$\frac{\partial^2 g(A_0,\Theta_0)}{\partial A \partial \Theta} = \frac{\partial^2 g(A_0,\Theta_0)}{\partial \Theta \partial A} = 2 * k * \sum a_i * \sin\left(k * (\alpha_i - \Theta_0)\right) \Rightarrow \tag{16}$$

$$\left|\text{from (10)} \Rightarrow \sum a_i * \sin\left(k * (\alpha_i - \Theta_0)\right) = 0\right| \Rightarrow$$

$$\frac{\partial^2 g(A_0,\Theta_0)}{\partial A \partial \Theta} = \frac{\partial^2 g(A_0,\Theta_0)}{\partial \Theta \partial A} = 0$$

$$\Delta = \begin{vmatrix} \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial^2 A} & \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial A \partial \Theta} \\ \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial \Theta \partial A} & \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial^2 \Theta} \end{vmatrix} = \begin{vmatrix} N & 0 \\ 0 & N * k^2 * A_0^2 \end{vmatrix} = \left(N * k * A_0\right)^2 \Rightarrow \tag{17}$$

$$\Delta = \left(N * k * A_0\right)^2 > 0 \Leftrightarrow A_0 \neq 0 \tag{18}$$

Equations given by (13) and (18) are sufficient conditions for minimum.
Q.E.D.

References

[1] Ditchburn, R.W. (1991) Light. Dover Publications Inc., New York.