

# On the Oscillation of Second-Order Nonlinear Neutral Delay Dynamic Equations on Time Scales

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## Abstract

Based on Riccati transformation and the inequality technique, we establish some new sufficient conditions for oscillation of the second-order neutral delay dynamic equations on time scales. Our results not only extend and improve some known theorems, but also unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation on time scales. At the end of this paper, we give an example to illustrate the main results.

## Keywords

Oscillation, Dynamic Equations, Neutral, Time Scale

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## 1. Introduction

The theory of time scales was first proposed by Hilger [1] in order to unify continuous and discrete analysis. Several researchers have made greater contributions to various aspects of this new theory; see [2]-[4]. The new theory of dynamic equations on time scales not only unifies the theories of differential equations and difference equations, but also extends these classical cases to cases “in between”, e.g., to so-called  $q$ -difference equations where  $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ .

In recent years, there has been much research involving the oscillation and nonoscillation of solutions of various equations on time scales such as [5]-[18]. In this paper we study and give the sufficient conditions for oscillation of the second-order neutral delay dynamic equation

$$\left( a(t) \left( [x(t) + r(t)x(\tau(t))]^\Delta \right)^\gamma \right)^\Delta + q(t)f(x(\tau(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \quad (1.1)$$

where  $t_0 > 0$  and  $\mathbb{T}$  is unbounded time scale. Besides that, we will have hypotheses as follows throughout the paper:

(H<sub>1</sub>)  $\gamma$  is the ratio of two positive odd integers and  $\gamma \geq 1$ .

(H<sub>2</sub>)  $a, r, q: \mathbb{T} \rightarrow \mathbb{R}$  are positive rd-continuous functions with  $r(t)$  satisfying  $0 \leq r(t) < 1$ .

(H<sub>3</sub>)  $\tau \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$  is a strictly increasing and differentiable function such that  $\tau(t) \leq t$ ,  $\tau(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and  $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$ .

(H<sub>4</sub>)  $f \in C(\mathbb{R}, \mathbb{R})$  is a continuous function which satisfies  $\frac{f(x)}{x^\gamma} \geq L$  for all  $x \neq 0$ , where  $L$  is a positive constant.

In addition, for the sake of clearness and convenience, we will use the notation

$$z(t) := x(t) + r(t)x(\tau(t))$$

in the following narrative.

It is well known by reserchers in this field that an dynamic equation is called oscillatory in case all its solutions are oscillatory, and a solution of the equation is said to be oscillatory if it is neither eventually positive nor eventually negative. We only discuss those solutions  $x$  of Equation (1.1) that are not eventually zero in this paper. Moreover we refer to [3] [4] for general basic background, ideas and more details on dynamic equations.

Because of  $a(t) > 0$ , we shall consider Equation (1.1) respectively based on the case

$$\int_{t_0}^{\infty} \left( \frac{1}{a(t)} \right)^{\frac{1}{\gamma}} \Delta t = \infty, \tag{1.2}$$

and the other case

$$\int_{t_0}^{\infty} \left( \frac{1}{a(t)} \right)^{\frac{1}{\gamma}} \Delta t < \infty. \tag{1.3}$$

## 2. Several Lemmas

In this section, we present and prove three lemmas which play important roles in the proofs of the main results.

**Lemma 1.** ([16]) Assume that  $\tau: \mathbb{T} \rightarrow \mathbb{R}$  is strictly increasing,  $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$  is a time scale and  $\tau(\sigma(t)) = \sigma(\tau(t))$ . Let  $x: \tilde{\mathbb{T}} \rightarrow \mathbb{R}$ . If  $\tau^\Delta(t)$  and  $x^\Delta(\tau(t))$  exist for  $t \in \mathbb{T}^k$ , then  $(x(\tau(t)))^\Delta$  exists, and

$$(x(\tau(t)))^\Delta = x^\Delta(\tau(t))\tau^\Delta(t). \tag{2.1}$$

**Lemma 2.** ([3]) Assume that  $x$  is delta-differentiable and eventually positive or eventually negative, then

$$\left( (x(t))^\gamma \right)^\Delta = \gamma \int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{\gamma-1} x^\Delta(t) dh. \tag{2.2}$$

We give the below lemma and prove it similar to that of Q. Zhang and X. Song ([17], Lemma 3.5).

**Lemma 3.** Based on (1.2), assume that (H<sub>1</sub>)-(H<sub>4</sub>) hold. If  $x$  is an eventually positive solution of (1.1), there exists  $t_1 \in [t_0, \infty)_{\mathbb{T}}$  such that

$$z^\Delta(t) = [x(t) + r(t)x(\tau(t))]^\Delta > 0, \left( a(t)(z^\Delta(t))^\gamma \right)^\Delta < 0, t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.3}$$

*Proof.* Assume  $x(t)$  is an eventually positive solution of (1.1). That is, there exists  $t_1 \in [t_0, \infty)_{\mathbb{T}}$  such that  $x(t) > 0$  and  $x(\tau(t)) > 0$  for  $t \in [t_1, \infty)_{\mathbb{T}}$ . Because of  $z(t) = x(t) + r(t)x(\tau(t))$  and  $0 \leq r(t) < 1$ , we get  $z(t) > 0$  esaily for  $t \in [t_1, \infty)_{\mathbb{T}}$ . At the same time for  $t \in [t_1, \infty)_{\mathbb{T}}$ , from equation (1.1) we obtain that

$$\left( a(t)(z^\Delta(t))^\gamma \right)^\Delta < 0, \tag{2.4}$$

so  $a(t)(z^\Delta(t))^\gamma$  is decreasing. From (2.4), we know that  $z^\Delta(t)$  is either eventually positive or eventually

negative. Now we assert that  $z^\Delta(t) > 0$ .

Suppose to the contrary that there exists  $t_2 \in [t_1, \infty)_{\mathbb{T}}$  such that  $z^\Delta(t) < 0$  for all  $t \in [t_2, \infty)_{\mathbb{T}}$ . Because  $a(t)(z^\Delta(t))^\gamma$  is decreasing,

$$a(t)(z^\Delta(t))^\gamma \leq a(t_2)(z^\Delta(t_2))^\gamma = -M^\gamma \tag{2.5}$$

for  $t \in [t_2, \infty)_{\mathbb{T}}$ , where  $M = [a(t_2)]^{\frac{1}{\gamma}} |z^\Delta(t_2)| > 0$ . Based on the above inequality (2.5), we get

$$z^\Delta(t) \leq -M \left[ \frac{1}{a(t)} \right]^{\frac{1}{\gamma}}, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.6}$$

After integrating the two sides of inequality (2.6) from  $t_2$  to  $t \in [t_2, \infty)_{\mathbb{T}}$ , we have

$$z(t) \leq z(t_2) - M \int_{t_2}^t \left[ \frac{1}{a(s)} \right]^{\frac{1}{\gamma}} \Delta s, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.7}$$

When  $t \rightarrow \infty$ , we get  $\lim z(t) = -\infty$  from (1.2) and the above (2.7), which is contradictory to  $z(t) > 0$ . So the above hypothesis of  $z^{\Delta}(t) < 0$  is false. In other words, we get  $z^\Delta(t) > 0$  for  $t \in [t_0, \infty)_{\mathbb{T}}$ . This completes the proof.  $\square$

### 3. Main Results

Now we state and prove our main results in this section.

**Theorem 1.** *Based on (1.2), assume that the conditions (H<sub>1</sub>)-(H<sub>4</sub>) hold. If there exists a positive nondecreasing  $\Delta$ -differentiable function  $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$  such that for every  $T \in [t_0, \infty)_{\mathbb{T}}$*

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ L\delta(s)q(s)(1-r(\tau(s)))^\gamma - \eta^\gamma(\tau(s))\delta^\Delta(s) \right] \Delta s = \infty, \tag{3.1}$$

where

$$\eta(\tau(t)) = \left( \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s \right)^{-1}, \tag{3.2}$$

then (1.1) is oscillatory on  $[t_0, \infty)_{\mathbb{T}}$ .

*Proof.* Assume that (1.1) has a nonoscillatory solution  $x$  on  $[t_0, \infty)_{\mathbb{T}}$ . We may assume that  $x(t) > 0$  and  $x(\tau(t)) > 0$  for all  $t \in [t_1, \infty)_{\mathbb{T}}, t_1 \in [t_0, \infty)_{\mathbb{T}}$ . By the definition of  $z(t)$ , it follows  $z(t) > 0$ . From (H<sub>3</sub>) we know  $\tau(t) \leq t$ , by Lemma 3 we have  $z^\Delta(t) > 0$ , so  $z(\tau(t)) \leq z(t)$ , and  $0 \leq r(t) < 1$ , we obtain

$$x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) - r(t)z(\tau(t)) \geq (1-r(t))z(t), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

The proof that  $x$  is eventually negative is similar. By Lemma 3 we have  $z^\Delta(t) > 0$  for all  $t \in [t_2, \infty)_{\mathbb{T}}, t_2 \in [t_1, \infty)_{\mathbb{T}}$ , and by Lemma 1 and (H<sub>3</sub>), there exists  $T \in [t_2, \infty)_{\mathbb{T}}$  such that  $(z(\tau(t)))^\Delta > 0$  for all  $t \in [T, \infty)_{\mathbb{T}}$ . Using (2.2) and (2.3), we have

$$\begin{aligned} \left( (z(\tau(t)))^\gamma \right)^\Delta &= \gamma \int_0^1 \left[ h(z(\sigma(\tau(t)))) + (1-h)z(\tau(t)) \right]^{\gamma-1} (z(\tau(t)))^\Delta dh \\ &\geq \gamma \int_0^1 \left[ h(z(\tau(t))) + (1-h)z(\tau(t)) \right]^{\gamma-1} (z(\tau(t)))^\Delta dh \\ &= \gamma (z(\tau(t)))^{\gamma-1} (z(\tau(t)))^\Delta. \end{aligned}$$

$\mathbb{T}$  is unbounded above, which implies  $\mathbb{T}^k = \mathbb{T}$ . Furthermore, from Lemma 1 we get

$$(z(\tau(t)))^\Delta = z^\Delta(\tau(t))\tau^\Delta(t).$$

Thus, by (H<sub>3</sub>),

$$\left( (z(\tau(t)))^\gamma \right)^\Delta \geq \gamma (z(\tau(t)))^{\gamma-1} z^\Delta(\tau(t)) \tau^\Delta(t) > 0. \tag{3.3}$$

Next we define the function  $W(t)$  by

$$W(t) = \delta(t) \frac{a(t)(z^\Delta(t))^\gamma}{(z(\tau(t)))^\gamma}, \quad t \in [T, \infty)_{\mathbb{T}}. \tag{3.4}$$

Then on  $[T, \infty)_{\mathbb{T}}$ , we have  $W(t) > 0$ . From the basic knowledge of the time scale calculus that you can see in [3], we obtain

$$W^\Delta(t) = \frac{\delta(t)}{(z(\tau(t)))^\gamma} \left( a(t)(z^\Delta(t))^\gamma \right)^\Delta + a(\sigma(t))(z^\Delta(\sigma(t)))^\gamma \frac{(z(\tau(t)))^\gamma \delta^\Delta(t) - \delta(t) \left( (z(\tau(t)))^\gamma \right)^\Delta}{(z(\tau(t)))^\gamma (z(\tau(\sigma(t))))^\gamma}.$$

From (1.1) and (H<sub>4</sub>), we get

$$\left( a(t)(z^\Delta(t))^\gamma \right)^\Delta = \left( a(t) \left[ [x(t) + r(t)x(\tau(t))]^\Delta \right]^\gamma \right)^\Delta \leq -q(t)f(x(\tau(t))),$$

i.e.,

$$W^\Delta(t) \leq -Lq(t)\delta(t) \left( \frac{x(\tau(t))^\gamma}{z(\tau(t))^\gamma} \right) + \frac{a(\sigma(t))(z^\Delta(\sigma(t)))^\gamma \delta^\Delta(t)}{(z(\tau(\sigma(t))))^\gamma} - \frac{\delta(t)a(\sigma(t))(z^\Delta(\sigma(t)))^\gamma \left( (z(\tau(t)))^\gamma \right)^\Delta}{(z(\tau(t)))^\gamma (z(\tau(\sigma(t))))^\gamma} \tag{3.5}$$

$$\leq -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \frac{a(\sigma(t))(z^\Delta(\sigma(t)))^\gamma \delta^\Delta(t)}{(z(\tau(\sigma(t))))^\gamma}$$

$$\leq -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \frac{a(\tau(t))(z^\Delta(\tau(t)))^\gamma \delta^\Delta(t)}{(z(\tau(t)))^\gamma}$$

$$= -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + a(\tau(t))\delta^\Delta(t) \left( \frac{z^\Delta(\tau(t))}{z(\tau(t))} \right)^\gamma. \tag{3.6}$$

On the other hand, because

$$\begin{aligned} z(\tau(t)) &= z(\tau(T)) + \int_T^t z^\Delta(\tau(s)) \tau^\Delta(s) \Delta s \\ &= z(\tau(T)) + \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} (a(\tau(s)))^{\frac{1}{\gamma}} z^\Delta(\tau(s)) \tau^\Delta(s) \Delta s \\ &\geq \left( \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s \right) \left( a(\tau(t))(z^\Delta(\tau(t)))^\gamma \right)^{\frac{1}{\gamma}}, \end{aligned}$$

we get

$$\left(\frac{z^\Delta(\tau(t))}{z(\tau(t))}\right)^\gamma \leq \frac{1}{a(\tau(t))} \left(\int_T^t \left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s\right)^{-\gamma} = \frac{\eta^\gamma(\tau(t))}{a(\tau(t))}, \quad t \in [T, \infty)_\mathbb{T}. \tag{3.7}$$

Using (3.7) in (3.6), we have

$$W^\Delta(t) \leq -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \eta^\gamma(\tau(t))\delta^\Delta(t), \quad [T, \infty)_\mathbb{T}. \tag{3.8}$$

At last, integrating (3.8) from  $T$  to  $t$ , we obtain

$$0 < W(t) \leq W(T) - \int_T^t \left[ L\delta(s)q(s)(1-r(\tau(s)))^\gamma - \eta^\gamma(\tau(s))\delta^\Delta(s) \right] \Delta s,$$

which creates a contradiction to (3.1). This completes the proof. □

**Remark 1.** From Theorem 1, we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of  $\delta(t)$ .

Next, we give the conditions that guarantee every solution of (1.1) oscillates when (1.3) holds.

**Theorem 2.** Based on (1.3), assume that the conditions  $(H_1)$ - $(H_4)$ , (3.1) and (3.2) hold. If for every  $T \in [t_0, \infty)_\mathbb{T}$

$$\int_T^\infty \left[ \frac{1}{a(s)} \int_T^s \theta^\gamma(u)(1-r(\tau(u)))^\gamma q(u) \Delta u \right]^{\frac{1}{\gamma}} \Delta s = \infty \tag{3.9}$$

where

$$\theta(t) = \int_t^\infty \left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}} \Delta s, \tag{3.10}$$

then (1.1) is oscillatory on  $[t_0, \infty)_\mathbb{T}$ .

*Proof.* Assume that (1.1) has a nonoscillatory solution  $x$  on  $[t_0, \infty)_\mathbb{T}$ , then it is neither eventually positive nor eventually negative. Without loss of generality, we may assume that  $x(t) > 0$ , then  $x(\tau(t)) > 0$  for all  $t \in [t_1, \infty)_\mathbb{T}$ ,  $t_1 \in [t_0, \infty)_\mathbb{T}$ , it follows  $z(t) = x(t) + r(t)x(\tau(t)) > 0$  and

$$x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) - r(t)z(\tau(t)) \geq (1-r(t))z(t), \quad t \in [t_1, \infty)_\mathbb{T}.$$

The proof is similar when  $x$  is eventually negative. Since  $a(t)(z^\Delta(t))^\gamma$  is decreasing for all  $t \in [T, \infty)_\mathbb{T}$  and  $T \in [t_1, \infty)_\mathbb{T}$ , it is eventually of one sign and hence  $z^\Delta(t)$  is eventually of one sign. So we shall distinguish the following two cases to discuss:

(I)  $z^\Delta(t) > 0$  for  $t \geq T$ ; and

(II)  $z^\Delta(t) < 0$  for  $t \geq T$ .

Case (I). The proof that  $z^\Delta(t)$  is eventually positive is similar to that in Theorem 1, so it is omitted here.

Case (II). For  $s \geq t \geq T$ , we have

$$a(s)(-z^\Delta(s))^\gamma \geq a(t)(-z^\Delta(t))^\gamma,$$

then

$$-z^\Delta(s) \geq \left(\frac{a(t)}{a(s)}\right)^{\frac{1}{\gamma}} (-z^\Delta(t)). \tag{3.11}$$

Integrating (3.11) from  $t$  ( $t \geq T$ ) to  $u$  ( $u \geq t$ ) and letting  $u \rightarrow \infty$ , we have

$$z(t) \geq \left[ \int_t^\infty \left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}} \Delta s \right] (a(t))^{\frac{1}{\gamma}} (-z^\Delta(t)) = -\theta(t) a^{\frac{1}{\gamma}}(t) z^\Delta(t), \quad t \in [T, \infty)_\mathbb{T},$$

and thus

$$(z(t))^\gamma \geq -(\theta(t))^\gamma a(t)(z^\Delta(t))^\gamma \geq -(\theta(t))^\gamma a(T)(z^\Delta(T))^\gamma = b(\theta(t))^\gamma, \quad t \in [T, \infty)_{\mathbb{T}}, \tag{3.12}$$

where  $b = -a(T)(z^\Delta(T))^\gamma > 0$ . Applying (3.12) to Equation (1.1), we find

$$\begin{aligned} -\left(a(t)(z^\Delta(t))^\gamma\right)^\Delta &\geq Lq(t)(x(\tau(t)))^\gamma \geq Lq(t)(1-r(\tau(t)))^\gamma (z(\tau(t)))^\gamma \\ &\geq Lq(t)(1-r(\tau(t)))^\gamma (z(t))^\gamma \geq bL\theta^\gamma(t)q(t)(1-r(\tau(t)))^\gamma, \quad t \in [T, \infty)_{\mathbb{T}}. \end{aligned} \tag{3.13}$$

Integrating (3.13) from  $T$  to  $t$ , we have

$$\begin{aligned} -a(t)(z^\Delta(t))^\gamma &\geq -a(T)(z^\Delta(T))^\gamma + bL \int_T^t \theta^\gamma(s)q(s)(1-r(\tau(s)))^\gamma \Delta s \\ &\geq bL \int_T^t \theta^\gamma(s)q(s)(1-r(\tau(s)))^\gamma \Delta s. \end{aligned}$$

Therefore,

$$-z^\Delta(t) \geq \left[ \frac{bL}{a(t)} \int_T^t \theta^\gamma(s)(1-r(\tau(s)))^\gamma q(s) \Delta s \right]^{\frac{1}{\gamma}}. \tag{3.14}$$

Next integrating (3.14) from  $T$  to  $t$ , we obtain

$$-z(t) + z(T) \geq \int_T^t \left[ \frac{bL}{a(s)} \int_T^s \theta^\gamma(u)(1-r(\tau(u)))^\gamma q(u) \Delta u \right]^{\frac{1}{\gamma}} \Delta s \rightarrow \infty, \quad t \rightarrow \infty.$$

By (3.9), we have  $\lim_{t \rightarrow \infty} z(t) = -\infty$ , which contradicts  $z(t) > 0$ . This completes the proof.  $\square$

**Remark 2.** By Theorem 2, we get the sufficient condition of oscillation for Equation (1.1) when the condition (1.3) is satisfied, while the usual result existing is that the conditions (1.3) was established, then every solution of the Equation (1.1) is either oscillatory or converges to zero on  $[t_0, \infty)_{\mathbb{T}}$ .

**Theorem 3.** Based on (1.2), assume  $(H_1)$ - $(H_4)$  hold and  $\tau(\sigma(t)) = \sigma(\tau(t))$ . If there exists a positive  $\Delta$ -differentiable function  $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$  such that for every  $T \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ Lq(s)\delta(s)(1-r(\tau(s)))^\gamma - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta^\Delta(s))^2}{4\gamma\delta(s)\tau^\Delta(s)} (\eta^\sigma(\tau(s)))^{\gamma-1} \right] \Delta s = \infty, \tag{3.15}$$

where  $\eta$  is as the same as that in (3.2), then (1.1) is oscillatory on  $[t_0, \infty)_{\mathbb{T}}$ .

*Proof.* Assume that (1.1) has a nonoscillatory solution  $x$  on  $[t_0, \infty)_{\mathbb{T}}$ . Without loss of generality, we can assume that  $x(t) > 0$  and  $x(\tau(t)) > 0$  for all  $t \in [t_1, \infty)_{\mathbb{T}}$ ,  $t_1 \in [t_0, \infty)_{\mathbb{T}}$ . By the definition of  $z(t)$ , it follows  $z(t) > 0$ . The proof when  $x$  is eventually negative is similar. Proceeding as the proof of Theorem 1, we obtained (3.3) and (3.5). Using (3.3) in (3.5), we have that on  $[T, \infty)_{\mathbb{T}}$

$$W^\Delta(t) \leq -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \frac{\delta^\Delta(t)}{\delta(\sigma(t))} W(\sigma(t)) - \frac{\gamma\delta(t)a(\sigma(t))(z^\Delta(\sigma(t)))^\gamma z^\Delta(\tau(t))\tau^\Delta(t)}{(z(\tau(\sigma(t))))^{\gamma+1}}. \tag{3.16}$$

Also, since  $\left(a(t)(z^\Delta(t))^\gamma\right)^\Delta < 0$ , we have

$$a(\tau(t))(z^\Delta(\tau(t)))^\gamma \geq a(\sigma(t))(z^\Delta(\sigma(t)))^\gamma,$$

i.e.,

$$z^\Delta(\tau(t)) \geq \frac{(a(\sigma(t)))^{\frac{1}{\gamma}}}{(a(\tau(t)))^{\frac{1}{\gamma}}} z^\Delta(\sigma(t)). \tag{3.17}$$

Substituting (3.17) into (3.16), we obtain on  $[T, \infty)_{\mathbb{T}}$

$$W^\Delta(t) \leq -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \frac{\delta^\Delta(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^\Delta(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^{\frac{\gamma+1}{\gamma}}}(W(\sigma(t)))^{\frac{\gamma+1}{\gamma}}.$$

i.e.,

$$W^\Delta(t) \leq -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \frac{\delta^\Delta(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^\Delta(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^{\frac{\gamma+1}{\gamma}}}(W(\sigma(t)))^{\frac{1-\gamma}{\gamma}}(W(\sigma(t)))^2. \tag{3.18}$$

Now using inequality (3.7), we get

$$\begin{aligned} z(\tau(t)) &\geq \left( \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s \right) \left( a(\tau(t))(z^\Delta(\tau(t)))^\gamma \right)^{\frac{1}{\gamma}}, \\ &\geq \left( \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s \right) \left( a(t)(z^\Delta(t))^\gamma \right)^{\frac{1}{\gamma}}. \end{aligned}$$

Hence, we have

$$\frac{z(\tau(t))}{z^\Delta(t)} \geq \frac{a^{\frac{1}{\gamma}}(t)}{\eta(\tau(t))}.$$

This implies that on  $[T, \infty)_{\mathbb{T}}$

$$(W(t))^{\frac{1-\gamma}{\gamma}} = (\delta(t)a(t))^{\frac{1-\gamma}{\gamma}} \left( \frac{z(\tau(t))}{z^\Delta(t)} \right)^{\gamma-1} \geq (\delta(t)a(t))^{\frac{1-\gamma}{\gamma}} \frac{a^{\frac{\gamma-1}{\gamma}}(t)}{\eta^{\gamma-1}(\tau(t))} = \delta^{\frac{1-\gamma}{\gamma}}(t)\eta^{1-\gamma}(\tau(t)). \tag{3.19}$$

Using (3.19) in (3.18), we have on  $[T, \infty)_{\mathbb{T}}$  that

$$\begin{aligned} W^\Delta(t) &\leq -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \frac{\delta^\Delta(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ &\quad - \frac{\gamma\delta(t)\tau^\Delta(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^2}(\eta(\sigma(\tau(t))))^{1-\gamma}(W(\sigma(t)))^2 \\ &= -Lq(t)\delta(t)(1-r(\tau(t)))^\gamma + \frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^\Delta(t))^2}{4\gamma\delta(t)\tau^\Delta(t)}(\eta(\sigma(\tau(t))))^{\gamma-1} \\ &\quad - \left[ \frac{\sqrt{\gamma\delta(t)\tau^\Delta(t)(\eta(\sigma(t)))^{1-\gamma}}}{\delta(\sigma(t))\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}}W(\sigma(t)) - \frac{\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}\delta^\Delta(t)}}{2\sqrt{\gamma\delta(t)\tau^\Delta(t)(\eta(\sigma(\tau(t))))^{1-\gamma}}} \right]^2 \\ &\leq - \left[ Lq(t)\delta(t)(1-r(\tau(t)))^\gamma - \frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^\Delta(t))^2}{4\gamma\delta(t)\tau^\Delta(t)}(\eta(\sigma(\tau(t))))^{\gamma-1} \right]. \end{aligned}$$

Integrating both sides of this inequality from  $T$  to  $t$ , taking the limsup of the resulting inequality as  $t \rightarrow \infty$  and applying condition (3.15), we obtain a contradiction to the fact that  $W(t) > 0$  for  $t \in [T, \infty)_{\mathbb{T}}$ . This completes the proof.  $\square$

Using the same ideas as in the proof of Theorem 2, we can now obtain the following result based on (1.3).

**Theorem 4.** *Under the condition (1.3), assume that the conditions  $(H_1)$ - $(H_4)$ , (3.9) and (3.15) hold, then (1.1) is oscillatory on  $[t_0, \infty)_{\mathbb{T}}$ .*  $\square$

### 4. Application

Now we shall reformulate the above conditions which are sufficient for the oscillation of (1.1) when (1.2) holds on different time scales:

If  $\mathbb{T} = \mathbb{R}$ , Equation (1.1) becomes

$$\left( a(t)(z'(t))^\gamma \right)' + q(t)f(x(\tau(t))) = 0, \quad t \in [t_0, \infty), \tag{4.1}$$

and then conditions (3.1) and (3.15), respectively, become

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ L\delta(s)q(s)(1-r(\tau(s)))^\gamma - \delta'(s) \left( \int_{\tau(t_1)}^{\tau(s)} \left( \frac{1}{a(u)} \right)^{\frac{1}{\gamma}} du \right)^{-\gamma} \right] ds = \infty \tag{4.2}$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ Lq(s)(1-r(\tau(s)))^\gamma \delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta'(s))^2}{4\gamma\delta(s)\tau'(s)} \left( \int_{\tau(t_1)}^{\tau(s)} \left( \frac{1}{a(u)} \right)^{\frac{1}{\gamma}} du \right)^{1-\gamma} \right] ds = \infty. \tag{4.3}$$

The conditions (4.2) and (4.3) are new.

If  $\mathbb{T} = \mathbb{Z}$ , Equation (1.1) becomes

$$\Delta \left( a_n (\Delta z_n)^\gamma \right) + q_n f(x_{n-\sigma}) = 0, \quad n = 0, 1, 2, \dots, \tag{4.4}$$

and conditions (3.1) and (3.15), respectively, become

$$\limsup_{n \rightarrow \infty} \sum_{l=n_0}^n \left[ Lq_l \delta_l (1-r_{l-\sigma})^\gamma - \Delta \delta_l \left( \sum_{k=n_0-\sigma}^{l-\sigma-1} \left( \frac{1}{a_k} \right)^{\frac{1}{\gamma}} \right)^{-\gamma} \right] = \infty \tag{4.5}$$

and

$$\limsup_{n \rightarrow \infty} \sum_{l=n_0}^n \left[ Lq_l \delta_l (1-r_{l-\sigma})^\gamma - \frac{(a_{l-\sigma})^{\frac{1}{\gamma}} (\Delta \delta_l)^2}{4\gamma\delta_l} \left( \sum_{k=n_0-\sigma}^{l-\sigma} \left( \frac{1}{a_k} \right)^{\frac{1}{\gamma}} \right)^{1-\gamma} \right] = \infty. \tag{4.6}$$

At same time, the Theorems 1 and 3 are new for the case  $\mathbb{T} = \mathbb{Z}$ .

**Example 1.** Consider the second-order nonlinear delay 2-difference equations

$$\left( t^{\frac{2}{3}} \left( z^\Delta(t) \right)^{\frac{5}{3}} \right)^\Delta + t^{\frac{8}{5}} \left( x \left( \frac{t}{2} \right) \right)^{\frac{5}{3}} \left( 1 + x^2 \left( \frac{t}{2} \right) \right) = 0, \quad t \in \overline{2\mathbb{Z}}, \quad t \geq t_0 := 2, \tag{4.7}$$

where  $z(t) = x(t) + \frac{1}{2}x\left(\frac{t}{2}\right)$ . This gives

$$a(t) = t^{\frac{2}{3}}, \quad q(t) = t^{\frac{8}{5}}, \quad f(x) = x^{\frac{5}{3}}(1+x^2), \quad r(t) = \frac{1}{2}, \quad \tau(t) = \frac{t}{2}, \quad \gamma = \frac{5}{3}.$$



The conditions (H<sub>1</sub>)-(H<sub>3</sub>) are clearly satisfied, and (H<sub>4</sub>) holds with  $L = 1$ . Because

$$\int_2^t \left( \frac{1}{a(s)} \right)^{\frac{1}{\gamma}} \Delta s = \int_2^t s^{-\frac{2}{5}} \Delta s = \frac{t^{\frac{3}{5}} - 2^{\frac{3}{5}}}{\frac{3}{5} - 1} \rightarrow \infty \text{ as } t \rightarrow \infty, t \geq 2,$$

(1.2) is satisfied. Now let  $\delta(t) = t^{\frac{3}{5}}$  for all  $t > s \geq 2$ , and then

$$\eta(\tau(t)) = \left( \int_2^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s \right)^{-1} = \left( \frac{1}{2^{\frac{3}{5}}} \int_2^t s^{-\frac{2}{5}} \Delta s \right)^{-1} = 2^{\frac{3}{5}} \frac{2^{\frac{3}{5}} - 1}{t^{\frac{3}{5}} - 2^{\frac{3}{5}}}.$$

Thus when  $t \rightarrow \infty$ , we have

$$\int_2^t \left[ L\delta(s)q(s)(1-r(\tau(s)))^\gamma - \eta^\gamma(\tau(s))\delta^\Delta(s) \right] \Delta s = \int_2^t \left[ \frac{1}{2^{\frac{3}{5}}s} - 2 \left( \frac{2^{\frac{3}{5}} - 1}{s^{\frac{3}{5}} - 2^{\frac{3}{5}}} \right)^{\frac{5}{3}} \left( s^{\frac{3}{5}} \right)^\Delta \right] \Delta s \rightarrow \infty.$$

It is easy to see that (3.1) is satisfied as well. Altogether, the Equation (4.7) is oscillatory by Theorem 1.

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