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A New Conjugate Gradient Projection Method for Solving Stochastic Generalized Linear Complementarity Problems

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Abstract

In this paper, a class of the stochastic generalized linear complementarity problems with finitely many elements is proposed for the first time. Based on the Fischer-Burmeister function, a new conjugate gradient projection method is given for solving the stochastic generalized linear complementarity problems. The global convergence of the conjugate gradient projection method is proved and the related numerical results are also reported.

Keywords

Stochastic Generalized Linear Complementarity Problems, Fischer-Burmeister Function, Conjugate Gradient Projection Method, Global Convergence

1. Introduction

Suppose (Ω_1, F, G, P) is a probability space with $\Omega_1 \subseteq \Re^n$; P is a known probability distribution. The stochastic generalized linear complementarity problems (denoted by SGLCP) is to find $x \in \Re^n$, such that

$$F(x,\omega) := M_1(\omega)x + q_1(\omega) \geq 0, G(x,\omega) := M_2(\omega)x + q_2(\omega) \geq 0, F^{\mathrm{T}}(x,\omega)G(x,\omega) = 0, \tag{1}$$

where $M_1(\omega), M_2(\omega) \in \Re^{n\times n}$ and $q_1(\omega), q_2(\omega) \in \Re^n$ for $\omega \in \Omega_1$, are random matrices and vectors. When $G(x,\omega) = x$, stochastic generalized linear complementarity problems reduce to the classic Stochastic Linear Complementarity Problems (SLCP), which has been studied in [1]-[7]. Generally, they usually apply the Expected Value (EV) method and Expected Residual Minimization (ERM) method to solve this kind of problem.

If Ω_1 only contains a single realization, then (1) reduces to the following standard Generalized Linear Complementarity Problem (GLCP), which is to find a vector $x \in \Re^n$ such that

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$$F(x) := M_1 x + q_1 \ge 0, G(x) := M_2 x + q_2 \ge 0, F^{\mathrm{T}}(x) G(x) = 0,$$

where $M_1, M_2 \in \Re^{n\times n}$ and $q_1, q_2 \in \Re^n$.

In this paper, we consider the following generalized stochastic linear complementarity problems. Denote $\Omega_1 = \{\omega_1, \omega_2, \cdots, \omega_m\}$, to find an $x \in \Re^n$ such that

$$\begin{aligned} &F(x,\omega_i) := M_1(\omega_i)x + q_1(\omega_i) \ge 0, \\ &G(x,\omega_i) := M_2(\omega_i)x + q_2(\omega_i) \ge 0, \quad i = 1,\cdots,m,\ m > 1. \\ &F^{\mathrm{T}}(x,\omega_i)\cdot G(x,\omega_i) = 0. \end{aligned} \tag{2}$$

Let $\bar{M}_j = \sum_{i=1}^{m} p_i M_j(\omega_i), \bar{q}_j = \sum_{i=1}^{m} p_i q_j(\omega_i)$, where $p_i = P(\omega_i \in \Omega_1) > 0$, $i = 1,\cdots,m$, $j = 1,2$. Then (2) is equivalent to (3) and (4)

$$\bar{M}_1 x + \bar{q}_1 \ge 0, \bar{M}_2 x + \bar{q}_2 \ge 0, \left(\bar{M}_1 x + \bar{q}_1\right)^{\mathrm{T}} \cdot \left(\bar{M}_2 x + \bar{q}_2\right) = 0, \tag{3}$$

$$\begin{aligned} &M_1(\omega_i)x + q_1(\omega_i) \ge 0, \\ &M_2(\omega_i)x + q_2(\omega_i) \ge 0, \quad i = 1,\cdots,m. \end{aligned} \tag{4}$$

In the following of this paper, we consider to give a new conjugate gradient projection method for solving (2). The method is based on a suitable reformulation. Base on the Fischer-Burmeister function, x is a solution of (3) $\Leftrightarrow \phi(x) = 0$, where

$$\phi(x) = \begin{pmatrix} \phi\left(\left(\bar{M}_1 x + \bar{q}_1\right)_1, \left(\bar{M}_2 x + \bar{q}_2\right)_1\right) \\ \vdots \\ \phi\left(\left(\bar{M}_1 x + \bar{q}_1\right)_n, \left(\bar{M}_2 x + \bar{q}_2\right)_n\right) \end{pmatrix}.$$

Define

$$\Psi(x) = \frac{1}{2}\|\phi(x)\|^2.$$

Then solving (3) is equivalent to find a global solution of the minimization problem

$$\min_{x\in\Re^n} \Psi(x).$$

So, (3) and (4) can be rewritten as

$$H(x,y) = 0, \quad y \ge 0, \tag{5}$$

where

$$H(x,y) = \begin{pmatrix} \phi(x) \\ M_1(\omega_1)x + q_1(\omega_1) - y_1 \\ \vdots \\ M_1(\omega_m)x + q_1(\omega_m) - y_m \\ M_2(\omega_1)x + q_2(\omega_1) - y_{m+1} \\ \vdots \\ M_2(\omega_m)x + q_2(\omega_m) - y_{2m} \end{pmatrix},$$

$y = \left[y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \cdots, y_{2m}^{\mathrm{T}}\right]^{\mathrm{T}} \in \Re^{2m\times n}$ is slack variable with $y_i \in \Re^n$, $i = 1,\cdots,2m$.

Let $x = x' - x''$, where $x', x'' \in \Re^n$ and $x', x'' \ge 0$. Then we know that $H(x', x'', y) = 0$ has $(2m+2)n$ equations with $(2m+2)n$ variables.

Let $t = (x', x'', y) \in \Re_+^{(2m+2)n}$ and define a merit function of (5) by

$$\theta(t) = \frac{1}{2}\|H(t)\|^2 .$$

If (2) has a solution, then solving (5) is equivalent to find a global solution of the following minimization problem

$$\min \theta(t) \quad s.t. \quad t \in \Omega \tag{6}$$

where $\Omega = \left\{ t \middle| t \in \Re_+^{(2m+2)n} \right\}$.

2. Preliminaries

In this section, we give some Lemmas, which are taken from [8]-[10].

Lemma 1. Let P be the projection onto Ω, let $t(s) = P(t+s)$ for given $t \in \Omega$ and $s \in \Re^{(2m+2)n}$, then
1) $\langle t(s) - (t+s), y - t(s) \rangle \geq 0$, for all $y \in \Omega$.
2) P is a non-expansive operator, that is, $\|P(y) - P(x)\| \leq \|y - x\|$ for all $x, y \in \Re^{(2m+2)n}$.
3) $\langle -s, t - t(s) \rangle \geq \|t(s) - t\|^2$.

Lemma 2. Let $\nabla_\Omega \theta(t)$ be the projected gradient of θ at $t \in \Omega$.
1) $\min\{\nabla\theta(t), v : v \in T(t), \|v\| \leq 1\} = -\|\nabla_\Omega \theta(t)\|$.
2) The mapping $\|\nabla_\Omega \theta(\cdot)\|$ is lower semicontinuous on Ω, that is, if $\lim_{k\to\infty} t_k \to t$, then

$$\|\nabla_\Omega \theta(t)\| \leq \liminf_{k\to\infty} \|\nabla_\Omega \theta(t_k)\| .$$

3) The point $t^* \in \Omega$ is a stationary point of problem (6) ⇔ $\nabla_\Omega \theta(t^*) = 0$.

3. The Conjugate Gradient Projection Method and Its Convergence Analysis

In this section, we give a new conjugate gradient projection method and give some discussions about this method.

Given an iterate $t_k \in \Omega = \left\{ t \middle| t \in \Re_+^{(2m+2)n} \right\}$, we let $\overline{t_k}(s_k) = P[t_k - \nabla\theta(t_k)]$,

$$t_{k+1} = t_k(s_k) = P[t_k + s_k], \tag{7}$$

where $s_k = \begin{cases} -\nabla\theta(t_k) & k = 1 \\ -\nabla\theta(t_k) + \beta_k d_{k-1} & k > 1 \end{cases}$. Inspired by the literature [8]-[11], we take

$$|\beta_k| = \frac{\|\overline{t_k}(s_k) - t_k\|^2}{(1+\lambda)\|\nabla\theta(t_k)\|\|d_{k-1}\|}, \tag{8}$$

with $\lambda > 0$.

And d_k is defined by

$$d_k = t_k(s_k) - t_k . \tag{9}$$

Method 1. Conjugate Gradient Projection Method (CGPM)

Step 0: Let $t_1 \in \Omega$, $0 \leq \varepsilon \leq 1$, $\sigma_1, \sigma_2 \in (0,1)$, $\beta_1 = 0$, $d_0 = 0$, set $k = 1$.

Step 1: Compute α_k, such that

$$\theta(t_k + \alpha_k d_k) \leq \theta(t_k) + \sigma_1 \alpha_k \nabla\theta(t_k)^{\mathrm{T}} d_k ,$$

$$\nabla\theta(t_k + \alpha_k d_k)^{\mathrm{T}} d_k \geq \sigma_2 \nabla\theta(t_k)^{\mathrm{T}} d_k .$$

Set $t_{k+1} = t_k + \alpha_k d_k$.

Step 2: If $\|t_k - t_k(s_k)\| \leq \varepsilon$, stop, $t^* = t_k(s_k)$.

Step 3: Let $k := k+1$, and go to Step 1.

In order to prove the global convergence of the Method 1, we give the following assumptions.

Assumptions 1

1) $\theta(t)$ has a lower bound on the level set $L_0=\left\{t_1\in\Re^{(2m+2)n}\,\middle|\,\theta(t)\le\theta(t_1)\right\}$, where t_1 is initial point.

2) $\theta(t)$ is continuously differentiable on the L_0, and its gradient is Lipschitz continuous, that is, there exists a positive constant L such that

$$\|g(u)-g(v)\|\le L\|u-v\|\quad \forall u,v\in L_0 .$$

Lemma 3. If t_k is not the stability point of (6), $t_k\ne t_k(s_k)$, then search direction d_k generated by (9) descent direction, which is $\langle\nabla\theta(t_k),d_k\rangle\le-\frac{\lambda}{1+\lambda}\|\nabla\theta(t_k)\|^2<0$.

Proof. From (7), Lemma 1, and (8), we have

$$\begin{aligned}\langle\nabla\theta(t_k),d_k\rangle&=\langle\nabla\theta(t_k),t_k(s_k)-t_k\rangle\\&=\left[\langle\nabla\theta(t_k),t_k(s_k)-\overline{t_k}(s_k)\rangle+\langle\nabla\theta(t_k),\overline{t_k}(s_k)-t_k\rangle\right]\\&\le\|\nabla\theta(t_k)\|\|t_k(s_k)-\overline{t_k}(s_k)\|-\langle\nabla\theta(t_k),t_k-\overline{t_k}(s_k)\rangle\\&\le|\beta_k|\|\nabla\theta(t_k)\|\|d_{k-1}\|-\|\overline{t_k}(s_k)-t_k\|^2\\&\le\left(\frac{1}{1+\lambda}-1\right)\|\overline{t_k}(s_k)-t_k\|^2\\&\le\frac{-\lambda}{1+\lambda}\|\nabla\theta(t_k)\|^2<0.\end{aligned}$$

Lemma 4. [11] Suppose that Assumptions 1 holds. Let $\theta(t)$ continuously differentiable and lower bound on the Ω, $\nabla\theta(t)$ is uniformly continuous on the Ω and $\{\nabla\theta(t_k)\}$ is bounded, then $\{t_k\}$ generated by Method 1 are satisfied

$$\lim_{k\to\infty}\|t_k-t_k(s_k)\|=0,\quad \lim_{k\to\infty}\|t_k-\overline{t_k}(s_k)\|=0 .$$

Theorem 1. Let $\theta(t)$ continuously differentiable and lower bound on the Ω, $\nabla\theta(t)$ is uniformly continuous on the Ω, $\{t_k\}$ is a sequence generated by Method 1, then $\lim_{k\to\infty}\|\nabla_\Omega\theta(t_k)\|=0$, and any accumulation point of $\{t_k\}$ is a stationary point of (6).

Proof. By Lemma 2, we have $\forall\varepsilon>0$, $\exists v_k\in T_\Omega(t_k)$, $\|v_k\|\le1$, satisfy

$$\|\nabla_\Omega\theta(t_k)\|\le\langle-\nabla\theta(t_k),v_k\rangle+\varepsilon, \tag{10}$$

for $\forall z\in\Omega$, by Lemma 1, we know that $\langle t_k(s_k)-(t_k+s_k),z-t_k(s_k)\rangle\ge0$, and we have $\langle s_k,z-t_k(s_k)\rangle\le\langle t_k(s_k)-t_k,z-t_k(s_k)\rangle\le\|t_k(s_k)-t_k\|\|z-t_k(s_k)\|$, so,

$$\langle s_k,z-t_k(s_k)\rangle\le\|t_k(s_k)-t_k\|\|z-t_k(s_k)\|. \tag{11}$$

Let $v_{k+1}=z-t_k(s_k)\in T_\Omega(t_{k+1})$, $\|v_{k+1}\|\le1$, from (11), we have

$$\langle s_k,v_{k+1}\rangle=\langle-\nabla\theta(t_k)+\beta_kd_{k-1},v_{k+1}\rangle\le\|t_k(s_k)-t_k\| .$$

By the above formula, (8) and Lemma 1, we get

$$\begin{aligned}\langle-\nabla\theta(t_k),v_{k+1}\rangle&\le\|t_k(s_k)-t_k\|+|\beta_k|\|d_{k-1}\|\\&\le\|t_k(s_k)-t_k\|+\frac{1}{(1+\lambda)\|\nabla\theta(t_k)\|}\|\overline{t_k}(s_k)-t_k\|^2\\&\le\|t_k(s_k)-t_k\|+\frac{1}{1+\lambda}\|\overline{t_k}(s_k)-t_k\|.\end{aligned}$$

Taking limit on both sides and by Lemma 4, we know that

$$\lim_{k\to\infty}\sup\left\langle -\nabla\theta(t_k), v_{k+1}\right\rangle = 0. \quad (12)$$

Because

$$\begin{aligned}\left\langle -\nabla\theta(t_k(s_k)), v_{k+1}\right\rangle &= \left\langle \nabla\theta(t_k) - \nabla\theta(t_k(s_k)), v_{k+1}\right\rangle + \left\langle -\nabla\theta(t_k), v_{k+1}\right\rangle \\ &\le \left\|\nabla\theta(t_k) - \nabla\theta(t_k(s_k))\right\| + \left\langle -\nabla\theta(t_k), v_{k+1}\right\rangle\end{aligned} \quad (13)$$

and Lemma 4, we have

$$\lim_{k\to\infty}\left\|t_k - t_k(s_k)\right\| = 0. \quad (14)$$

By (12), (13), (14) and $\nabla\theta(t)$ is uniformly continuous on the Ω, we get

$$\lim_{k\to\infty}\sup\left\langle -\nabla\theta(t_k(s_k)), v_{k+1}\right\rangle = 0.$$

By (10), we know that

$$\lim_{k\to\infty}\left\|\nabla_\Omega\theta(t_k)\right\| = 0. \quad (15)$$

Let $\lim_{k\in N_0, k\to\infty} t_k = t$, where $N_0 \subseteq N$, by Lemma 2 and (15), we have

$$\left\|\nabla_\Omega\theta(t)\right\| \le \lim_{k\in N_0, k\to\infty}\inf\left\|\nabla_\Omega\theta(t_k)\right\| = 0.$$

From Lemma 2 3), we get any accumulation point of $\{t_k\}$ is a stationary point of (6).

4. Numerical Results

In this section, we give the numerical results of the conjugate gradient projection method for the following given test problems, which are all given for the first time. We present different initial point t_0, which indicates that Method 1 is global convergence.

Throughout the computational experiments, according to Method 1 for determining the parameters, we set the parameters as

$$\sigma_1 = 0.49,\ \sigma_2 = 0.5,\ \lambda = 1.067.$$

The stopping criterion for the method is $\|g_k\| \le 10^{-6}$ or $k_{\max} = 100000$.

In the table of the test results, t_0 denotes initial point, x^* denotes the solution, val denotes the final value of $\theta(t) = \frac{1}{2}\|H(t)\|^2$, Itr denotes the number of iteration.

Example 1. Considering SGLCP with

$$M_1(\omega) = \begin{pmatrix} \frac{3}{2}+\omega & -1 & 0 \\ -1 & \frac{3}{2}+\omega & -1 \\ 0 & -1 & \frac{3}{2}+\omega \end{pmatrix},\quad q_1(\omega) = \begin{pmatrix} \frac{1}{2}+\omega \\ \frac{1}{2}+\omega \\ \frac{1}{2}+\omega \end{pmatrix},$$

$$M_2(\omega) = \begin{pmatrix} \frac{5}{2}+\omega & -1 & 0 \\ -1 & \frac{5}{2}+\omega & -1 \\ 0 & -1 & \frac{5}{2}+\omega \end{pmatrix},\quad q_2(\omega) = \begin{pmatrix} 1+\omega \\ 1+\omega \\ 1+\omega \end{pmatrix},$$

$$\Omega_1 = \{\omega_1, \omega_2\} = \{0,1\} \text{ and } p_i = P(\omega_i \in \Omega_1) = 0.5,\ i = 1,2.$$

The test results are listed in "**Table 1**" using different initial points.

Table 1. Results of the numerical Example 1-2 using method 1.

Problem	t_0	x^*	val	Itr
Example 1	$0.5\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	1465
	$(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	1701
	$5\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	2670
	$10\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	3261
	$20\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	3847
	$50\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	4704
Example 2	$0.5\times(1,1,\cdots,1)$	$(-0.3747,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	62788
	$(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	65528
	$5\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	66962
	$10\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	100,000
	$20\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	100,000
	$50\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	100,000

Example 2. Considering SGLCP with

$$M_1(\omega)=\begin{pmatrix} \frac{1}{2}+\omega & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & \frac{1}{2}+\omega & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & \frac{1}{2}+\omega & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & \frac{1}{2}+\omega & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}+\omega & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+\omega & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+\omega \end{pmatrix},\quad q_1(\omega)=\begin{pmatrix} -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \end{pmatrix},$$

$$M_1(\omega)=\begin{pmatrix} \frac{3}{2}+\omega & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & \frac{3}{2}+\omega & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & \frac{3}{2}+\omega & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & \frac{3}{2}+\omega & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & \frac{3}{2}+\omega & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2}+\omega & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2}+\omega \end{pmatrix},\quad q_1(\omega)=\begin{pmatrix} -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \end{pmatrix},$$

$$\Omega_1 = \{\omega_1, \omega_2\} = \{0,1\} \quad \text{and} \quad p_i = P(\omega_i \in \Omega_1) = 0.5, \quad i = 1,2.$$

The test results are listed in "**Table 1**" using different initial points.

5. Conclusion

In this paper, we present a new conjugate gradient projection method for solving stochastic generalized linear complementarity problems. The global convergence of the method is analyzed and numerical results show that Method 1 is effective. In future work, large-scale stochastic generalized linear complementarity problems need to be studied and developed.

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