Properties of Solutions of Kolmogorov-Fisher Type Biological Population Task with Variable Density

Muhamediyeva Dildora

CDSPHSC under TUIT, Tashkent, Uzbekistan
Email: matematichka@inbox.ru

Abstract

In this paper, we discussed population model of two competing populations with non-linear double diffusion and variable density which described by nonlinear system of competing individuals. We identify new properties, such as finite speed of propagation, and localization of the outbreaks in a specific area.

Keywords

Model of Biological Population, Reaction-Diffusion, Double Nonlinearity, Self-Similar Solution, Variable Density, Fast Diffusion, Low Diffusion

1. Introduction

Population models are studied for a long time. The first such work was done by Gause G.F. and Fisher R.D., and mathematical studies were performed by Kolmogorov, Petrovskii (KPP) and Piskunov (1937) in the famous paper [1]-[4]. They were interested in the behavior of the speed of the wave solutions and the resulting estimate of the speed of wave propagation.

Then there were other models of the population [5]-[8]. In recent years, intensive study of nonlinear models was based on diffusion and revealed new properties of finite speed of propagation of diffusion waves (see [3] and the literature given there). We have proposed a population model of two competing populations with non-linear double diffusion and variable density that are described by nonlinear system of competing individuals. We identify new properties, such as finite speed of propagaton, and localization of the outbreaks in a specific area. In particular, in the critical case, the rate type CPT generalizes their result.

Statement of the Task

In this paper, we investigate the properties of solutions of biological population task of Fisher-Kolmogorov type in the case of variable density. The main research method is a self-similar approach. Considering in the field $Q = \left\{ (t,x) : 0 < t < \infty, x \in R^N \right\}$, there is a parabolic system of two quasilinear equations of reaction-diffusion

$$\begin{cases} \dfrac{\partial(\rho(x)u_1)}{\partial t} = div\left(D_1 |x|^n u_2^{m_1-1} |\nabla u_1|^{p-2} \nabla u_1 \right) + \rho(x)k_1 u_1 \left(1 - u_1^{\beta_1}\right), \\[2mm] \dfrac{\partial(\rho(x)u_2)}{\partial t} = div\left(D_2 |x|^n u_1^{m_2-1} |\nabla u_2|^{p-2} \nabla u_2 \right) + \rho(x)k_2 u_2 \left(1 - u_2^{\beta_2}\right), \end{cases} \tag{1}$$

$$u_1\big|_{t=0} = u_{10}(x), \quad u_2\big|_{t=0} = u_{20}(x), \tag{2}$$

which describes the process of biological population of Kolmogorov-Fisher in a nonlinear two-component environment, and mutual diffusion coefficients which are respectively equal to $D_1 |x|^n u_2^{m_1-1} |\nabla u_1|^{p-2} \nabla u_1$, $D_2 |x|^n u_1^{m_2-1} |\nabla u_2|^{p-2} \nabla u_2$. Numeric parameters $m_1, m_2, n, p, \beta_1, \beta_2$, D_1, D_2 are positive real numbers, and $\nabla(.) - grad(.)$, $\beta_1, \beta_2 \geq 1$, $\rho(x) = |x|^{-l}$, $x \in R^N$ $l > 0$; $u_1 = u_1(t,x) \geq 0$, $u_2 = u_2(t,x) \geq 0$ is desired solutions.

We study properties of solutions to problem (1), (2) based on the self-similar analysis of solutions of a system of equations constructed by the method of nonlinear splitting and a reference equations and bringing the system (1) for radially symmetric mind.

Note that replacing in (1)

$$u_1(t,x) = e^{-k_1 t} v_1\left(\tau(t), \varphi(x)\right), \quad u_2(t,x) = e^{-k_2 t} v_2\left(\tau(t), \varphi(x)\right)$$

leads to the form

$$\begin{cases} \dfrac{\partial(\rho(x)v_1)}{\partial \tau} = div\left(D_1 |x|^n v_2^{m_1-1} |\nabla v_1|^{p-2} \nabla v_1 \right) - \rho(x)k_1 e^{[(\beta_1-p+2)k_1 - (m_1-1)k_2]t} v_1^{\beta_1+1}, \\[2mm] \dfrac{\partial(\rho(x)v_2)}{\partial \tau} = div\left(D_2 |x|^n v_1^{m_2-1} |\nabla v_2|^{p-2} \nabla v_2 \right) - \rho(x)k_2 e^{[(\beta_2-p+2)k_2 + (m_2-1)k_1]t} v_2^{\beta_2+1}, \end{cases} \tag{3}$$

$$v_1\big|_{t=0} = v_{10}(x), \quad v_2\big|_{t=0} = v_{20}(x). \tag{4}$$

If $k_1\left(p - (m_1+1)\right) = k_2\left(p - (m_2+1)\right)$, choosing

$$\tau(t) = \frac{e^{[(m_1-1)k_2+(p-2)k_1]t}}{(m_1-1)k_2 + (p-2)k_1} = \frac{e^{[(m_2-1)k_1+(p-2)k_2]t}}{(m_2-1)k_1 + (p-2)k_2}, \quad \varphi(x) = |x|^{p_1}/p_1, \quad p_1 = \left(p - (n+l)\right)/p,$$

we get the following system of equations:

$$\begin{cases} \dfrac{\partial w_1}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_1 w_2^{m_1-1} |\nabla w_1|^{p-2} \nabla w_1 \right) - a_1 \tau^{b_1} w_1^{\beta_1+1}, \\[2mm] \dfrac{\partial w_2}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_2 w_1^{m_2-1} |\nabla w_2|^{p-2} \nabla w_2 \right) - a_2 \tau^{b_2} w_2^{\beta_2+1}, \end{cases} \tag{5}$$

where $a_1 = k_1\left((p-2)k_1 + (m_1-1)k_2\right)^{b_1}$, $b_1 = \dfrac{(\beta_1-(p-2))k_1 - (m_1-1)k_2}{(p-2)k_1 + (m_1-1)k_2}$,

$$a_2 = k_2\left((m_2-1)k_1 + (p-2)k_2\right)^{b_2}, \quad b_2 = \frac{(\beta_2-(p-2))k_2 - (m_2-1)k_1}{(m_2-1)k_1 + (p-2)k_2}.$$

If $b_i = 0$, and $a_i(t) = const$, $i = 1, 2$, system has the form:

$$\begin{cases} \dfrac{\partial w_1}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_1 w_2^{m_1-1} |\nabla w_1|^{p-2} \nabla w_1 \right) - a_1 w_1^{\beta_1+1}, \\[2mm] \dfrac{\partial w_2}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_2 w_1^{m_2-1} |\nabla w_2|^{p-2} \nabla w_2 \right) - a_2 w_2^{\beta_2+1}, \end{cases} \tag{6}$$

A significant role in the study of the Cauchy problem and boundary problems for Equations (1) has self-similar solutions. Under self-similar solution we will understand as particular solutions of Equation (1), depending on the combination of t and x. Knowledge of them plays a sometimes crucial role in the study of various properties of solutions of the original equations.

Below we describe one way of obtaining self-similar system for the system of Equations (5). It consists in the following. We find first the solution of a system of ordinary differential equations

$$\begin{cases} \dfrac{d\overline{w}_1}{d\tau} = -a_1 \overline{w}_1^{\beta_1+1}, \\ \dfrac{d\overline{w}_2}{d\tau} = -a_2 \overline{w}_2^{\beta_2+1}, \end{cases}$$

in the form

$$\overline{w}_1(\tau) = (\tau(t))^{-\gamma_1}, \ \gamma_1 = \frac{1}{\beta_1}, \ \overline{w}_2(\tau) = (\tau(t))^{-\gamma_2}, \ \gamma_2 = \frac{1}{\beta_2},$$

for the case of $b_i = 0$, and $a_i(t) = \text{const}$, $i = 1,2$. And in the case of $b_i \neq 0$, and $a_i(t) = \text{const}$, $i = 1,2$ we solve a system of ordinary differential equations

$$\begin{cases} \dfrac{d\overline{w}_1}{d\tau} = -a_1 \tau^{b_1} \overline{w}_1^{\beta_1+1}, \\ \dfrac{d\overline{w}_2}{d\tau} = -a_2 \tau^{b_2} \overline{w}_2^{\beta_2+1}, \end{cases}$$

in the form

$$\overline{w}_1(\tau) = (\tau(t))^{-\gamma_1}, \ \gamma_1 = \frac{b_1+1}{\beta_1}, \ \overline{w}_2(\tau) = (\tau(t))^{-\gamma_2}, \ \gamma_2 = \frac{b_2+1}{\beta_2},$$

then the solution of system (5) is sought in the form

$$\begin{aligned} v_1(t,x) &= \overline{w}_1(\tau) z_1(\tau(t), \varphi(|x|)), \\ v_2(t,x) &= \overline{w}_2(\tau) z_2(\tau(t), \varphi(|x|)), \end{aligned} \tag{7}$$

and $\tau = \tau(t)$ is selected so

$$\tau_1(\tau) = \int_0^\tau \overline{v}_1^{(p-2)}(t) \overline{v}_2^{(m_1-1)}(t) dt$$

$$= \begin{cases} \dfrac{1}{1-[\gamma_1(p-2)+\gamma_2(m_1-1)]}(T+\tau)^{1-[\gamma_1(p-2)+\gamma_2(m_1-1)]}, & \text{if } 1-[\gamma_1(p-2)+\gamma_2(m_1-1)] \neq 0, \\ \ln(T+\tau), & \text{if } 1-[\gamma_1(p-2)+\gamma_2(m_1-1)] = 0, \\ (T+\tau), & \text{if } p=2 \text{ and } m_1 = 1, \end{cases}$$

if $\gamma_1(p-2)+\gamma_2(m_1-1) = \gamma_2(p-2)+\gamma_1(m_2-1)$.

Then for $z_i(\tau, \varphi(|x|)), i=1,2$ we get the system of equations:

$$\begin{cases} \dfrac{\partial z_1}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_1 z_2^{m_1-1} |\nabla z_1|^{p-2} \nabla z_1\right) + \psi_1\left(z_1 - z_1^{\beta_1+1}\right), \\ \dfrac{\partial z_2}{\partial \tau} = \varphi^{1-s} div\left(\varphi^{s-1} D_2 z_1^{m_2-1} |\nabla z_2|^{p-2} \nabla z_2\right) + \psi_2\left(z_2 - z_2^{\beta_2+1}\right), \end{cases} \tag{8}$$

where

$$\psi_1 = \begin{cases} \dfrac{1}{\left(1-\left[\gamma_1\left(p-2\right)+\gamma_2\left(m_1-1\right)\right]\right)\tau}, & \text{if } 1-\left[\gamma_1\left(p-2\right)+\gamma_2\left(m_1-1\right)\right]>0, \\ \gamma_1 c_1^{-\left(1-\left[\gamma_1\left(p-2\right)+\gamma_2\left(m_1-1\right)\right]\right)}, & \text{if } 1-\left[\gamma_1\left(p-2\right)+\gamma_2\left(m_1-1\right)\right]=0, \end{cases} \tag{9}$$

$$\psi_2 = \begin{cases} \dfrac{1}{\left(1-\left[\gamma_2\left(p-2\right)+\gamma_1\left(m_2-1\right)\right]\right)\tau}, & \text{if } 1-\left[\gamma_2\left(p-2\right)+\gamma_1\left(m_2-1\right)\right]>0, \\ \gamma_2 c_1^{-\left(1-\left[\gamma_2\left(p-2\right)+\gamma_1\left(m_2-1\right)\right]\right)}, & \text{if } 1-\left[\gamma_2\left(p-2\right)+\gamma_1\left(m_2-1\right)\right]=0. \end{cases}$$

If $1-\left[\gamma_1\left(p-2\right)+\gamma_2\left(m_1-1\right)\right]=0$, self-similar solution of system (9) has the form

$$z_i\left(\tau(t),\varphi\right)=f_i\left(\xi\right),\ i=1,2,\ \xi=\varphi\left(|x|\right)\Big/\left[\tau(t)\right]^{1/p}, \tag{10}$$

Then substituting (10) into (8) with respect to $f_i\left(\xi\right)$ gets a self-similar system of equations

$$\begin{cases} \xi^{1-s}\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}f_2^{m_1-1}\left|\dfrac{\mathrm{d}f_1}{\mathrm{d}\xi}\right|^{p-2}\dfrac{\mathrm{d}f_1}{\mathrm{d}\xi}\right)+\dfrac{\xi}{p}\dfrac{\mathrm{d}f_1}{\mathrm{d}\xi}+\mu_1 f_1\left(1-f_1^{\beta_1}\right)=0, \\ \xi^{1-s}\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}f_1^{m_2-1}\left|\dfrac{\mathrm{d}f_2}{\mathrm{d}\xi}\right|^{p-2}\dfrac{\mathrm{d}f_2}{\mathrm{d}\xi}\right)+\dfrac{\xi}{p}\dfrac{\mathrm{d}f_2}{\mathrm{d}\xi}+\mu_2 f_2\left(1-f_2^{\beta_2}\right)=0. \end{cases} \tag{11}$$

where $\mu_1=\dfrac{1}{\left(1-\left[\gamma_1\left(p-2\right)+\gamma_2\left(m_1-1\right)\right]\right)}$ и $\mu_2=\dfrac{1}{\left(1-\left[\gamma_2\left(p-2\right)+\gamma_1\left(m_2-1\right)\right]\right)}$.

System (11) has an approximate solution of the form

$$\bar{f}_1=A\left(a-\xi^\gamma\right)^{n_1},\ \gamma=p/\left(p-1\right),\ \bar{f}_2=B\left(a-\xi^\gamma\right)^{n_2},$$

where *A and B* are constants and

$$n_1=\dfrac{\left(p-1\right)\left(p-\left(m_1+1\right)\right)}{\left(p-2\right)^2-\left(m_1-1\right)\left(m_2-1\right)},\quad n_2=\dfrac{\left(p-1\right)\left(p-\left(m_2+1\right)\right)}{\left(p-2\right)^2-\left(m_1-1\right)\left(m_2-1\right)}.$$

In this paper, on the basis of the aforesaid methods, we studied qualitative properties of solutions of the system (1), solved the problem of choosing the initial approximation for iterative, leading to fast convergence to the solution of the Cauchy problem (1), (2), depending on the values of numerical parameters and initial data. For this purpose, as the initial approximation was used, we found the asymptotic representation of the solution. This has allowed to perform numerical experiments and visualization of the process described by system (1), depending on the values included in the system of numeric parameters.

2. Construction of Upper Solutions

Let us build an upper solution for system (11).

Note that the functions $\bar{f}_1(\xi),\bar{f}_2(\xi)$ have properties

$$\bar{f}_2^{m_1-1}\left|\dfrac{\mathrm{d}\bar{f}_1}{\mathrm{d}\xi}\right|^{p-2}\dfrac{\mathrm{d}\bar{f}_1}{\mathrm{d}\xi}=-A^{p-1}B^{m_2-1}\left(\gamma\gamma_1\right)^{p-1}\xi\bar{f}_1\in C\left(0,\infty\right)$$

$$\bar{f}_1^{m_2-1}\left|\dfrac{\mathrm{d}\bar{f}_2}{\mathrm{d}\xi}\right|^{p-2}\dfrac{\mathrm{d}\bar{f}_2}{\mathrm{d}\xi}=-A^{m_2-1}B^{p-1}\left(\gamma\gamma_2\right)\xi\bar{f}_2\in C\left(0,\infty\right)$$

and

$$\begin{cases} \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f_2}^{\,m_1-1}\left|\dfrac{d\overline{f_1}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_1}}{d\xi}\right)=-\left|\gamma_1\right|^{p-1}\gamma\gamma_1 A^{p-1}B^{m_1-1}\left(s\overline{f_1}+\xi\dfrac{d\overline{f_1}}{d\xi}\right) \\[2mm] \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f_1}^{\,m_2-1}\left|\dfrac{df_2}{d\xi}\right|^{p-2}\dfrac{df_2}{d\xi}\right)=-\left|\gamma_2\right|^{p-1}\gamma\gamma_2 A^{m_2-1}B^{p-1}\left(s\overline{f_2}+\xi\dfrac{d\overline{f_2}}{d\xi}\right). \end{cases}$$

We choose A and B from the system of nonlinear algebraic equations

$$\left|\gamma_1\right|^{p-1}\gamma\gamma_1 A^{p-1}B^{m_1-1}=1/p$$
$$\left|\gamma_2\right|^{p-1}\gamma\gamma_2 A^{m_2-1}B^{p-1}=1/p.$$

Then functions $\overline{f_1},\overline{f_2}$ were the solution of the Zel'dovich-Kompanees for the system (1) and in the field $|\xi|<(a)^{(p-1)/p}$ they satisfy the system of equations

$$\begin{cases} \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f_2}^{\,m_1-1}\left|\dfrac{d\overline{f_1}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_1}}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f_1}}{d\xi}+\dfrac{s}{p}\overline{f_1}=0 \\[2mm] \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f_1}^{\,m_2-1}\left|\dfrac{d\overline{f_2}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_2}}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f_2}}{d\xi}+\dfrac{s}{p}\overline{f_2}=0 \end{cases}$$

in the classical sense.

Due to the fact that

$$\begin{cases} \xi^{s-1}\overline{f_2}^{\,m_1-1}\left|\dfrac{d\overline{f_1}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_1}}{d\xi}=\xi^{s}\overline{f_1} \\[2mm] \xi^{s-1}\overline{f_1}^{\,m_2-1}\left|\dfrac{d\overline{f_2}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_2}}{d\xi}=\xi^{s}\overline{f_2} \end{cases}$$

function $\overline{f_1}(\xi),\overline{f_2}(\xi)$ and the flows have the following smoothness properties

$$0\le\overline{f_1}(\xi),\ \xi^{s-1}\overline{f_2}^{\,m_1-1}\left|\dfrac{d\overline{f_1}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_1}}{d\xi}=\xi^{s}\overline{f_1}\in C(0,\infty),$$

$$0\le\overline{f_2}(\xi),\ \xi^{s-1}\overline{f_1}^{\,m_2-1}\left|\dfrac{d\overline{f_2}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_2}}{d\xi}=\xi^{s}\overline{f_2}\in C(0,\infty).$$

We choose A and B such that the inequality of inequality

$$\left|\gamma_1\right|^{p-1}\gamma\gamma_1 A^{p-1}B^{m_1-1}\ge 1/p$$
$$\left|\gamma_2\right|^{p-1}\gamma\gamma_2 A^{m_2-1}B^{p-1}\ge 1/p. \qquad (12)$$

Since then

$$\begin{cases} \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f_2}^{\,m_1-1}\left|\dfrac{d\overline{f_1}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_1}}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f_1}}{d\xi}=-\dfrac{s}{p}\overline{f_1} \\[2mm] \xi^{1-s}\dfrac{d}{d\xi}\left(\xi^{s-1}\overline{f_1}^{\,m_2-1}\left|\dfrac{d\overline{f_2}}{d\xi}\right|^{p-2}\dfrac{d\overline{f_2}}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f_2}}{d\xi}=-\dfrac{s}{p}\overline{f_2}. \end{cases}$$

It is due to the fact that

$$\frac{\mathrm{d}\overline{f_1}}{\mathrm{d}\xi} \le 0, \ \frac{\mathrm{d}\overline{f_2}}{\mathrm{d}\xi} \le 0 \ npu \ \xi \in (0,\infty)$$

from (12) we have

$$\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1-}\overline{f_2}^{m_1-1}\left|\frac{\mathrm{d}\overline{f_1}}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\overline{f_1}}{\mathrm{d}\xi}\right) + \frac{\xi}{p}\frac{\mathrm{d}\overline{f_1}}{\mathrm{d}\xi} \le 0,$$

$$\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}\overline{f_1}^{m_1-1}\left|\frac{\mathrm{d}\overline{f_2}}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\overline{f_2}}{\mathrm{d}\xi}\right) + \frac{\xi}{p}\frac{\mathrm{d}\overline{f_2}}{\mathrm{d}\xi} \le 0, \qquad \xi \in (0,\infty).$$

Then in the field Q according to the comparison principle of solutions have

Theorem 1. Let $u_i(0,x) \le u_{i\pm}(0,x)$, $x \in R$. Then for the solution of problem (1) Q is the evaluation

$$u_1(t,x) \le u_{1+}(t,x) = e^{k_1 t}\tau^{-\alpha_1}\overline{f_1}(\xi),$$
$$u_2(t,x) \le u_{2+}(t,x) = e^{k_2 t}\tau^{-\alpha_2}\overline{f_2}(\xi), \qquad \xi = \varphi(|x|)/[\tau(t)]^{1/p}$$

where $\overline{f_1}(\xi)$, $\overline{f_2}(\xi)$ и $\tau(t)$—above-defined functions.

Note that the solution of system (1) when $\beta_i = \dfrac{(p-2)^2 - (m_1-1)(m_2-1)}{(p-1)(p-(m_i+1))}$ has the following representation at

$$a = \left(P_1\gamma\Big/B\left(\frac{1}{\gamma},1+n_1\right)\right)^{\frac{\gamma}{n_1}} = \left(P_2\gamma\Big/B\left(\frac{1}{\gamma},1+n_2\right)\right)^{\frac{\gamma}{n_2}}$$

where $B(a, b)$-Euler Beta function.

It is proved that this view is the self-similar asymptotics of solutions of systems (1).

$$\begin{cases} \tau^{-\frac{1}{\mu_1}}\displaystyle\int_{-\infty}^{\infty}\left(a-\xi_1^{\gamma}\right)_{+}^{n_1}\mathrm{d}x = P_1 \\[2mm] \tau^{-\frac{1}{\mu_2}}\displaystyle\int_{-\infty}^{\infty}\left(a-\xi_2^{\gamma}\right)_{+}^{n_2}\mathrm{d}x = P_2 \end{cases}$$

$$\begin{cases} \tau^{-\frac{1}{\mu_1}}\displaystyle\int_{-\infty}^{\infty}\left(a-\xi_1^{\gamma}\right)_{+}^{n_1}\mathrm{d}x = a^{\frac{n_1}{\gamma}}\frac{1}{\gamma}\int_0^1 \eta^{\frac{1}{\gamma}-1}(1-\eta)^{n_1}\mathrm{d}\eta = a^{\frac{n_1}{\gamma}}\frac{1}{\gamma}B\left(\frac{1}{\gamma},1+n_1\right) = P_1 \\[3mm] \tau^{-\frac{1}{\mu_2}}\displaystyle\int_{-\infty}^{\infty}\left(a-\xi_2^{\gamma}\right)_{+}^{n_2}\mathrm{d}x = a^{\frac{n_2}{\gamma}}\frac{1}{\gamma}\int_0^1 \eta^{\frac{1}{\gamma}-1}(1-\eta)^{n_2}\mathrm{d}\eta = a^{\frac{n_2}{\gamma}}\frac{1}{\gamma}B\left(\frac{1}{\gamma},1+n_2\right) = P_2. \end{cases}$$

Here

$$a = \left[P_1\gamma\Big/B\left(\frac{1}{\gamma},1+n_1\right)\right]^{\frac{\gamma}{n_1}} = \left[P_2\gamma\Big/B\left(\frac{1}{\gamma},1+n_2\right)\right]^{\frac{\gamma}{n_2}}.$$

3. Slow Diffusion

Case $n_1 > 0, n_2 > 0, n > 0$ (slow diffusion). Applying the method of [1] to solve Equation (11) will receive the following features

$$\overline{\theta_1}(\xi) = (a-\xi)_{+}^{n_1}, \overline{\theta_2}(\xi) = (a-\xi)_{+}^{n_2},$$

where $a > 0$, $(y)_{+} = \max(y,0)$, $\xi < a$. It is known [1] [2] that for the global existence of solutions of prob-

lem (1) function $f(\xi)$ must satisfy the following inequality:

$$\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}\,\overline{f_2}^{\,m_1-1}\left|\frac{\mathrm{d}\overline{f_1}}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\overline{f_1}}{\mathrm{d}\xi}\right)+\frac{\xi}{p}\frac{\mathrm{d}\overline{f_1}}{\mathrm{d}\xi}+\mu_1 f_1\left(1-f_1^{\beta_1}\right)\le 0,$$

$$\xi\in(0,\infty).$$

$$\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}\,\overline{f_1}^{\,m_1-1}\left|\frac{\mathrm{d}\overline{f_2}}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\overline{f_2}}{\mathrm{d}\xi}\right)+\frac{\xi}{p}\frac{\mathrm{d}\overline{f_2}}{\mathrm{d}\xi}+\mu_1 f_1\left(1-f_1^{\beta_1}\right)\le 0,$$

and

$$\beta_1 = 1/n_2,\ \beta_2 = 1/n_1.$$

Let's take the function $\overline{\theta}_i(\xi), \overline{\theta}_i(\xi)$, and show that they are asymptotic finite solutions of the system (11).

Theorem 2. The finite solution of system (11) when $\xi\to a_-$ has asymptotic $f_i(\xi)\sim\vartheta_i(\xi)$.

Proof. We seek a solution of Equation (8) in the following form

$$f_i = \overline{\vartheta}_i(\xi)\, y_i(\eta),\ i=1,2, \tag{13}$$

where $\eta = -\ln(a-\xi)$, and $\eta\to+\infty$ at $\xi\to a_-$, to explore the asymptotic stability of the solution of problem (11) when $\eta\to+\infty$. Substituting (13) into (11) for $y_i(\eta)$ gets the following equation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(y_{3-i}^{m_i-1}\left|\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right|^{p-2}\left(\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right)\right)+\left(\frac{s}{\gamma}\frac{\mathrm{e}^{-\eta}}{a-\mathrm{e}^{-\eta}}-n_i\right)\left(y_{3-i}^{m_i-1}\left|\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right|^{p-2}\left(\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right)\right)$$

$$+\frac{\xi}{p}\left(\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right)-\mu_i\frac{\mathrm{e}^{-\eta}}{a-\mathrm{e}^{-\eta}}\,y_i(\eta)\left(1+\mathrm{e}^{-n_{3-i}\beta_i\eta}y_{3-i}^{\beta_i}\right)=0 \tag{14}$$

where η above-defined function.

Note that the study of the solution of the last equation is equivalent to examining the solution of Equation (11), each of which in a certain period $[\eta_0,+\infty)$ satisfies the inequality:

$$y_i(\eta)>0,\ \frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\ne 0.$$

Let us show first of all that decision $y_i(\eta)$ Equations (14) have a finite limit y_{0i} at $\eta\to+\infty$. We introduce the notation

$$\omega_i(\eta) = y_{3-i}^{m_i-1}\left|\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right|^{p-2}\left(\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right).$$

Then for the Equation (14) has the form

$$\omega_i' = -\left(\frac{s}{\gamma}\frac{\mathrm{e}^{-\eta}}{a-\mathrm{e}^{-\eta}}-n_i\right)\omega_i-\frac{\xi}{p}\left(\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right)-\mu_i\frac{\mathrm{e}^{-\eta}}{a-\mathrm{e}^{-\eta}}\,y_i(\eta)\left(1+\mathrm{e}^{-n_i\beta_i\eta}y_{3-i}^{\beta_i}\right).$$

To analyze the last expression we introduce a new helper function

$$\phi(\tau,\eta) = -\left(\frac{s}{\gamma}\frac{\mathrm{e}^{-\eta}}{a-\mathrm{e}^{-\eta}}-n_i\right)\tau-\frac{\xi}{p}\left(\frac{\mathrm{d}y_i}{\mathrm{d}\eta}-n_i y_i\right)-\mu_i\frac{\mathrm{e}^{-\eta}}{a-\mathrm{e}^{-\eta}}\,y_i(\eta)\left(1+\mathrm{e}^{-n_i\beta_i\eta}y_{3-i}^{\beta_i}\right),$$

where τ —real numbers. Hence it is easy to see that each value τ function $\phi(\tau,\eta)$ stores the sign on a certain interval $[\eta_1,+\infty)\subset[\eta_0,+\infty)$ and for all $\eta\in[\eta_1,+\infty)$ either of the inequalities

$$\omega_i'(\eta)>0,\ \omega_i'(\eta)<0.$$

And so for the function $\omega_i(\eta)$ there is a limit when $\eta\in[\eta_1,+\infty)$. From the expression for $\omega_i(\eta)$ it follows that

$$\lim_{\eta \to +\infty} \omega_i'(\eta) = \lim_{\eta \to +\infty} \left\{ -\left(\frac{s}{\gamma} \frac{e^{-\eta}}{a - e^{-\eta}} - n_i \right) \omega_i - \frac{\xi}{p} \left(\frac{dy_i}{d\eta} - n_i y_i \right) - \mu_i \frac{e^{-\eta}}{a - e^{-\eta}} y_i(\eta) \left(1 + e^{-n_i \beta_i \eta} y_{3-i}^{\beta_{3-i}} \right) \right\} = 0.$$

Hence, given that

$$\xi \to a, \quad \lim_{\eta \to +\infty} e^{-\eta} \to 0, \quad \lim_{\eta \to +\infty} a - e^{-\eta} \to a, \quad \omega_i' = 0$$

get the following algebraic equation

$$n_1^{p-1} y_2^{m_1-1} |y_1|^{p-1} = \frac{1}{p\gamma^{p-1}}$$

$$n_2^{p-1} y_1^{m_2-1} |y_2|^{p-1} = \frac{1}{p\gamma^{p-1}}.$$

The latter system gives $y_i = 1$ and because (14) $f_i(\xi) \sim \overline{\mathcal{G}}_i(\xi)$.

Theorem 2 is proved.

4. Fast Diffusion

Case $n_1 > 0, n_2 > 0, n < 0$ (fast diffusion case). For (11), we have

$$\chi_1(\xi) = (a + \xi)^{n_1}, \chi_2(\xi) = (a + \xi)^{n_2},$$

where $a > 0$.

Theorem 3. At $\xi \to +\infty$ vanishing at infinity solution of problem (11) has the asymptotics $f_i(\xi) \sim \chi_i(\xi), i = 1, 2$.

Proof. In the proof of theorem used the transform

$$f_i = \chi_i(\xi) y(\eta), \quad i = 1, 2, \tag{15}$$

where $\eta = \ln(a + \xi)$, which leads (11) to the following form.

Substituting (15) into (11) for $y_i(\eta)$ gets the following equation

$$\frac{d}{d\eta} \left(y_{3-i}^{m_i-1} \left| \frac{dy_i}{d\eta} - n_i y_i \right|^{p-2} \left(\frac{dy_i}{d\eta} - n_i y_i \right) \right) + \left(\frac{s}{\gamma} \frac{e^\eta}{a + e^\eta} + n_i \right) \left(y_{3-i}^{m_i-1} \left| \frac{dy_i}{d\eta} - n_i y_i \right|^{p-2} \left(\frac{dy_i}{d\eta} - n_i y_i \right) \right)$$

$$+ \frac{\xi}{p} \left(\frac{dy_i}{d\eta} + n_i y_i \right) - \mu_i \frac{e^\eta}{a + e^\eta} y_i(\eta) \left(1 + e^{-n_i \beta_i \eta} y_{3-i}^{\beta_i} \right) = 0, \tag{16}$$

where η is above-defined function.

Note that the study of the solution of the last equation is equivalent to examining the solution of Equation (11), each of which in a certain period $[\eta_0, +\infty)$ satisfies the inequality:

$$y_i(\eta) > 0, \quad \frac{dy_i}{d\eta} - n_i y_i \neq 0.$$

Let us show first of all that solution $y_i(\eta)$ of the Equation (16) has a finite limit y_{0i} at $\eta \to +\infty$. We introduce the notation $\omega_i(\eta) = y_{3-i}^{m_i-1} \left| \frac{dy_i}{d\eta} - n_i y_i \right|^{p-2} \left(\frac{dy_i}{d\eta} - n_i y_i \right)$. Then Equation (15) has the form

$$\omega_i' = -\left(\frac{s}{\gamma} \frac{e^\eta}{a + e^\eta} + n_i \right) \omega_i - \frac{\xi}{p} \left(\frac{dy_i}{d\eta} + n_i y_i \right) - \mu_i \frac{e^\eta}{a + e^\eta} y_i(\eta) \left(1 + e^{-n_i \beta_i \eta} y_{3-i}^{\beta_{3-i}} \right).$$

To analyze the last expression we introduce a new helper function

$$\phi(\tau,\eta) = -\left(\frac{e^{\eta}}{a+e^{\eta}} + n_{3-i}\right)\tau - c\left(\frac{dy_i}{d\eta} + n_i y_i\right) - \mu_i \frac{e^{\eta}}{a+e^{\eta}} y_i(\eta)\left(1 + e^{-n_{3-i}\beta_{3-i}\eta} y_{3-i}^{\beta_{3-i}}\right),$$

where τ-real number. Hence it is easy to see that each value τ function $\phi(\tau,\eta)$ stores the sign on a certain interval $[\eta_1,+\infty) \subset [\eta_0,+\infty)$ and at all $\eta \in [\eta_1,+\infty)$ satisfied either of the inequalities

$$\omega_i'(\eta) > 0, \quad \omega_i'(\eta) < 0.$$

Therefore for the function $\omega_i(\eta)$ there is a limit when $\eta \in [\eta_1,+\infty)$. From the expression for $\omega_i(\eta)$ follows that

$$\lim_{\eta\to+\infty} \omega_i'(\eta) = \lim_{\eta\to+\infty}\left\{-\left(\frac{s}{\gamma}\frac{e^{\eta}}{a-e^{\eta}} - n_i\right)\omega_i - \frac{\xi}{p}\left(\frac{dy_i}{d\eta} - n_i y_i\right) - \mu_i \frac{e^{\eta}}{a-e^{\eta}} y_i(\eta)\left(1 + e^{-n_i\beta_i\eta} y_{3-i}^{\beta_i}\right)\right\} = 0.$$

Hence, when $\xi \to \infty$, $\beta_i > 1, i = 1,2$ we get the following algebraic equation

$$(-n_1)^{p-1} y_2^{m_1-1} y_1^{p-1} = \frac{1}{p\gamma^{p-1}},$$

$$(-n_2)^{p-1} y_1^{m_2-1} y_2^{p-1} = \frac{1}{p\gamma^{p-1}}.$$

The calculation of the last equation gives $y_i = 1$ and because (15) $f(\xi) \sim \chi_i(\xi)$.
Theorem 3 is proved.

5. Computational Experiment

Investigation of qualitative properties of system (1) has allowed to perform numerical experiment depending on the values included in the system of numeric parameters. For this purpose, the initial approximation was used to construct asymptotic solutions. The numerical solution of the problem for the linearization of system (2) was used linearization methods of Newton and Picard. To build self-similar system of equations of biological population used the method of nonlinear splitting [1] [6].

For the numerical solution of the problem (1) we will construct a uniform grid

$$\omega_h = \{x_i = ih, h > 0, i = 0,1,\cdots,n, \ hn = l\},$$

and temporal grid

$$\omega_{h_1} = \{t_j = jh_1, h_1 > 0, j = 0,1,\cdots,n, \tau m = T\}.$$

Replace the problem (1) implicit difference scheme and receive differential task with the error $O(h^2 + h_1)$.

It is known that the main problem for the numerical solution of nonlinear problems is the appropriate choice of the initial approximation and the method of linearization of system (1).

Consider the function:

$$v_{10}(t,x) = v_1(t)\cdot(a-\xi)_+^{\gamma_1},$$

$$v_{20}(t,x) = v_2(t)\cdot(a-\xi)_+^{\gamma_2},$$

where $v_1(t) = e^{kt}\bar{v}_1(t)$ и $v_2(t) = e^{kt}\bar{v}_2(t)$ above-defined functions,

Record $(a)_+$ means $(a)_+ = \max(0,a)$. These functions have the property of finite speed of propagation of perturbations [1] [6]. Therefore, for the numerical solution of problem (1) when $\beta_1 > \sigma_1$ as an initial approximation of the proposed function $v_{i0}(t,x)$, $i = 1,2$.

Created on input language Matlab the program allows you to visually trace the evolution process for different values of the parameters and data.

Numerical calculations show that in the case of arbitrary values $\sigma > 0, \beta > 0$ qualitative properties of solutions do not change. Below are the results of numerical experiments for various values of the parameters (**Figures 1-4**).
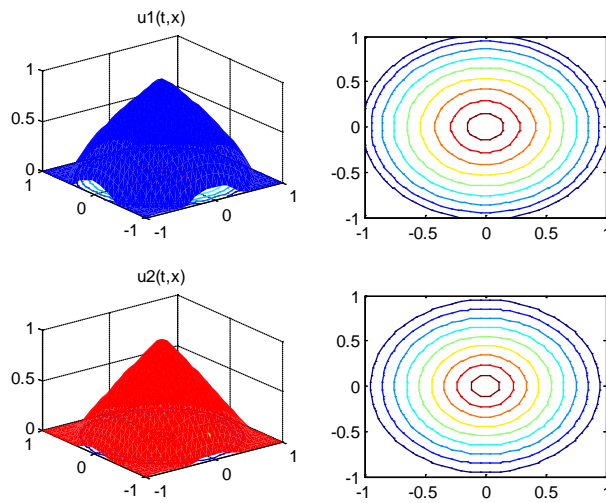
Figure 1. Results of numerical simulations at $n_i < 1, i = 1, 2$, $x_1 = 1; x_2 = 1$, $n_1 = 0.87, n_2 = 0.77, m_1 = 3, m_2 = 2, p = 4.2$.
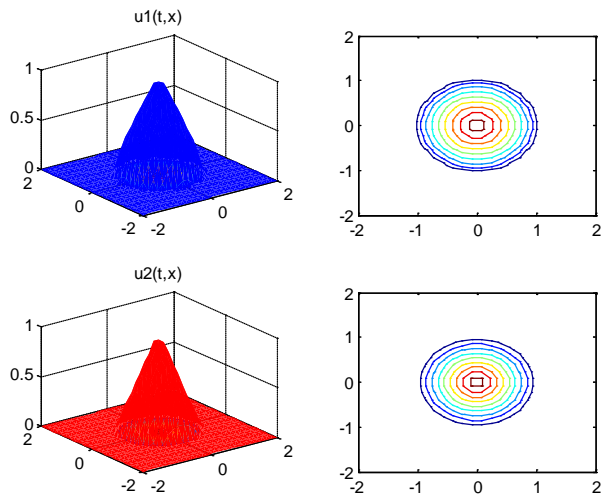

Figure 2. The results of numerical simulations at $n_i < 1, i = 1, 2$, $x_1 = 2; x_2 = 2$, $n_1 = 0.87, n_2 = 0.77, m_1 = 3, m_2 = 2, p = 4.2$.
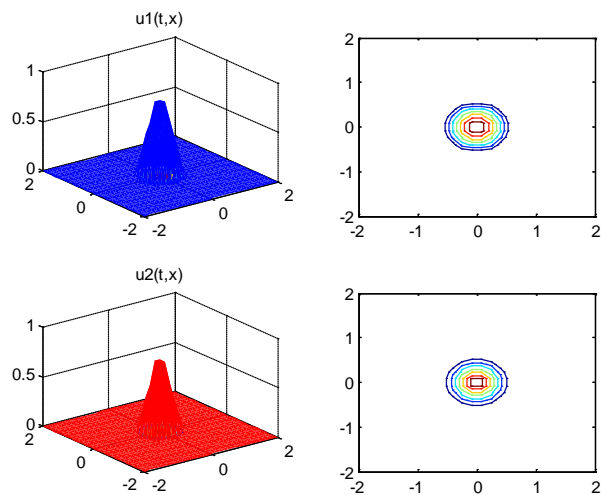

Figure 3. The results of numerical simulations at $n_i > 1, i = 1, 2$, $x_1 = 1; x_2 = 1$, $n_1 = 1.87, n_2 = 1.77, m_1 = 3, m_2 = 2, p = 4.2$.
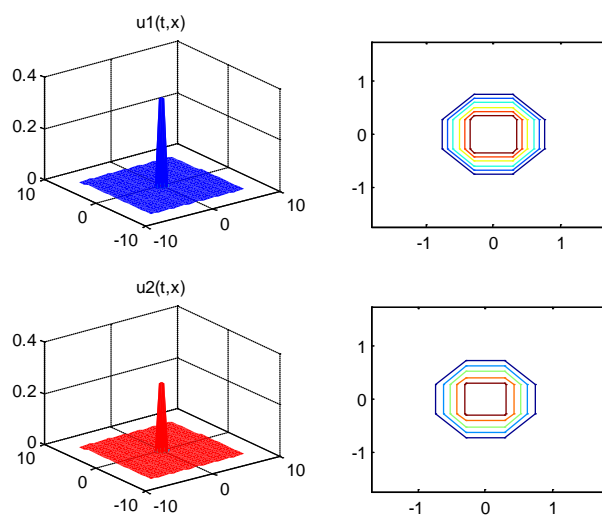
Figure 4. The results of numerical simulations at $n_i > 1, i = 1, 2$, $x_1 = 2; x_2 = 2$, $n_1 = 1.87, n_2 = 1.77, m_1 = 3, m_2 = 2, p = 4.2$.

6. Conclusions

Thus, the proposed nonlinear mathematical model of biological populations with double nonlinearity and variable density properly describes the studied process. Numerical study of nonlinear processes described by equations with a double nonlinearity and analysis results on the basis of evaluation solutions provides a comprehensive picture of the process in two-component systems competing biological population with the preservation of localization properties in the target area and the size of the flash.

Results in future will provide an opportunity to evaluate the speed of propagation of diffusive waves.

References

[1] Aripov, M. (1988) The Method of Standard Equations for the Solution of Nonlinear Boundary Value Problems. Tashkent: Fan Publishing, Uzbekstan, 137.

[2] Belotelov, N.V. and Lobanov, A.I. (1997) The Population Model with Nonlinear Diffusion. *Mathematical Modeling*, No. 12, 43-56.

[3] Volterra, V. (1976) Mathematical Theory of Struggle for Existence. Nauka, Moscow, 288 p.

[4] Gause, G.F. (1934) Processes of Destruction of One Species by another in Populations of Ciliates. *Zoology Journal*, **13**, No. 1.

[5] Aripov, M. and Muhammadiev, J. (1999) Asymptotic Behaviour of Automodel Solutions for One System of Quasilinear Equations of Parabolic Type. *Buletin Stiintific-Universitatea din Pitesti*, *Seria Matematica si Informatica*, No. 3, 19-40.

[6] Aripov, M.M. and Muhamediyeva, D.K. (2013) To the Numerical Modeling of Self-Similar Solutions of Reaction-Diffusion System of the One Task of Biological Population of Kolmogorov-Fisher Type. *International Journal of Engineering and Technology*, **2**, No. 11.

[7] Aripov, M.M. and Muhamedieva, D.K. (2013) Approaches to the Solution of One Task of Biological Population. *Issues of Computational and Applied Mathematics*, **129**, 22-31.

[8] Murray James, D. (1983) Diffusion of Nonlinear Equations in Biology. Clarendon Press, Oxford, 397 p.