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On Henstock-Stieltjes Integrals of Interval-Valued Functions and Fuzzy-Number-Valued Functions

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Abstract

In this paper we introduce the notion of the Henstock-Stieltjes (HS) integrals of interval-valued functions and fuzzy-number-valued functions and discuss some of their properties.

Keywords

Fuzzy Numbers, Henstock-Stieltjes (HS) Integrals of Interval-Valued Functions, Henstock-Stieltjes (HS) Integrals of Fuzzy-Number-Valued Functions

1. Introduction

As it is well known, the Henstock (H) integral for a real function was first defined by Henstock [1] in 1963. The Henstock (H) integral is a lot powerful and easier than the Lebesgue, Wiener and Richard Phillips Feynman integrals. Furthermore, it is also equal to the Denjoy and the Perron integrals [1] [2]. In 2000, Congxin Wu and Zengtai Gong [3] introduced the notion of the Henstock (H) integrals of interval-valued functions and fuzzy-number-valued functions and obtained a number of their properties. In 2016, Yoon [4] introduced the interval-valued Henstock-Stieltjes integral on time scales and investigated some properties of these integrals. In 1998, Lim *et al*. [5] introduced the notion of the Henstock-Stieltjes (HS) integral of real-valued function which was a generalization of the Henstock (H) integral and obtained its properties.

In this paper, we tend to introduce the notion of the Henstock-Stieltjes (HS) integrals of interval-valued functions and fuzzy-number-valued functions and discuss some of their properties.

The paper is organized as follows. In Section two, we tend to give the preliminary terminology used in the

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present paper. Section three is dedicated to discussing the Henstock-Stieltjes (HS) integral of interval-valued functions. In Section four, we tend to introduce the Henstock-Stieltjes (HS) integral of fuzzy-number-valued functions. The last section provides conclusions.

2. Preliminaries

Definition 2.1 [1] [2] *Let* $\delta:[a,b]\to\mathbf{R}^+$ *be a positive real-valued function.* $P=\{[x_{i-1},x_i];\xi_i\}_{i=1}^n$ *is called a δ-fine division, if the subsequent conditions are satisfied*:

1) $a=x_0<x_1<x_2<\cdots<x_n=b$,

2) $\xi_i\in[x_{i-1},x_i]\subset(\xi_i-\delta(\xi_i),\xi_i+\delta(\xi_i))(i=1,2,\cdots,n)$.

For brevity, we write $P=\{[u,v];\xi\}$, wherever $[u,v]$ denotes a typical interval in *P* and ξ is that the associated point of $[u,v]$.

Definition 2.2 [1] [2] *A real-valued function* $f(t)$ *is called Henstock (H) integrable to A on* $[a,b]$ *if for each* $\varepsilon>0$, *there exists a function* $\delta(t)>0$ *such that for any δ-fine division* $P=\{[u_i,v_i];\xi_i\}_{i=1}^n$ *of* $[a,b]$, *we have*

$$\left|\sum_{i=1}^n f(\xi_i)(v_i-u_i)-A\right|<\varepsilon. \tag{1}$$

where the sum $\sum$ is understood to be over *P*, we write $(H)\int_a^b f(t)\mathrm{d}t=A$, and $f\in H[a,b]$.

Definition 2.3 [5] *Let* $\alpha:[a,b]\to\mathbf{R}$ *be an increasing function. A real-valued function* $f:[a,b]\to R$ *is Henstock-Stieltjes (HS) integrable to* $I\in R$ *with respect to* α *on* $[a,b]$ *if for each* $\varepsilon>0$, *there exists a function* $\delta(t)>0$, *such that for any δ-fine division* $P=\{[u_i,v_i];\xi_i\}_{i=1}^n$ *we have*

$$\left|\sum_{i=1}^n f(\xi_i)[\alpha(v_i)-\alpha(u_i)]-I\right|<\varepsilon. \tag{2}$$

We write $(HS)\int_a^b f(t)\mathrm{d}\alpha=I$, and $f\in HS_\alpha[a,b]$.

Lemma 2.1 [5] *Let* $\alpha:[a,b]\to\mathbf{R}$ *be an increasing function and let f, g are Henstock-Stieltjes (HS) integrable with respect to* α *on* $[a,b]$. *If* $\alpha\in C^1[a,b]$ *and* $f\le g$ *almost everywhere on* $[a,b]$, *then*

$$\int_a^b f\mathrm{d}\alpha\le\int_a^b g\mathrm{d}\alpha. \tag{3}$$

3. The Henstock-Stieltjes (HS) Integrals of Interval-Valued Functions

Definition 3.1 [3] *Let* $I_{\mathbf{R}}=\{I=[I^-,I^+]:I \text{ is the closed bounded interval on the real line } \mathbf{R}\}$.

For $A,B\in I_{\mathbf{R}}$, we define $A\le B$ if and only if $A^-\le B^-$ and $A^+\le B^+$, $A+B=C$ if and only if $C^-=A^-+B^-$ and $C^+=A^++B^+$, and $A\cdot B=\{a\cdot b:a\in A,b\in B\}$, wherever $(A\cdot B)^-=\min\{A^-\cdot B^-,A^-\cdot B^+,A^+\cdot B^-,A^+\cdot B^+\}$ and $(A\cdot B)^+=\max\{A^-\cdot B^-,A^-\cdot B^+,A^+\cdot B^-,A^+\cdot B^+\}$. Define $d(A,B)=\max(|A^--B^-|,|A^+-B^+|)$ as the distance between intervals *A* and *B*.

Definition 3.2 [3] *Let* $F:[a,b]\to I_{\mathbf{R}}$ *be an interval-valued function.* $I_0\in I_{\mathbf{R}}$, *for each* $\varepsilon>0$ *there exists a* $\delta(t)>0$ *such that for any δ-fine division* $P=\{[u_i,v_i];\xi_i\}_{i=1}^n$ *we have*

$$d\left(\sum_{i=1}^n F(\xi_i)(v_i-u_i),I_0\right)<\varepsilon, \tag{4}$$

then $F(t)$ is called the Henstock (H) integrable over $[a,b]$ and write $(IH)\int_a^b F(t)\mathrm{d}t=I_0$. Also, we write $F(t)\in IH[a,b]$.

Definition 3.3 *Let* $\alpha:[a,b]\to\mathbf{R}$ *be an increasing function. An interval-valued function* $F:[a,b]\to I_{\mathbf{R}}$ *is Henstock-Stieltjes (HS) integrable to* $I_0\in I_{\mathbf{R}}$ *with respect to* α *on* $[a,b]$, *if for each* $\varepsilon>0$ *there exists a* $\delta(t)>0$ *such that for any* δ*-fine division* $P=\{[u_i,v_i];\xi_i\}_{i=1}^n$, *we have*

$$d\left(\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)],I_0\right)<\varepsilon. \tag{5}$$

We write $(IHS)\int_a^b F(t)\,\mathrm{d}\alpha=I_0$ and $F(t)\in IHS_\alpha[a,b]$.

Theorem 3.1 *Let* $\alpha:[a,b]\to\mathbf{R}$ *be an increasing function. If* $F(t)\in IHS_\alpha[a,b]$, *then there exists a unique integral value.*

Proof Let the integral value is not unique and let $A_1=(IHS)\int_a^b F(t)\,\mathrm{d}\alpha$ and $A_2=(IHS)\int_a^b F(t)\,\mathrm{d}\alpha$. If $\varepsilon>0$ is given. Then there exists a $\delta(t)>0$ such that for any δ- fine division $P=\{[u_i,v_i];\xi_i\}_{i=1}^n$, we have

$$d\left(\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)],A_1\right)<\frac{\varepsilon}{2}, \tag{6}$$

$$d\left(\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)],A_2\right)<\frac{\varepsilon}{2} \tag{7}$$

$$\begin{aligned}d(A_1,A_2)&=d\left(\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)],A_1\right)+d\left(\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)],A_2\right)\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Since for all $\varepsilon>0$, there exists a $\delta(t)>0$ as above then $A_1=A_2$. □

Theorem 3.2 *Let* $\alpha:[a,b]\to\mathbf{R}$ *be an increasing function. Then an interval-valued function* $F(t)\in IHS_\alpha[a,b]$ *iff* $F^-(t),F^+(t)\in HS_\alpha[a,b]$ *and*

$$(IHS)\int_a^b F(t)\,\mathrm{d}\alpha=\left[(HS)\int_a^b F^-(t)\,\mathrm{d}\alpha,(HS)\int_a^b F^+(t)\,\mathrm{d}\alpha\right]. \tag{8}$$

Proof If $F(t)\in IHS_\alpha[a,b]$, by Definition 3.3 there exists a unique interval number $I_0=[I_0^-,I_0^+]$ with the property, for any $\varepsilon>0$ there exists a $\delta(t)>0$ such that for any δ- fine division $P=\{[u_i,v_i];\xi_i\}_{i=1}^n$, we have

$$d\left(\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)],I_0\right)<\varepsilon, \tag{9}$$

that is

$$\max\left(\left|\left[\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)]\right]^- -I_0^-\right|,\left|\left[\sum_{i=1}^{n}F(\xi_i)[\alpha(v_i)-\alpha(u_i)]\right]^+ -I_0^+\right|\right)<\varepsilon. \tag{10}$$

Since $\alpha(v_i)-\alpha(u_i)\ge 0$ for $1\le i\le n$, we have

$$\left|\sum_{i=1}^{n}F^-(\xi_i)[\alpha(v_i)-\alpha(u_i)]-I_0^-\right|<\varepsilon, \tag{11}$$

$$\left|\sum_{i=1}^{n}F^+(\xi_i)[\alpha(v_i)-\alpha(u_i)]-I_0^+\right|<\varepsilon. \tag{12}$$

Therefore, by Definition 2.3 we can obtain $F^{-}(t), F^{+}(t) \in HS_{\alpha}[a,b]$ and

$$I_0^{-} = (HS)\int_a^b F^{-}(t)\,\mathrm{d}\alpha, \tag{13}$$

$$I_0^{+} = (HS)\int_a^b F^{+}(t)\,\mathrm{d}\alpha. \tag{14}$$

Conversely, let $F^{-}(t) \in HS_{\alpha}[a,b]$, then there exists a unique $H_1 \in R$ with the property, given $\varepsilon > 0$ there exists a $\delta_1(t) > 0$ such that for any δ_1-fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^{n}$, we have

$$\left|\sum_{i=1}^{n} F^{-}(\xi_i)\left[\alpha(v_i) - \alpha(u_i)\right] - H_1\right| < \varepsilon. \tag{15}$$

It is similar to find $\delta_2(t) > 0$ such that for any δ_2-fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^{n}$, we have

$$\left|\sum_{i=1}^{n} F^{+}(\xi_i)\left[\alpha(v_i) - \alpha(u_i)\right] - H_2\right| < \varepsilon. \tag{16}$$

If $F^{-}(t) \le F^{+}(t)$, then $H_1 \le H_2$. We define $\delta(t) = \min(\delta_1(t), \delta_2(t))$ and $I_0 = [H_1, H_2]$, then for any δ- fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^{n}$, we have

$$d\left(\sum_{i=1}^{n} F(\xi_i)\left[\alpha(v_i) - \alpha(u_i)\right], I_0\right) < \varepsilon. \tag{17}$$

Hence $F:[a,b] \to I_R$ is Henstock-Stieltjes (HS) integrable with respect to α on $[a,b]$. □

Theorem 3.3 *If* $F, G \in IHS_{\alpha}[a,b]$ *and* $\beta, \gamma \in R$. *Then*

i) $\beta F + \gamma G \in IHS_{\alpha}[a,b]$ and

$$(IHS)\int_a^b (\beta F + \gamma G)\,\mathrm{d}\alpha = \beta(IHS)\int_a^b F\mathrm{d}\alpha + \gamma(IHS)\int_a^b G\mathrm{d}\alpha. \tag{18}$$

ii) Let $F(t) = G(t)$ almost everywhere on $[a,b]$. Then

$$(IHS)\int_a^b F(t)\,\mathrm{d}\alpha = (IHS)\int_a^b G(t)\,\mathrm{d}\alpha. \tag{19}$$

Proof i) If $F, G \in IHS_{\alpha}[a,b]$, then $F^{-}, F^{+}, G^{-}, G^{+} \in HS[a,b]$ by Theorem 3.2. Hence $\beta F^{-} + \gamma G^{-}, \beta F^{-} + \gamma G^{+}, \beta F^{+} + \gamma G^{-}, \beta F^{+} + \gamma G^{+} \in HS_{\alpha}[a,b]$.

1) If $\beta > 0$ and $\gamma > 0$, then

$$\begin{aligned}
(HS)\int_a^b (\beta F + \gamma G)^{-}\,\mathrm{d}\alpha &= (HS)\int_a^b \left(\beta F^{-} + \gamma G^{-}\right)\mathrm{d}\alpha \\
&= \beta(HS)\int_a^b F^{-}\mathrm{d}\alpha + \gamma(HS)\int_a^b G^{-}\mathrm{d}\alpha \\
&= \beta\left((IHS)\int_a^b F\mathrm{d}\alpha\right)^{-} + \gamma\left((IHS)\int_a^b G\mathrm{d}\alpha\right)^{-} \\
&= \left(\beta(IHS)\int_a^b F\mathrm{d}\alpha + \gamma(IHS)\int_a^b G\mathrm{d}\alpha\right)^{-}.
\end{aligned}$$

2) If $\beta < 0$ and $\gamma < 0$, then

$$(HS)\int_a^b(\beta F+\gamma G)^- \mathrm{d}\alpha=(HS)\int_a^b(\beta F^+ +\gamma G^+)\mathrm{d}\alpha$$

$$=\beta(HS)\int_a^b F^+\mathrm{d}\alpha+\gamma(HS)\int_a^b G^+\mathrm{d}\alpha$$

$$=\beta\left((IHS)\int_a^b F\mathrm{d}\alpha\right)^+ +\gamma\left((IHS)\int_a^b G\mathrm{d}\alpha\right)^+$$

$$=\left(\beta(IHS)\int_a^b F\mathrm{d}\alpha+\gamma(IHS)\int_a^b G\mathrm{d}\alpha\right)^-.$$

3) If $\beta>0$ and $\gamma<0$, (or $\beta<0$ and $\gamma>0$), then

$$(HS)\int_a^b(\beta F+\gamma G)^- \mathrm{d}\alpha=(HS)\int_a^b(\beta F^- +\gamma G^+)\mathrm{d}\alpha$$

$$=\beta(HS)\int_a^b F^-\mathrm{d}\alpha+\gamma(HS)\int_a^b G^+\mathrm{d}\alpha$$

$$=\beta\left((IHS)\int_a^b F\mathrm{d}\alpha\right)^- +\gamma\left((IHS)\int_a^b G\mathrm{d}\alpha\right)^+$$

$$=\left(\beta(IHS)\int_a^b F\mathrm{d}\alpha+\gamma(IHS)\int_a^b G\mathrm{d}\alpha\right)^-.$$

Similarly, for four cases above we have

$$(HS)\int_a^b(\beta F+\gamma G)^+ \mathrm{d}\alpha=\left(\beta(IHS)\int_a^b F\mathrm{d}\alpha+\gamma(IHS)\int_a^b G\mathrm{d}\alpha\right)^+. \quad (20)$$

Hence by Theorem 3.2 $\beta F+\gamma G\in IHS_\alpha[a,b]$ and

$$(IHS)\int_a^b(\beta F+\gamma G)\mathrm{d}\alpha=\beta(IHS)\int_a^b F\mathrm{d}\alpha+\gamma(IHS)\int_a^b G\mathrm{d}\alpha. \quad (21)$$

ii) The proof is similar to Theorem 2.8 in [5]. □

Theorem 3.4 *Let* $F\in IHS_\alpha[a,c]$ *and let* $F\in IHS_\alpha[c,b]$. *Then* $F\in IHS_\alpha[a,b]$ *and*

$$(IHS)\int_a^b F(t)\mathrm{d}\alpha=(IHS)\int_a^c F(t)\mathrm{d}\alpha+(IHS)\int_c^b F(t)\mathrm{d}\alpha. \quad (22)$$

Proof If $F\in IHS_\alpha[a,c]$ and $F\in IHS_\alpha[c,b]$, then by Theorem 3.2 $F^-,F^+\in HS_\alpha[a,c]$ and $F^-,F^+\in HS_\alpha[c,b]$. Hence $F^-,F^+\in HS_\alpha[a,b]$ and

$$(HS)\int_a^b F^-\mathrm{d}\alpha=(HS)\int_a^c F^-\mathrm{d}\alpha+(HS)\int_c^b F^-\mathrm{d}\alpha$$

$$=\left((IHS)\int_a^c F d\alpha+(IHS)\int_c^b F\mathrm{d}\alpha\right)^-.$$

Similarly, $(HS)\int_a^b F^+\mathrm{d}\alpha=\left((IHS)\int_a^c F\mathrm{d}\alpha+(IHS)\int_c^b F\mathrm{d}\alpha\right)^+$. Hence by Theorem 3.2 $F\in IHS_\alpha[a,b]$ and

$$(IHS)\int_a^b F\mathrm{d}\alpha=(IHS)\int_a^c F\mathrm{d}\alpha+(IHS)\int_c^b F\mathrm{d}\alpha. \quad (23)$$

□

Theorem 3.5 *Let* $\alpha:[a,b]\to\mathbf{R}$ *be an increasing function such that* $\alpha\in C^1([a,b])$. *If* $F(t)\le G(t)$ *nearly everywhere on* $[a,b]$ *and* $F,G\in IHS_\alpha[a,b]$, *then*

$$(IHS)\int_a^b F(t)\,\mathrm{d}\alpha\le(IHS)\int_a^b G(t)\,\mathrm{d}\alpha. \tag{24}$$

Proof Let $F(t)\le G(t)$ nearly everywhere on $[a,b]$ and $F,G\in IHS_\alpha[a,b]$. Then $F^-,F^+,G^-,G^+\in HS_\alpha[a,b]$ and $F^-\le G^-$, $F^+\le G^+$ nearly everywhere on $[a,b]$. By Lemma 2.1 $(HS)\int_a^b F^-(t)\,\mathrm{d}\alpha\le(HS)\int_a^b G^-(t)\,\mathrm{d}\alpha$ and $(HS)\int_a^b F^+(t)\,\mathrm{d}\alpha\le(HS)\int_a^b G^+(t)\,\mathrm{d}\alpha$. Hence

$$(IHS)\int_a^b F(t)\,\mathrm{d}\alpha\le(IHS)\int_a^b G(t)\,\mathrm{d}\alpha, \tag{25}$$

by Theorem 3.2. □

Theorem 3.6 *Let* $F,G\in IHS_\alpha[a,b]$ *and* $d(F,G)$ *is Lebesgue-Stieltjes* (*LS*) *integrable on* $[a,b]$. *Then*

$$d\left((IHS)\int_a^b F\mathrm{d}\alpha,(IHS)\int_a^b G\mathrm{d}\alpha\right)\le(LS)\int_a^b d(F,G)\,\mathrm{d}\alpha. \tag{26}$$

Proof By definition of distance,

$$\begin{aligned}
&d\left((IHS)\int_a^b F\mathrm{d}\alpha,(IHS)\int_a^b G\mathrm{d}\alpha\right)\\
&=\max\left(\left|\left((HS)\int_a^b F\mathrm{d}\alpha\right)^- -\left((HS)\int_a^b G\mathrm{d}\alpha\right)^-\right|,\left|\left((HS)\int_a^b F\mathrm{d}\alpha\right)^+ -\left((HS)\int_a^b G\mathrm{d}\alpha\right)^+\right|\right)\\
&=\max\left(\left|(HS)\int_a^b\left(F^- -G^-\right)\mathrm{d}\alpha\right|,\left|(HS)\int_a^b\left(F^+ -G^+\right)\mathrm{d}\alpha\right|\right)\\
&\le\max\left((LS)\int_a^b\left|F^- -G^-\right|\mathrm{d}\alpha,(LS)\int_a^b\left|F^+ -G^+\right|\mathrm{d}\alpha\right)\\
&\le(LS)\int_a^b\max\left(\left|F^- -G^-\right|,\left|F^+ -G^+\right|\right)\mathrm{d}\alpha\\
&=(LS)\int_a^b d(F,G)\,\mathrm{d}\alpha.
\end{aligned} \tag{27}$$

□

4. The Henstock-Stieltjes (HS) Integral of Fuzzy-Number-Valued Functions

Definition 4.1 [6]-[8] *If* $\tilde{A}\in F(\mathbf{R})$ *is a fuzzy subset on* $\mathbf{R}$. *If for any* $\lambda\in[0,1]$, $A_\lambda=\left[A_\lambda^-,A_\lambda^+\right]$ *and* $A_1\ne\phi$, *wherever* $A_\lambda=\{t:\tilde{A}(t)\ge\lambda\}$, *then* $\tilde{A}$ *is called a fuzzy number. If* $\tilde{A}$ *satisfy the following conditions*: 1) *convex*, 2) *normal*, 3) *upper semi-continuous*, 4) *has the compact support, then* $\tilde{A}$ *is called a compact fuzzy number*.

Let $\tilde{\mathbf{R}}$ denote the set of all fuzzy numbers and $\tilde{\mathbf{R}}^C$ denote the set of all compact fuzzy numbers.

Definition 4.2 [6] *Let* $\tilde{A},\tilde{B}\in\tilde{\mathbf{R}}$, *we define* $\tilde{A}\le\tilde{B}$ *if and only if* $A_\lambda\le B_\lambda$ *for all* $\lambda\in(0,1]$, $\tilde{A}+\tilde{B}=\tilde{C}$ *if and only if* $A_\lambda+B_\lambda=C_\lambda$ *for any* $\lambda\in(0,1]$, $\tilde{A}\cdot\tilde{B}=\tilde{D}$ *if and only if* $A_\lambda\cdot B_\lambda=D_\lambda$ *for any* $\lambda\in(0,1]$.

For $\tilde{A},\tilde{B}\in\tilde{\mathbf{R}}^C$, $D(\tilde{A},\tilde{B})=\sup_{\lambda\in[0,1]}d(A_\lambda,B_\lambda)$ is called the distance between $\tilde{A}$ and $\tilde{B}$.

Lemma 4.1 [9] *If a mapping* $H:[0,1]\to I_{\mathbf{R}}$, $\lambda\to H(\lambda)=[m_\lambda,n_\lambda]$, *satisfies* $\left[m_{\lambda_1},n_{\lambda_1}\right]\supset\left[m_{\lambda_2},n_{\lambda_2}\right]$ *when* $\lambda_1<\lambda_2$, *then*

$$\tilde{A} := \bigcup_{\lambda\in(0,1]} \lambda H(\lambda) \in \tilde{\mathbf{R}} \tag{28}$$

and

$$A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n), \tag{29}$$

where $\lambda_n = \left[1 - \frac{1}{(n+1)}\right]\lambda.$

Definition 4.3 [3] *Let* $\tilde{F}:[a,b]\to\tilde{\mathbf{R}}$ *and let the interval-valued function* $F_\lambda(t) = \left[F_\lambda^-(t), F_\lambda^+(t)\right]$ *is Henstock (H) integrable on* $[a,b]$ *for any* $\lambda\in(0,1]$, *then* $\tilde{F}(t)$ *is called Henstock (H) integrable on* $[a,b]$ *and the integral value is defined by*

$$\begin{aligned}(FH)\int_a^b \tilde{F}(t)\,\mathrm{d}t &:= \bigcup_{\lambda\in(0,1]} \lambda (IH)\int_a^b F_\lambda(t)\,\mathrm{d}t \\ &= \bigcup_{\lambda\in(0,1]} \lambda\left[(H)\int_a^b F_\lambda^-\mathrm{d}t, (H)\int_a^b F_\lambda^+\mathrm{d}t\right].\end{aligned}$$

We write $\tilde{F}(t)\in FH[a,b]$.

Definition 4.4 *Let* $\alpha:[a,b]\to\mathbf{R}$ *be an increasing function and let* $\tilde{F}:[a,b]\to\tilde{\mathbf{R}}$. *If the interval-valued function* $F_\lambda(t) = \left[F_\lambda^-(t), F_\lambda^+(t)\right]$ *is Henstock-Stieltjes (HS) integrable with respect to* α *on* $[a,b]$ *for any* $\lambda\in(0,1]$, *then* $\tilde{F}(t)$ *is called Henstock-Stieltjes (HS) integrable with respect to* α *on* $[a,b]$ *and the integral value is defined by*

$$\begin{aligned}(FHS)\int_a^b \tilde{F}(t)\,\mathrm{d}\alpha &:= \bigcup_{\lambda\in(0,1]} \lambda (IHS)\int_a^b F_\lambda(t)\,\mathrm{d}\alpha \\ &= \bigcup_{\lambda\in(0,1]} \lambda\left[(HS)\int_a^b F_\lambda^-\mathrm{d}\alpha, (HS)\int_a^b F_\lambda^+\mathrm{d}\alpha\right].\end{aligned}$$

We write $\tilde{F}(t)\in FHS_\alpha[a,b]$.

Theorem 4.1 $\tilde{F}\in FHS_\alpha[a,b]$, *then* $(FHS)\int_a^b \tilde{F}(t)\,\mathrm{d}\alpha\in\tilde{\mathbf{R}}$ *and*

$$\left[(FHS)\int_a^b \tilde{F}(t)\,\mathrm{d}\alpha\right]_\lambda = \bigcap_{n=1}^{\infty}(IHS)\int_a^b F_{\lambda_n}(t)\,\mathrm{d}\alpha, \tag{30}$$

where $\lambda_n = \left[1 - \frac{1}{(n+1)}\right]\lambda.$

Proof Let $H:(0,1]\to I_R$, be defined by $H(\lambda) = \left[(HS)\int_a^b F_\lambda^-(t)\,\mathrm{d}\alpha, (HS)\int_a^b F_\lambda^+(t)\,\mathrm{d}\alpha\right].$

Since $F_\lambda^-(t)$ and $F_\lambda^+(t)$ are increasing and decreasing on λ respectively, therefore, when $0<\lambda_1\le\lambda_2\le 1$, we have $F_{\lambda_1}^-(t)\le F_{\lambda_2}^-(t)$, $F_{\lambda_1}^+(t)\ge F_{\lambda_2}^+(t)$, on $[a,b]$. From Theorem 3.5 we have

$$\left[(HS)\int_a^b F_{\lambda_1}^-(t)\,\mathrm{d}\alpha, (HS)\int_a^b F_{\lambda_1}^+(t)\,\mathrm{d}\alpha\right] \supset \left[(HS)\int_a^b F_{\lambda_2}^-(t)\,\mathrm{d}\alpha, (HS)\int_a^b F_{\lambda_2}^+(t)\,\mathrm{d}\alpha\right]. \tag{31}$$

From Theorem 3.2 and Lemma 4.1 we have

$$(FHS)\int_a^b \tilde{F}(t)\,\mathrm{d}\alpha := \bigcup_{\lambda\in(0,1]} \lambda\left[(HS)\int_a^b F_\lambda^-\mathrm{d}\alpha, (HS)\int_a^b F_\lambda^+\mathrm{d}\alpha\right] \in \tilde{R} \tag{32}$$

and $\forall\lambda\in(0,1]$, $\left[(FHS)\int_a^b \tilde{F}(t)\,\mathrm{d}\alpha\right]_\lambda = \bigcap_{n=1}^{\infty}(IHS)\int_a^b F_{\lambda_n}(t)\,\mathrm{d}\alpha$, wherever $\lambda_n = \left[1 - \frac{1}{(n+1)}\right]\lambda.$ □

Using Theorem 4.1 and the properties of (IHS) integral, we are able to get the properties of (FHS) integral, for example, 1) the linear, 2) monotone, 3) interval additive properties of (FHS) integral.

5. Conclusion

In this paper, we proposed the definition of the Henstock-Stieltjes (HS) integrals of interval-valued functions and fuzzy-number-valued functions and investigated some properties of those integrals.

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