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Non-Negative Integer Solutions of Two Diophantine Equations $2^x + 9^y = z^2$ and $5^x + 9^y = z^2$

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Abstract

In this paper, we study two Diophantine equations of the type $p^x + 9^y = z^2$, where p is a prime number. We find that the equation $2^x + 9^y = z^2$ has exactly two solutions (x, y, z) in non-negative integer *i.e.*, $\{(3,0,3),(4,1,5)\}$ but $5^x + 9^y = z^2$ has no non-negative integer solution.

Keywords

Exponential Diophantine Equation, Integer Solutions

1. Introduction

Recently, there have been a lot of studies about the Diophantine equation of the type $a^x + b^y = c^z$. In 2012, B. Sroysang [1] proved that $(1,0,2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2013, B. Sroysang [2] showed that the Diophantine equation $3^x + 17^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1,0,2)$. In the same year, B. Sroysang [3] found all the solutions to the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. The solutions (x, y, z) are $(0,1,2)$, $(3,0,3)$ and $(4,2,5)$. In 2013, Rabago [4] showed that the solutions (x, y, z) of the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ where x, y and z are non-negative integers are $\{(1,0,2),(4,1,10)\}$ and $\{(1,0,2),(2,1,10)\}$, respectively. Different examples of Diophantine equations have been studied (see for instance [5]-[11]).

In this study, we consider the Diophantine equation of the type $p^x + 9^y = z^2$ where p is prime. Particularly, we show that $2^x + 9^y = z^2$ has exactly two solutions in non-negative integer and $5^x + 9^y = z^2$ has no

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non-negative integer solution.

2. Main Results

Theorem 2.1. (Catalan's Conjecture [12]) The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $a, b, x, y > 1$, has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$.

Theorem 2.2. The Diophantine equation $2^x + 1 = z^2$ has a unique non-negative integer solution $(x, z) = (3, 3)$.

Proof: Let x and z be non-negative integers such that $2^x + 1 = z^2$. For $x = 0$, $z^2 = 2$ which is impossible. Suppose $x \geq 1$. Then, $2^x = z^2 - 1 = (z+1)(z-1)$. Let $(z+1) = 2^\xi$ and $(z+1) = 2^\eta$, where $\eta < \xi, \xi + \eta = x$. Thus, $2^\xi - 2^\eta = 2$ or, $2^\eta\left(2^{\xi-\eta} - 1\right) = 2$. Now we have two possibilities.

Case-1: If $2^\eta = 2$, then $2^{\xi-\eta} - 1 = 1$. These give us $\eta = 1$ and $\xi = 2$. Then $x = 3$ and $z = 3$. Thus $(x, z) = (3, 3)$ is a solution of $2^x + 1 = z^2$.

Case-2: If $2^\eta = 1$, then $2^{\xi-\eta} - 1 = 2$. These give us $\eta = 0$ and $2^\xi = 3$ which is impossible.

Hence, $(x, z) = (3, 3)$ is a unique non-negative integer solution for the equation $2^x + 1 = z^2$.

Theorem 2.3. *The Diophantine equation* $p^x + 1 = z^2$, *where p is an odd prime number*, *has exactly one non-negative integers solution* $(x, z, p) = (1, 2, 3)$.

Proof: Let x and z be non-negative integers such that $p^x + 1 = z^2$, where p be an odd prime. If $x = 0$, then $z^2 = 2$. It is impossible. If $z = 0$, then $p^x = -1$, which is also impossible. Now for $x, z > 0$,

$$p^x + 1 = z^2$$

or $p^x = z^2 - 1 = (z-1)(z+1)$.

Let $z + 1 = p^\xi$ and $z - 1 = p^\psi$, where $\psi < \xi$, $\psi + \xi = x$. Then,

$$p^\xi - p^\psi = 2$$

or $p^\psi\left(p^{\xi-\psi} - 1\right) = 2$.

Thus, $p^\psi = 1 \Rightarrow p^\psi = p^0 \Rightarrow \psi = 0$ and $p^{\xi-\psi} - 1 = 2 \Rightarrow p^\xi = 3$, which is possible only for $p = 3$ and $\xi = 1$. So $x = \psi + \xi = 0 + 1 = 1$, $z = p^\xi - 1 = 3^1 - 1 = 2$.

Therefore, $(x, z, p) = (1, 2, 3)$ is the solution of $p^x + 1 = z^2$. This proves the theorem.

Corollary 2.4. *The Diophantine equation* $5^x + 1 = z^2$ *has no non-negative integers solution*.

Theorem 2.5. The Diophantine equation $1 + 9^y = z^2$ has no unique non-negative integer solution.

Proof: Suppose x and z be non-negative integers such that $1 + 9^x = z^2$. For $x = 0$, we have $z^2 = 2$. It is impossible. Let $x \geq 1$. Then $1 + 9^x = z^2$ gives us $3^{2x} = (z-1)(z+1)$. Let $z + 1 = 3^{\Pi_1}$ and $z - 1 = 3^{\Pi_2}$, where $\Pi_2 < \Pi_1$, $\Pi_1 + \Pi_2 = 2x$. Therefore,

$$3^{\Pi_1} - 3^{\Pi_2} = 2$$

or $3^{\Pi_2}\left(3^{\Pi_1-\Pi_2} - 1\right) = 2$.

Thus, $3^{\Pi_2} = 1$ or $\Pi_2 = 0$ and $3^{\Pi_1-\Pi_2} - 1 = 2$ or $\Pi_1 = 1$. So $2x = 1 \Rightarrow x = \frac{1}{2}$, which is not acceptable since x is a non-negative integer. This completes the proof.

Theorem 2.6. The Diophantine equation $2^x + 9^y = z^2$ has exactly two solutions (x, y, z) in non-negative integer *i.e.*, $\{(3,0,3), (4,1,5)\}$.

Proof: Suppose x, y and z are non-negative integers for which $2^x + 9^y = z^2$. If $x = 0$, we have $1 + 9^y = z^2$ which has no solution by theorem 2.5. For $y = 0$, by theorem 2.2 we have $x = 3$ and $y = 3$. Hence $(x, y, z) = (3, 0, 3)$ is a solution to $2^x + 9^y = z^2$. If $z = 0$, then $2^x + 9^y = 0$ which is not possible for any non-negative integers x and y.

Now we consider the following remaining cases.

Case-1: $x = 1$. If $x = 1$, then $2 + 9^y = z^2$ or $2 = \left(z + 3^y\right)\left(z - 3^y\right)$. We have two possibilities. If $z + 3^y = 1$ and $z - 3^y = 2$, then $2z = 3$ or $z = \frac{3}{2}$ but which is not acceptable. On the other hand, if $z + 3^y = 2$ and $z - 3^y = 1$ same thing is occurred.

Case-2: $y = 1$. If $y = 1$, then $2^x + 9 = z^2$ or $2^x = (z+3)(z-3)$. Let $z + 3 = 2^\xi$ and $z - 3 = 2^\eta$, where $\eta < \xi, \xi + \eta = x$. Then $2^\xi - 2^\eta = 2.3$ or $2^\eta\left(2^{\xi-\eta} - 1\right) = 2.3$. Thus, $2^\eta = 2$ and $2^{\xi-\eta} - 1 = 3$, then this implies that $\eta = 1$ and $\xi - 1 = 2$ or $\xi = 3$. So $x = 4$ and $z = 5$. Here we obtain the solution $(x, y, z) = (4, 1, 5)$.

Case-3: $z = 1$. If $z = 1$, then $2^x + 9^y = 1$ which is not possible for any for any non-negative integers *x* and *y*.

Case-4: $x, y, z > 1$. Now

$$2^x + 9^y = z^2 \text{ or } 2^x = \left(z + 3^y\right)\left(z - 3^y\right).$$

Let $z + 3^y = 2^{\Pi_1}$ and $z - 3^y = 2^{\Pi_2}$, where $\Pi_2 < \Pi_1, \Pi_1 + \Pi_2 = x$. So $2^{\Pi_1} - 2^{\Pi_2} = 2.3^y$ or $2^{\Pi_2}\left(2^{\Pi_1 - \Pi_2} - 1\right) = 2.3^y$. Thus, $2^{\Pi_2} = 2$ and $2^{\Pi_1 - \Pi_2} - 1 = 3^y$ then these imply that $\Pi_2 = 1$ and $2^{\Pi_1 - 1} - 1 = 3^y$. So we get

$$2^{\Pi_1 - 1} - 3^y = 1 \tag{1}$$

The Diophantine Equation (1) is a Diophantine equation by Catalan's type $a^x - b^y = 1$ because for $y > 1$, the value of $\Pi_1 - 1$ must be grater that 1. So by the Catalan's conjecture Equation (1) has no solution. This proves the theorem.

Theorem 2.7. The Diophantine equation $5^x + 9^y = z^2$ has no non-negative integer solution.

Proof: Suppose x, y and z are non-negative integers for which $5^x + 9^y = z^2$. If $x = 0$, we have $1 + 9^y = z^2$ which has no solution by Theorem 2.5. For $y = 0$ we use corollary 2.4. If $z = 0$, then $5^x + 9^y = 0$ which is not possible for any non-negative integers *x* and *y*.

Now we consider the following remaining cases.

Case-1: $x = 1$. If $x = 1$, then $5 + 9^y = z^2$ or $5 = \left(z + 3^y\right)\left(z - 3^y\right)$. We have two possibilities. If $z + 3^y = 5$ and $z - 3^y = 1$, it follows that $2z = 6$ or $z = 3$ and $3^y = 2$, a contradiction. On the other hand, $z + 3^y = 1$ and $z - 3^y = 5$, it follows that $2z = 6$ or $z = 3$ and $3^y = -2$ which is impossible.

Case-2: $y = 1$. If $y = 1$, then $5^x + 9 = z^2$ or $5^x = (z + 3)(z - 3)$. Let $z + 3 = 5^\xi$ and $z - 3 = 5^\eta$, where $\eta < \xi, \xi + \eta = x$. Then $5^\xi - 5^\eta = 2.3$ or $5^\eta\left(5^{\xi - \eta} - 1\right) = 2.3$. Thus, $5^\eta = 1$ and $5^{\xi - \eta} - 1 = 6$, then this implies that $\eta = 0$ and $5^\xi = 7$, a contradiction.

Case-3: $z = 1$. If $z = 1$, then $5^x + 9^y = 1$ which is not possible for any for any non-negative integers *x* and *y*.

Case-4: $x, y, z > 1$. Now

$$5^x + 9^y = z^2 \text{ or } 5^x = \left(z + 3^y\right)\left(z - 3^y\right)$$

Let $z + 3^y = 5^{\Pi_1}$ and $z - 3^y = 5^{\Pi_2}$, where $\Pi_2 < \Pi_1, \Pi_1 + \Pi_2 = x$. So $5^{\Pi_1} - 5^{\Pi_2} = 2.3^y$ or $5^{\Pi_2}\left(5^{\Pi_1 - \Pi_2} - 1\right) = 2.3^y$. Thus, $5^{\Pi_2} = 1$ and $5^{\Pi_1 - \Pi_2} - 1 = 2.3^y$ then these imply that $\Pi_2 = 0$ and $5^{\Pi_1} - 1 = 2.3^y$. Since $5 \equiv 1 \pmod 4$, it follows that $5^{\Pi_1} \equiv 1 \pmod 4$ *i.e.*, $5^{\Pi_1} - 1 \equiv 0 \pmod 4$. But we see that $2.3^y \not\equiv 0 \pmod 4$. This is impossible.

3. Conclusion

In the paper, we have discussed two Diophantine equation of the type $p^x + 9^y = z^2$, where *p* is a prime number. We have found that $(3, 0, 3)$ and $(4, 1, 5)$ are the exact solutions to $2^x + 9^y = z^2$ in non-negative integers. On the contrary, we have also found that the Diophantine equation $5^x + 9^y = z^2$ has no non-negative integer solution.

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