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On the Strongly Damped Wave Equations with Critical Nonlinearities

Qinghua Zhang

Department of Mathematics, Nantong University, Nantong, China
Email: zhangqh1971@126.com

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Abstract

We study the strongly damped wave equations with critical nonlinearities. By choosing suitable state spaces, we prove sectorial property of the operator matrix $\begin{bmatrix} 0 & -I \\ A & \eta A^{\theta} \end{bmatrix}$ together with its adjoint operator, investigate the associated interpolation and extrapolation spaces, analysis the criticality of the nonlinearity with critical growth, and study the higher spatial regularity of the *Y*-regular solution by bootstrapping.

Keywords

Negative Laplacian, Wave Equation, Strong Damping, Sectorial Operator, Fractional Power, Global Attractor

1. Introduction

This paper deals with a class of wave equations with strong damping

$$\begin{cases} u_{tt} + \eta(-\Delta)^{\theta} u_t + (-\Delta)u = f(u), & t > 0, x \in \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), & x \in \Omega, \\ u(t,x) = 0, & t \geq 0, x \in \partial\Omega. \end{cases} \tag{1}$$

Here $\Omega \subseteq \mathbf{R}^N$ $(N \geq 3)$ is a bounded domain with C^2 boundary, and $\eta > 0$ is the coefficient of strong damping. Let $X = L^2(\Omega)$, then the negative Laplacian $-\Delta$, denoted by *A*, is a positive definite and self-adjoint operator defined in *X* with compact inverse. For each $\alpha \in \mathbb{R}$, there define A^{α} and X_{α} as the fractional power of *A* and its domain endowed with the graph norm respectively. Evidently, in this setting, $X_{1/2} = H_0^1(\Omega)$, $X_1 = H^2(\Omega) \cap H_0^1(\Omega)$, $X_{-1/2} = H^{-1}(\Omega)$, and for all $\alpha > 0$, we have $X_{-\alpha} = (X_{\alpha})'$.

Introduce the energy space $Y = X_{1/2} \times X$ as our work space, and let $v = u_t$, $\boldsymbol{A}_\theta = \begin{bmatrix} 0 & -I \\ A & \eta A^\theta \end{bmatrix}$, $\mathcal{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix}$, then Equation (1) turns to be an abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} + \boldsymbol{A}_\theta \begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right), \ t > 0, \tag{2}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{3}$$

and we can treat it in the framework of semigroup of operators.

Recall that, the operator matrix $\boldsymbol{A}_\theta$ itself is not closed in *Y*, and consequently its negative is not a generator of any C_0-semigroups except $\theta = 1$. But its closure, which is still denoted by $\boldsymbol{A}_\theta$, is a sectorial operator whenever $1/2 \le \theta < 1$, and its negative generates an analytic and exponential decaying semigroup (see [1]-[3] for references).

By using the notation of ε-regular solution introduced in [4] [5] together with interpolation and extrapolation spaces, and under the Lipschitz condition,

$$\left|f(u) - f(u')\right| \le C\left|u - u'\right|\left(1 + |u|^{\rho-1} + |u'|^{\rho-1}\right), \quad \forall u, u' \in \mathbf{R} \tag{4}$$

$$\left(1 \le \rho \le (N+2)/(N-2)\right).$$

Carvalho-Cholewa in [1] and lately Carvalho-Cholewa-Dlotko in [2] studied the local existence and regularity of the ε-regular (or *Y*-regular in this paper) solution of Equation (1). Under the dissipative condition,

$$\limsup_{|s| \to \infty} \frac{f(s)}{s} \le 0,. \tag{5}$$

Carvalho-Cholewa in [6] investigate the global existence of ε-regular solutions in the subcritical case $1 \le \rho < (N+2)/(N-2)$, together with the existence and regularity of the universal attractors. As for the critical case $\rho = (N+2)/(N-2)$, there are few references except $\theta = 1$. According to the general theory of the ε-regular solutions, in this case, the related nonlinear map $\mathcal{F}: Y_\varepsilon \to Y_{\eta-1}$ is critical (*i.e.* $0 \le \varepsilon < \rho^{-1}$, and η can only take the value $\rho\varepsilon$), consequently for a ε-regular solution arising in the energy space, boundedness of the *Y*-norm on its maximal existence interval could not guarantee the global existence (see [1] [2]).

Here we are concerned with the higher regularity and global existence of the *Y*-regular solution of Equation (1). By introducing a new state space $E_{\theta,s} = X_s \times X_{s-\theta}$ $(1/2 \le s \le 1)$ weak than *Y* somewhat, we will reveal that, the operator matrix $\boldsymbol{A}_\theta$ is also sectorial, together with its dual operators $\boldsymbol{A}_\theta^\#$. Moreover, all the interpolation and extrapolation spaces $\left(E_{\theta,s}\right)_\alpha$ $(\alpha \in [-1,1])$ can be expressed by the Cartesian products. And consequently, for $\rho = (N+2)/(N-2)$ and $1/2 < \theta \le 1$, the corresponding nonlinearity $\mathcal{F}$ turns to be subcritical. Using these properties, we will prove by bootstrapping that every $E_{\theta,s}$-regular solution of (2) with the initial value taken in *Y* is a strong one exactly. Moreover, this solution exists on the whole interval $\mathbb{R}^+$, or its *Y*-norm blows up in finite time. Results obtained here, which can be viewed as useful supplements to the references listed above, tell us that in a semilinear parabolic equation, substitution of phase spaces may change the criticality of the nonlinear perturbation attached to it. In other words, criticality is not absolute for the parabolic systems in many concrete situations.

2. Main Results and Proofs

Lemma 2.1 *Suppose that X and Y are two Banach spaces, A is a sectorial operators defined in X, and B is a linear operator densely defined in Y. Suppose also there is a homeomorphism* $Q: X \to Y$ *satisfying* $QA = BQ$, *then B is also sectorial together with* $\sigma(A) = \sigma(B)$ *and* $\mathrm{e}^{-tB} = Q\mathrm{e}^{-tA}Q^{-1}$ *(see* [7], *§5.2).*

Lemma 2.2 *The operator matrix* A_θ *is sectorial in the new space* $E_\theta := E_{\theta,\theta} = X^\theta \times X$, *and* $Re\left(\sigma(A_\theta)\right) > 0$. *Moreover, the domain* $\mathcal{F}(A_\theta) =: (E_\theta)_1$ *equipped with the graph norm is equivalent to the product space* $X^1 \times X^\theta$ (cf. [8]).

For the Hilbert space $X = L^2(\Omega)$ and the operator $A = -\Delta$ introduced above, consider the interpolation-extrapolation Hilbert scale $\{(X_\alpha, A_\alpha):\alpha\in\mathbb{R}\}$, where $X_\alpha=\mathcal{D}(A^\alpha)$ if $\alpha\ge 0$, $X_\alpha=\overline{(X,\|A^\alpha\cdot\|)}$ if $\alpha<0$, and A_α is the realization of A in the space X_α. For the real and complex interpolation methods, please refer to [9], Ch.1, and for the extrapolation method, see [10], Ch. V for references. Recall that, for every $\alpha\in\mathbb{R}$, A_α is also a sectorial operator in X_α, and $A^\alpha\in\mathcal{L}$ is (X_α, X), $(A_{-\alpha})^\alpha\in\mathcal{L}$ is $(X, X_{-\alpha})$ for all $\alpha\ge 0$ (cf. [10], § 5.1.3).

Define the realization of $\boldsymbol{A}_\theta$ in $E_{\theta,s}$ as follows:

$$\boldsymbol{A}_{\theta,s}=\begin{bmatrix}0 & -I\\ (A_{s-\theta})^{\theta-s}A^{1+s-\theta} & \eta(A_{s-\theta})^{\theta-s}A^s\end{bmatrix},\ \text{if } 1/2<s\le\theta;$$

$$\boldsymbol{A}_{\theta,s}=\begin{bmatrix}0 & -I\\ A^{\theta-s}A^{1+s-\theta} & \eta A^{\theta-s}A^s\end{bmatrix},\ \text{if } \theta<s\le 1.$$

It is easy to check that, for all $s\in[1/2,1]$, $(E_{\theta,s})_1:=\mathcal{D}(\boldsymbol{A}_{\theta,s})=X_{1+s-\theta}\times X_s$ in the sense of equivalent norms. Furthermore, we have

Lemma 2.3 *$\boldsymbol{A}_{\theta,s}$ is sectorial in the state space $E_{\theta,s}$ with the same spectrum as $\boldsymbol{A}_\theta$ has.*

Proof: This lemma can be easily verified by Lemma 2.1, together with the fact that the following operator

$$\boldsymbol{Q}_{\theta,s}=\begin{bmatrix}(A_{s-\theta})^{\theta-s} & 0\\ 0 & (A_{s-\theta})^{\theta-s}\end{bmatrix},\ \text{if } 1/2<s\le\theta,$$

$$\boldsymbol{Q}_{\theta,s}=\begin{bmatrix}A^{\theta-s} & 0\\ 0 & A^{\theta-s}\end{bmatrix},\ \text{if } \theta<s\le 1.$$

is an isomorphism between E_θ and $E_{\theta,s}$, satisfying $\boldsymbol{Q}_{\theta,s}\boldsymbol{A}_{\theta,s}=\boldsymbol{A}_\theta\boldsymbol{Q}_{\theta,s}$. □

Consider another operator matrix $\boldsymbol{A}^{\#}_{\theta,s}$ defined below,

$$\boldsymbol{A}^{\#}_{\theta,s}=\begin{bmatrix}0 & A^{1-2\theta}\\ -(A_{s-\theta})^{\theta-s}A^{\theta+s} & \eta(A_{s-\theta})^{\theta-s}A^s\end{bmatrix},\ \text{if } 1/2<s\le\theta,$$

$$\boldsymbol{A}^{\#}_{\theta,s}=\begin{bmatrix}0 & A^{1-2\theta}\\ -A^{\theta-s}A^{1+s-\theta} & \eta A^{\theta-s}A^s\end{bmatrix},\ \text{if } \theta<s\le 1.$$

Evidently, $\boldsymbol{A}^{\#}_{\theta,s}$ is closed in the space $E^{\#}_{\theta,s}:=E'_{\theta,s}=X_s\times X_{s-\theta}$ with domain $\mathcal{D}(\boldsymbol{A}^{\#}_{\theta,s})=X_{s+\theta}\times X_s$. And for all $\begin{bmatrix}u\\ v\end{bmatrix}=\mathcal{D}(\boldsymbol{A}_{\theta,s})$ and $\begin{bmatrix}\varphi\\ \psi\end{bmatrix}=\mathcal{D}(\boldsymbol{A}^{\#}_{\theta,s})$, we have

$$\begin{aligned}
&\left(\boldsymbol{A}_{\theta,s}\begin{bmatrix}u\\ v\end{bmatrix}\middle|\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right)_{E_{\theta,s}}\\
&=\left(\begin{bmatrix}-v\\ (A_{s-\theta})^{\theta-s}A^{1+s-\theta}u+\eta(A_{s-\theta})^{\theta-s}A^s v\end{bmatrix}\middle|\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right)_{E_{\theta,s}}\\
&=-(v\mid\varphi)_{X_s}+\left((A_{s-\theta})^{\theta-s}A^{1+s-\theta}u\mid\psi\right)_{X_{s-\theta}}+\left(\eta(A_{s-\theta})^{\theta-s}A^s v\mid\psi\right)_{X_{s-\theta}}\\
&=-\left(A^s v\mid A^s\varphi\right)_X+\left(A^{1+s-\theta}u\mid A^{s-\theta}\psi\right)_X+\left(A^s v\mid\eta A^{s-\theta}\psi\right)_X\\
&=-\left(v\mid(A_{s-\theta})^{\theta-s}A^{\theta+s}\varphi\right)_{X_{s-\theta}}+\left(u\mid A^{1-2\theta}\psi\right)_{X_s}+\left(v\mid\eta(A_{s-\theta})^{\theta-s}A^s\psi\right)_{X_{s-\theta}}\\
&=\left(\begin{bmatrix}u\\ v\end{bmatrix}\middle|\begin{bmatrix}A^{1-2\theta}\psi\\ -(A_{s-\theta})^{\theta-s}A^{\theta+s}\varphi+\eta(A_{s-\theta})^{\theta-s}A^s\psi\end{bmatrix}\right)_{E_{\theta,s}}\\
&=\left(\begin{bmatrix}u\\ v\end{bmatrix}\middle|\boldsymbol{A}^{\#}_{\theta,s}\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right)_{E_{\theta,s}}.
\end{aligned}$$

This tell us that, $\boldsymbol{A}_{\theta,s}^{\#}$ is contained in $\boldsymbol{A}_{\theta,s}'$, the adjoint operator of $\boldsymbol{A}_{\theta,s}$. In order to show the equality $\boldsymbol{A}_{\theta,s}^{\#}=\boldsymbol{A}_{\theta,s}'$, it suffices to check that $\sigma\left(\boldsymbol{A}_{\theta,s}^{\#}\right)\cap\sigma\left(\boldsymbol{A}_{\theta,s}'\right)\neq\varnothing$, which is a consequence of the following lemma.

Lemma 2.4 *$\boldsymbol{A}_{\theta,s}^{\#}$ is sectorial in $E_{\theta,s}$ with the spectrum $\sigma\left(\boldsymbol{A}_{\theta,s}^{\#}\right)\supseteq\{Rez>0\}$.*

Proof of this lemma is much similar to that of Lemma 2.3, and here we omit it.

Denote $\left(E_{\theta,s}^{\#}\right)_1:=\mathcal{D}\left(\boldsymbol{A}_{\theta,s}^{\#}\right)$, which is isomorphic to the product space $X_{s+\theta}\times X_s$ according to the graph norm.

Now we can give some representations for the interpolation and extrapolation spaces attached to $\boldsymbol{A}_{\theta,s}$. For each $\alpha\in[0,1]$, we have

$$\left(E_{\theta,s}\right)_\alpha=\left[E_{\theta,s},\left(E_{\theta,s}\right)_1\right]_\alpha=X_{s+\alpha(1-\theta)}\times X_{s-(1-\alpha)\theta},\tag{6}$$

and

$$\left(E_{\theta,s}^{\#}\right)_\alpha=\left[E_{\theta,s}^{\#},\left(E_{\theta,s}^{\#}\right)_1\right]_\alpha=X_{s+\alpha\theta}\times X_{s-(1-\alpha)\theta}.$$

Thus by the dual principle (refer to [10], Ch. V, thm. 1.5.12), we obtain

$$\begin{aligned}\left(E_{\theta,s}\right)_{-\alpha}&=\left(\left(E_{\theta,s}^{\#}\right)_\alpha\right)'=\left(\left(X_s\right)_{\alpha\theta}\right)'\times\left(\left(X_{s-\theta}\right)_{\alpha\theta}\right)'\\&=\left(X_s\right)_{-\alpha\theta}\times\left(X_{s-\theta}\right)_{-\alpha\theta}=X_{s-\alpha\theta}\times X_{s-(1+\alpha)\theta}.\end{aligned}$$

Hence, for each $\gamma\in[0,1]$, we have that

$$\left(\left(E_{\theta,s}\right)_{-1}\right)_\gamma=\left(E_{\theta,s}\right)_{-1+\gamma}=X_{s-(1-\gamma)\theta}\times X_{s-(2-\gamma)\theta}\tag{7}$$

in the sense of isomorphism.

Let us study the nonlinear operator $\mathcal{F}$ in the case $\rho=(N+2)/(N-2)$ and $\theta\in(1/2,1]$ in new state spaces.

Theorem 2.5 *Take $\gamma_0=2-\theta^{-1}\in(0,1]$, then under the assumption* (4), *for each $\gamma\in(0,\gamma_0]$, $\mathcal{F}:E_{\theta,s}\to\left(E_{\theta,s}^{-1}\right)_\gamma$ is bounded and locally Lipschitz. More precisely, $\mathcal{F}$ verifies*

$$\left\|\mathcal{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right)-\mathcal{F}\left(\begin{bmatrix}u'\\v'\end{bmatrix}\right)\right\|_{\left(\left(E_{\theta,s}\right)_{-1}\right)_\gamma}\leq C\left\|\begin{bmatrix}u\\v\end{bmatrix}-\begin{bmatrix}u'\\v'\end{bmatrix}\right\|_{E_{\theta,s}}\left(1+\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\|_{E_{\theta,s}}^{\rho-1}+\left\|\begin{bmatrix}u'\\v'\end{bmatrix}\right\|_{E_{\theta,s}}^{\rho-1}\right).\tag{8}$$

Proof: Firstly using the embedding $X_s\to H^{2s}(\Omega)$, we can easily deduce that $X_s\to C\left(\overline{\Omega}\right)$ if $s>N/4$, and $X_s\to L^\sigma(\Omega)$ for all $\sigma\in[1,\infty)$ if $s=N/4$. Notice that $s-(2-\gamma)\theta\leq 0$ for all $\gamma\in(0,\gamma_0]$. Hence for the number *s* satisfying $N/4\leq s\leq 1$, by invoking (4), we find that the Nemytskij operator of *f*, denoted also by *f* verifies

$$\left\|f(u)-f(v)\right\|_{X_{s-(2-\gamma)\theta}}\leq C\left\|u-v\right\|_{X_s}\left(\left\|u\right\|_{X_s}^{\rho-1}+\left\|v\right\|_{X_s}^{\rho-1}\right).$$

This inequality, together with the definition of $\mathcal{F}$ and (7) leads to the desired inequality (8).

If $s<N/4$, then we have the following embedding

$$X_s\to L^\sigma(\Omega),\text{ if }1\leq\sigma\leq\frac{2N}{N-4s};\tag{9}$$

$$X_{s-(2-\gamma)\theta}\leftarrow L^r(\Omega),\text{ if }r\geq\max\left\{1,\frac{2N}{N-4s+4(2-\gamma)\theta}\right\}.\tag{10}$$

And simple calculations show that in case $N\geq 3$, for all $s\in[1/2,N/4)$ and $\gamma\in(0,\gamma_0]$, inequalities

$$\frac{2N}{N-4s}\geq\frac{N+2}{N-2}$$

and

$$\frac{N-4s+4(2-\gamma)\theta}{N-4s} \geq \frac{N+2}{N-2}$$

hold simultaneously. Thus for the number r verifying the restriction in (10), the other number $\sigma = \rho r$ satisfies the restriction in (9). Hence by invoking (9), (10) and (4), we obtain

$$\begin{aligned}\|f(u)-f(v)\|_{X_{s-(2-\gamma)\theta}} &\leq \|f(u)-f(v)\|_{L^r(\Omega)} \\ &\leq C\|u-v\|_{L^{\rho r}(\Omega)}\left(\|u\|_{L^{\rho r}(\Omega)}^{\rho-1}+\|v\|_{L^{\rho r}(\Omega)}^{\rho-1}\right) \\ &\leq C\|u-v\|_{X_s}\left(\|u\|_{X_s}^{\rho-1}+\|v\|_{X_s}^{\rho-1}\right),\end{aligned}$$

which means that inequality (8) still holds in the case $s < N/4$. This complete the proof. □

Theorem 2.6 *Let* $\alpha_0 = \max\left\{0, 1-2(n+2)^{-1}(1-\theta)^{-1}\right\}$, *then under the assumption* (4), *for all* $\alpha \in (\alpha_0, 1)$, *the operator* $\mathcal{F}$ *satisfies*

$$\left\|\mathcal{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right)-\mathcal{F}\left(\begin{bmatrix}u'\\v'\end{bmatrix}\right)\right\|_{E_\theta} \leq C\left\|\begin{bmatrix}u\\v\end{bmatrix}-\begin{bmatrix}u'\\v'\end{bmatrix}\right\|_{(E_\theta)_\alpha}\left(1+\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\|_{(E_\theta)_\alpha}^{\rho-1}+\left\|\begin{bmatrix}u'\\v'\end{bmatrix}\right\|_{(E_\theta)_\alpha}^{\rho-1}\right). \tag{11}$$

Similar to Thm. 2.5, core of the proof for this theorem is to check the validity of the following inequality

$$\|f(u)-f(u')\|_X \leq C\|u-u'\|_{X_{\theta+\alpha(1-\theta)}}\left(1+\|u\|_{X_{\theta+\alpha(1-\theta)}}^{\rho-1}+\|u'\|_{X_{\theta+\alpha(1-\theta)}}^{\rho-1}\right)$$

under condition (4). Here we omit the whole process.

Remark 2.7 *In the new state spaces*, *the nonlinearity* $\mathcal{F}$ *turns to be a subcritical map* (*please compare to* [1] [2]).

Now we can investigate higher regularity and global existence of solutions of the abstract Cauchy problem (2) + (3) for the critical growth exponent $\rho = (N+2)/(N-2)$ in the case $1/2 < \theta \leq 1$. In view of [4] and [1] [2], we know that for the initial point $\begin{bmatrix}u_0\\v_0\end{bmatrix} \in Y$, there exists a unique ε-regular (or in other words Y-regular) solution $\begin{bmatrix}u\\v\end{bmatrix}(\cdot, u_0, v_0)$ defined on an interval $[0, T]$ for some $T > 0$, s.t.

$$\begin{bmatrix}u\\v\end{bmatrix} \in C([0,\tau_0), Y) \cap C((0,\tau_0), Y_{\rho\varepsilon}) \cap C^1((0,\tau_0), Y_\eta) \quad \forall \eta \in [0, \rho\varepsilon) \tag{12}$$

for some $\varepsilon \in (0, 1/2]$, and Equtaion (2) is satisfied in the space $Y_{-1+\rho\varepsilon}$. If $\begin{bmatrix}u_0\\v_0\end{bmatrix}$ lies in the space $E_{\theta,s}$, then thanks to (8), there exists another interval $[0, T]$, on which there is a unique $E_{\theta,s}$-regular solution $\begin{bmatrix}u_0\\v_0\end{bmatrix}$ satisfying

$$\begin{bmatrix}u\\v\end{bmatrix} \in C([0,\tau), E_{\theta,s}) \cap C\left((0,\tau), (E_{\theta,s})_{\gamma_0}\right) \cap C^1\left((0,\tau), (E_{\theta,s})_\beta\right) \tag{13}$$

for all $\beta \in [0, \gamma_0)$ together with Equation (2) satisfied in the space $(E_{\theta,s})_{-1+\gamma_0}$ (see [7], Ch. 6 or [11], Ch. 3 for references).

Take $s = 1/2$, then by the uniqueness and regularity mentioned above, we can easily find that an $E_{\theta,1/2}$-regular solution is equal to a Y-regular one on the common existing interval if they have the same initial value. Denote by $[0,\tau)$ and $[0,\tau_0)$ respectively the maximal intervals of $\begin{bmatrix}u\\v\end{bmatrix}(\cdot, u_0, v_0)$ existing as a Y-regular solution and as an $E_{\theta,1/2}$-regular one with $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y$. In the following paragraph, we will prove that $\tau_0 = \tau$. Evidently $\tau_0 \le \tau$ since $Y \to E_{\theta,1/2}$. For the inverse inequality, it suffices to show that $\begin{bmatrix} u \\ v \end{bmatrix} \in C([0,T],Y)$ for arbitrary $T \in (0,\tau)$ (cf. [12]). This can be done by bootstrapping.

Taking any $t_1 \in (0,\tau)$, and using (13) and (6), we obtain

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}(t_1,u_0,v_0) \in \left(E_{\theta,1/2}\right)_{\gamma_0} &= X_{1/2+\gamma_0(1-\theta)} \times X_{1/2-(1-\gamma_0)\theta} \\ &\to X_{1/2+\gamma_0(1-\theta)} \times X_{1/2+\gamma_0(1-\theta)-\theta} = E_{\theta,1/2+\gamma_0(1-\theta)}. \end{aligned} \tag{14}$$

Regard t_1 and $E_{\theta,1/2+\gamma_0(1-\theta)}$ as the initial time and space respectively, then by invoking the local existence and uniqueness of the $E_{\theta,1/2+\gamma_0(1-\theta)}$-regular solution, we can find a time $\delta > 0$, such that

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &\in C\left([t_1,t_1+\delta],E_{\theta,1/2+\gamma_0(1-\theta)}\right) \cap C\left((t_1,t_1+\delta],\left(E_{\theta,1/2+\gamma_0(1-\theta)}\right)_{\gamma_0}\right) \\ &\cap C^1\left((t_1,t_1+\delta],\left(E_{\theta,1/2+\gamma_0(1-\theta)}\right)_{\beta}\right) \quad \forall \beta \in [0,\gamma_0). \end{aligned}$$

Here the time δ depends on the norm $\left\| \begin{bmatrix} u \\ v \end{bmatrix}(t_1,u_0,v_0) \right\|_{E_{\theta,1/2+\gamma_0(1-\theta)}}$ due to the subcriticality of $\mathcal{F}$ (8). Notice that $\begin{bmatrix} u \\ v \end{bmatrix}$ is uniformly continuous in $E_{\theta,1/2+\gamma_0(1-\theta)}$ on any bounded interval $[t_1,T] \subseteq [t_1,\tau)$ thanks to (13) and (14), therefore it can be extended to the whole interval $[t_1,\tau)$ as an $E_{\theta,1/2+\gamma_0(1-\theta)}$-regular solution. And similar to (14), for any $t_2 \in (t_1,\tau)$, we have that

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}(t_2,u_0,v_0) \in \left(E_{\theta,1/2+\gamma_0(1-\theta)}\right)_{\gamma_0} &= X_{1/2+2\gamma_0(1-\theta)} \times X_{1/2+\gamma_0(1-\theta)-(1-\gamma_0)\theta} \\ \to X_{1/2+2\gamma_0(1-\theta)} \times X_{1/2+2\gamma_0(1-\theta)-\theta} &= E_{\theta,1/2+2\gamma_0(1-\theta)}. \end{aligned}$$

The above inclusion is valid for all $t \in (0,\tau)$ due to the arbitrariness of t_1. Thus using the procedure performed above, we can deduce that, as an $E_{\theta,1/2+2\gamma_0(1-\theta)}$-regular solution,

$$\begin{bmatrix} u \\ v \end{bmatrix} \in C\left((0,\tau),\left(E_{\theta,1/2+2\gamma_0(1-\theta)}\right)_{\gamma_0}\right) \cap C^1\left((0,\tau),\left(E_{\theta,1/2+2\gamma_0(1-\theta)}\right)_{\beta}\right).$$

for all $\beta \in [0,\gamma_0)$.

Select $k \in \mathbb{N}$ so that $1/2 + (k+1)\gamma_0(1-\theta) \ge \theta$, and repeat the above step k times, we finally obtain

$$\begin{bmatrix} u \\ v \end{bmatrix} \in C\left((0,\tau),\left(E_{\theta,1/2+k\gamma_0(1-\theta)}\right)_{\gamma_0}\right) \cap C^1\left((0,\tau),\left(E_{\theta,1/2+k\gamma_0(1-\theta)}\right)_{\beta}\right). \tag{15}$$

for all $\beta \in [0,\gamma_0)$, and $\left(E_{\theta,1/2+k\gamma_0(1-\theta)}\right)_{\gamma_0} \to E_{\theta,1/2+(k+1)\gamma_0(1-\theta)} \to E_\theta \to Y$. Thus, for any $T \in (0,\tau)$, we can conclude that $\begin{bmatrix} u \\ v \end{bmatrix} \in C([0,T],Y)$, which leads to the desired conclusion $\tau \le \tau_0$.

Theorem 2.8 *Every Y-regular solution* $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot,u_0,v_0)$ *of the problem* (2) + (3) *with* $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y$ *is exactly the* *strong one on its maximal interval of existence* $[0,\tau)$. *More precisely,* $\begin{bmatrix} u \\ v \end{bmatrix}$ *verifies all the following properties*

- $\begin{bmatrix} u \\ v \end{bmatrix} \in C([0,\tau),Y) \cap C((0,\tau),(E_\theta)_1) \cap C^1((0,\tau),(E_\theta)_\beta)$ for all $\beta \in [0,1)$,
- Equation (2) holds in E_θ for all $t \in (0,\tau)$, and
- either $\limsup_{t\to\tau^-} \left\| \begin{bmatrix} u \\ v \end{bmatrix}(t,u_0,v_0) \right\|_Y = \infty$, *i.e.* $\begin{bmatrix} u \\ v \end{bmatrix}$ blows up in finite time, or $\tau = +\infty$, *i.e.* $\begin{bmatrix} u \\ v \end{bmatrix}$ exists globally.

Proof: Choose $k \in \mathbb{N}$ so that $1/2 + k\gamma_0(1-\theta) \ge 1$, then the inclusion (15) and the imbedding $(E_{\theta,1})_{\gamma_0} = X_{4-2\theta-\theta^{-1}} \times X_\theta \to X_1 \times X_\theta = (E_\theta)_1$ jointly produce 1). Moreover, thanks to (11), if we regard $(E_\theta)_\alpha$ ($\alpha \in (\alpha_0,1)$) as the initial space, and use the existence and uniqueness of the $(E_\theta)_\alpha$-regular solution, we can derive 2). Suppose that condition

$$\sup_{t\in[0,\tau)} \left\| \begin{bmatrix} u \\ v \end{bmatrix}(t,u_0,v_0) \right\|_Y < \infty \tag{16}$$

holds, then as an $E_{\theta,1/2}$-regular solution, $\begin{bmatrix} u \\ v \end{bmatrix}$ can be extended onto the whole interval $\mathbb{R}^+$ since $\mathcal{F} : E_{\theta,1/2} \to \left(\left(E_{\theta,1/2}\right)_{-1}\right)_{\gamma_0}$ is subcritical and $Y \to E_{\theta,1/2}$. Therefore $\tau_0 = \tau = +\infty$, and $\begin{bmatrix} u \\ v \end{bmatrix}$ exists globally as a Y-regular solution (it is a global strong solution indeed). This results means that (iii) holds. □

Remark 2.9 *From Thm.* 2.8(i), *one can conclude that the first component function* $u(\cdot,u_0,v_0)$ *of a Y-regular solution* $\begin{bmatrix} u \\ v \end{bmatrix}(t,u_0,v_0)$ *belongs to* $C^1((0,\tau),X_1) \cap C^2((0,\tau),X_\beta)$ *for all* $\beta \in [0,1)$, *and satisfies Equation* (1) *in the strong sense on its maximal existing interval* $(0,\tau)$ *definitely. In* [6], *the authors showed that,* $u(\cdot,u_0,v_0)$ *is the strong solution under the extra conditions* $3 \le N \le 5$ *and* $\theta \in (1/2, 4/(N+2))$. *And in* [2], *the authors proved that* $u(\cdot,u_0,v_0)$ *is the classical one whenever* $1/2 \le \theta < 2/3$. *In this sense, Thm* 2.8 *is a useful supplement to the above two results.*

Remark 2.10 *Under the assumptions* (4) *and* (5), *the following estimate is valid for* $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot,u_0,v_0)$ (*see* [6] [13]):

$$\mathcal{L}\left(\begin{bmatrix} u \\ v \end{bmatrix}(t,u_0,v_0)\right) \le C\sqrt{1+\mathcal{L}\left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right)}$$

where

$$\mathcal{L}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \frac{1}{2}\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_Y^2 - \int_\Omega \int_0^{u(x)} f(s)\,\mathrm{d}s\mathrm{d}x$$

is the energy functional attached to (2). Thus for every $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y$, condition (2.11) holds, and consequently $\tau = \infty$, $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot,u_0,v_0)$ is globally defined.

3. Further Discussions

By introducing some new state spaces, we investigate the higher regularity and global existence of the weak solution of the wave Equation (1) for the critical growth exponent $\rho = (N+2)/(N-2)$ in the case

$1/2<\theta\le 1$. Results obtained here show that criticality of the nonlinearity attached to a semilinear parabolic system is not absolutely. It depends on the state spaces selected in many concrete situations. On the other hand, we have to admitted that, methods used here are inadequate for $\theta=1/2$, since criticality of $\mathcal{F}$ does not change anymore ($\gamma=0$), regardless of the space $E_{1/2,s}$ we selected. In this case, condition (2.11) does not guarantee the global existence of the *Y*-regular solution any more. In [14], the authors proved that, under hypotheses (4) and (5), every *Y*-regular solution $\begin{bmatrix}u\\v\end{bmatrix}(t,u_0,v_0)$ arising in *Y* can be extended onto the whole interval $\mathbb{R}^+$ as a Y_{-1}-regular solution ($Y_{-1}=X\times X_{-1/2}$) or a piece-wise ε-regular solution in other words (see [12] for references). More precisely, $\begin{bmatrix}u\\v\end{bmatrix}$ verifies

1) $\begin{bmatrix}u\\v\end{bmatrix}\in C\left(\mathbb{R}^+,Y_{-1}\right)\cap BWC\left(\left[0,T\right],Y\right)$ for every $T\in(0,\infty)$,

2) $\left(\mathrm{d}/\mathrm{d}t\right)\begin{bmatrix}u\\v\end{bmatrix}\in WC\left(\mathbb{R}^+,Y_{-1}\right)$, and

3) there is a sequence of singular times $\{\tau_i\}$ with $\tau_i\to\infty$, s.t. on each $\left[\tau_{i-1},\tau_i\right)$ ($\tau_0=0$), $\begin{bmatrix}u\\v\end{bmatrix}$ is a *Y*-regular solution, and $\limsup_{t\to\tau_i^-}\left\|\begin{bmatrix}u(t)\\v(t)\end{bmatrix}\right\|_{Y_\varepsilon}=\infty$ for each $\varepsilon\in\left(0,(2\rho)^{-1}\right]$.

Thus, we can also consider the existence and regularity of the universal attractors.

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