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Schur Convexity and the Dual Simpson's Formula

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Abstract

In this paper, we show that some functions related to the dual Simpson's formula and Bullen-Simpson's formula are Schur-convex provided that f is four-convex. These results should be compared to that of Simpson's formula in Applied Math. Lett. (24) (2011), 1565-1568.

Keywords

Schur Convexity, 4-Convex Function, Dual Simpson's Formula, Bullen-Simpson's Formula

1. Introduction

Schur convexity is an important notion in the theory of convex functions, which were introduced by Schur in 1923 ([1] [2]), its definition is stated in what follows. Let $R^n_{\geq}$ be denoted as,

$$R^n_{\geq} = \left\{ x = \left(x_1, x_2, \cdots, x_n\right) \in R^n; x_1 \geq x_2 \geq \cdots \geq x_n \right\},$$

and $(R^n_{\geq})^+$ be defined by,

$$(R^n_{\geq})^+ = \left\{ y \in R^n; \sum_{i=1}^{j} y_i \geq 0 \ for\ all\ j = 1, 2, \cdots, n-1\ and\ \sum_{i=1}^{n} y_i = 0 \right\}.$$

Then we recall (see, e.g., [3]-[5]) that a function $f: R^n \to R$ is Schur convex if

$$\forall x, y \in R^n_{\geq}; y - x \in (R^n_{\geq})^+\ f(x) \leq f(y).$$

Every Schur-convex function $f: D \in R^n \to R$ is a symmetric function, and if I is an open interval and $f: I^n \to R$ is symmetric and of class C^1, then f is Schur-convex if and only if

$$\left(x_i - x_j\right)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0, \text{ on } I^n \tag{1.1}$$

for all $i, j \in \{1, 2, \cdots, n\}$.

Let $f: I \subseteq R \to R$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

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$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

holds. This double inequality is called Hermite-Hadamard inequality for convex functions. Hermite-Hadamard inequality is improved though Schur convexity, c.f., [6]-[10]. Among these paper, it is proven that if $I \in \mathrm{R}$ is an interval and $f: \mathrm{I} \to \mathrm{R}$ is continuous, then f is convex if and only if the mapping

$$S_1(a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx, \ \ if \ \ b \ne a$$

(Here and what follows, we use the mapping convention $S_i(\mathrm{a},\mathrm{a}) = \lim_{\mathrm{b}\to\mathrm{a}} S_i(\mathrm{a},\mathrm{b})$ for $\mathrm{b}=\mathrm{a}$ case, which is no longer stated.) is Schur convex, and in this case, $S_1(\mathrm{a},\mathrm{b})$ is convex. If $I \in \mathrm{R}$ is an interval and $f: \mathrm{I} \to \mathrm{R}$ is continuous, then *f* is convex if and only if one of the following mappings

$$S_2(a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right), \ \ if \ \ b \ne a,$$

$$S_3(a,b) = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx, \ \ if \ \ b \ne a$$

is Schur convex. Some exciting results on Schur's majorization inequality can be found in [11]-[13].

Let $f:[\mathrm{a},\mathrm{b}] \to \mathrm{R}$ be a four times continuously differentiable mapping on [a, b]. Then the following quadrature rule is well-known:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{2880}f^{(4)}(\xi)(b-a)^4, \ \xi \in (a,b), \tag{1.3}$$

which is called Simpson's formula, c.f. [14] and [15]. For $I \in \mathrm{R}$ is an interval and $f: \mathrm{I} \to \mathrm{R}$ is called four-convex, if $f^{(4)}(\mathrm{t}) \ge 0$, for all $\mathrm{t} \in [\mathrm{a},\mathrm{b}]$. In [15], the authors proved that if $f^{(4)}: I \to \mathrm{R}$ is continuous, then f is four-convex is equivalent to the mappings defined by

$$S_4(a,b) = \frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx, \ \ if \ \ b \ne a$$

is Schur-convex, this is an improvement of the Simpson's formula.

On the other hand, the dual Simpson's formula ([14]) is stated as follows: if $f^{(4)}$ is continuous, there exist $\eta \in (\mathrm{a},\mathrm{b})$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)\right] + \frac{1}{23040}f^{(4)}(\eta)(b-a)^4, \ \eta \in (a,b). \tag{1.4}$$

In [16], Bullen proved that, if f is four-convex, then the dual Simpson's quadrature formula is more accurate than Simpson's formula. That is, it holds that

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{12}\left[f(a)+4f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+4f\left(\frac{a+3b}{4}\right)+f(b)\right],$$

provided that f is four-convex.

Now we can state our main results. In view of the dual Simpson's formula and the above Bullen-Simpson formula, we construct two mappings as follows: for $\mathrm{b} \ne \mathrm{a}$, we set

$$S_5(a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)\right],$$

$$S_6(a,b) = \frac{1}{12}\left[f(a)+4f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+4f\left(\frac{a+3b}{4}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We shall show that if $f^{(4)}: I \to \mathrm{R}$ is continuous, then f is four-convex if and only if the mapping $S_5(a,b)$ or $S_6(a,b)$ is Schur-convex. Obviously our results improve the dual-Simpson's formula and the Bullen-Simpson's formula, and hence complement the main result in [15].

2. Main Results

We now present our main theorem.

Theorem 2.1. Let $I \subseteq \mathrm{R}, f \in C^4(I)$ be a mapping on *I*, then the following statements are equivalent:

(a) The function $S_4(a,b)$ is Schur-convex on I^2.

(b) The function $S_5(a,b)$ is Schur-convex on I^2.

(c) The function $S_6(a,b)$ is Schur-convex on I^2.

(d) For any $a,b \in I$ with $a < b$, we have the Simpson inequality holds, *i.e.*:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right].$$

(e) For any $a,b \in I$ with $a < b$, we have the dual Simpson inequality holds, *i.e.*:

$$\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)\right] \le \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(f) For any $a,b \in I$ with $a < b$, we have the Bullen-Simpson inequality holds, *i.e.*:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{12}\left[f(a)+4f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+4f\left(\frac{a+3b}{4}\right)+f(b)\right].$$

(g) The function f is four-convex on *I*.

Proof:

The equivalence of (a) (d) (g) was already proven in [15]. Suppose that item (g) holds, then by the definition of the function $S_5(a,b)$, we have

$$(\mathrm{b}-\mathrm{a})\left(\frac{\partial S_5}{\partial b}-\frac{\partial S_5}{\partial a}\right) = f(\mathrm{a})+f(b)-\frac{2}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}[f\left(\frac{a+3b}{4}\right)-f(\frac{3a+b}{4})](b-a).$$

$$\ge f(\mathrm{a})+f(b)-\frac{1}{3}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{3}[f'\left(\frac{a+3b}{4}\right)-f'(\frac{3a+b}{4})](b-a),$$

(by Simpson's formula (1.4) and four-convexity of f) hence,

$$\frac{\partial S_5}{\partial b}-\frac{\partial S_5}{\partial a} = \frac{2}{3}\left[\left(\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f'(x)\,dx-\frac{1}{2}f'\left(\frac{a+3b}{4}\right)\right)-\left(\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f'(x)\,dx-\frac{1}{2}f'\left(\frac{3a+b}{4}\right)\right)\right]$$

$$=\frac{1}{3}\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f'\left(x+\frac{b-a}{2}\right)-f'(x)\,dx-\left(f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right)\right]$$

$$=\frac{1}{3}\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}} h(x)\,dx-h\left(\frac{3a+b}{4}\right)\right].$$

Here we denote $h(x)=f'\left(x+\frac{b-a}{2}\right)-f'(x)$, for $x \in \left[\mathrm{a},\frac{a+b}{2}\right]$. Since f is four-convex, $h(x)$ is convex.

Thus Hermite-Hadamard (1.2) holds for $h(x)$ in $\left[a,\frac{a+b}{2}\right]$, this gives that $(b-a)\left(\frac{\partial S_5}{\partial b}-\frac{\partial S_5}{\partial a}\right)\geq 0$, so by the criteria (1.1) S_5 is Schur-convex, item (b) is a consequence of item (g).

Now suppose that item (b) holds. Since $\left(\frac{a+b}{2},\frac{a+b}{2}\right)\eth(a,b)$, Schur-convexity of S_5 gives that $0=S_5\left(\frac{a+b}{2},\frac{a+b}{2}\right)\leq S_5(a,b)$, *i.e.*, item (e) is valid if item (b) holds.

Next we prove item (e) implies item (g). By item (e) and the dual Simpson's formula (1.6), we get

$$0\leq S_5(a,b)=\frac{1}{23040}f^{(4)}(\eta)(b-a)^4,\eta\in(a,b).$$

Since $f\in C^4(I)$, and a, b are arbitrary, it follows that f is four-convex. Now the equivalence of (b) (e) (g) is proven. We follow the same pattern to show the equivalence of (c) (f) (g). If item (c) holds, then $0=S_6\left(\frac{a+b}{2},\frac{a+b}{2}\right)\leq S_6(a,b)$, *i.e.*, item (f) is valid. Suppose that item (f) is valid. By the definitions and formulas (1.3) and (1.4), we get

$$0\leq 2S_6(a,b)=S_4(a,b)-S_5(a,b)=\frac{1}{2880}\left(f^{(4)}(\xi)-\frac{1}{8}f^{(4)}(\eta)\right)(b-a)^4,\ \ \xi,\eta\in(a,b).$$

Since $f\in C^4(I)$, and a, b are arbitrary, item (g) follows again. It is only left to show that item (g) implies item (c). We give a lemma first.

Lemma 2.1. Let $I\subseteq R, f\in C^4(I)$ be four-convex on *I*, then the following inequalities hold for any $a,b\in I$ with $b\geq a$:

$$\frac{1}{b-a}\int_a^b f(x)dx\geq f(a)+\frac{1}{6}f'(a)(b-a)+\frac{1}{3}f'\left(\frac{a+b}{2}\right)(b-a).$$

$$\frac{1}{b-a}\int_a^b f(x)dx\geq f(b)-\frac{1}{6}f'(b)(b-a)-\frac{1}{3}f'\left(\frac{a+b}{2}\right)(b-a).$$

Proof:

We only prove the first inequality. Denote that

$$T(b):=\int_a^b f(x)dx-[f(a)(b-a)+\frac{1}{6}f'(a)(b-a)^2+\frac{1}{3}f'\left(\frac{a+b}{2}\right)(b-a)^2],$$

and that $g(x)=f''(x)$, then

$$T(a)=0;\ T'(a)=0;\ T''(a)=0. \tag{2.1}$$

$$T'(b)=f(b)-f(a)-\left(\frac{1}{3}f'(a)+\frac{2}{3}f'\left(\frac{a+b}{2}\right)\right)(b-a)-\frac{1}{6}f''\left(\frac{a+b}{2}\right)(b-a)^2.$$

$$\begin{aligned}T''(b)&=f'(b)-\left(\frac{1}{3}f'(a)+\frac{2}{3}f'\left(\frac{a+b}{2}\right)\right)-\frac{2}{3}f''\left(\frac{a+b}{2}\right)(b-a)-\frac{1}{12}f'''\left(\frac{a+b}{2}\right)(b-a)^2\\&=\frac{2}{3}\int_{\frac{a+b}{2}}^b g(x)dx+\frac{1}{3}\int_a^b g(x)dx-\frac{2}{3}g\left(\frac{a+b}{2}\right)(b-a)-\frac{1}{12}g'\left(\frac{a+b}{2}\right)(b-a)^2\\&=T_1(b)+T_2(b).\end{aligned}$$

Here,

$$T_1(b)=\frac{1}{3}[\int_a^b g(x)dx-g\left(\frac{a+b}{2}\right)(b-a)].$$

$$T_2(b)=\frac{2}{3}\int_{\frac{a+b}{2}}^{b}g(x)dx-\frac{1}{3}g\left(\frac{a+b}{2}\right)(b-a)-\frac{1}{12}g\left(\frac{a+b}{2}\right)(b-a)^2.$$

From the Hermite-Hadamard inequality for convex function $g(x)$, we see that $T_1(b)\ge 0$. Besides, it follows from convexity of $g(x)$ that for any $x\le y$:

$$g(y)\ge g(x)+g'(x)(y-x).$$

Take integration w.r.t *y*, we get

$$\int_x^y g(y)dy\ge g(x)(y-x)+\frac{1}{2}g'(x)(y-x)^2,$$

applying this inequality in $[\frac{a+b}{2},b]$, we see that $T_2(b)\ge 0$. It follows that $T''(b)\ge 0$ for any b ≥ a, hence by (2.1) we know $T(\mathrm{b})\ge 0$ for any b ≥ a. The second inequality in the lemma is just the first inequality with b ≤ a, we omit its proof. The lemma is proven.

Now we continue the proof of our main theorem. By the definition of $S_6(a,b)$, we have

$$(b-a)\left(\frac{\partial S_6}{\partial b}-\frac{\partial S_6}{\partial a}\right)=\frac{2}{b-a}\int_a^b f(x)dx-\left[f(\mathrm{a})+f(b)\right]$$
$$+\frac{1}{12}[f'(b)-f'(a)](b-a)+\frac{1}{6}[f'\left(\frac{a+3b}{4}\right)-f'(\frac{3a+b}{4})](b-a)$$
$$=K_1(b)+K_2(b),$$

here $K_1(b),K_2(b)$ is denoted as

$$K_1(b):=\frac{2}{b-a}\int_a^{\frac{a+b}{2}}f(x)dx-f(a)-\frac{1}{12}f'(a)(b-a)-\frac{1}{6}f'(\frac{3a+b}{4})(b-a)$$
$$K_2(b):=\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}f(x)dx-f(b)+\frac{1}{12}f'(b)(b-a)+\frac{1}{6}f'(\frac{a+3b}{4})(b-a)$$

Suppose that item (g) holds, by applying the lemma to f in $\left[a,\frac{a+b}{2}\right],[\frac{a+b}{2},b]$, we get both $K_1,K_2\ge 0$, thus $(b-a)\left(\frac{\partial S_6}{\partial b}-\frac{\partial S_6}{\partial a}\right)\ge 0$, so by the criteria (1.1) $S_6(a,b)$ is Schur-convex, item (c) follows.

Remark 2.1. From **Lemma 2.1**, we add the two inequalities together to see that the following holds for four-convex functions f:

$$\int_a^b f(x)dx\ge\frac{1}{2}\left[f(a)+f(b)\right]-\frac{1}{12}[f'(b)-f'(b)](b-a) \tag{2.2}$$

it is well-known, c.f., [14] or [15].

Starting from this inequality (2.2), we deduce some properties for four-convex functions. As in the above, we define a pair of mappings S_7,S_8 by

$$S_7(a,b)=\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[f(a)+f(b)\right]+\frac{1}{12}[f'(b)-f'(b)](b-a);$$

$$S_8(a,b)=\frac{1}{2}\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{12}f''\left(\frac{a+b}{2}\right)(b-a)^2.$$

Then we have

Theorem 2.2. Let $I \subseteq \mathrm{R}, \mathrm{f} \in \mathrm{C}^4(I)$ be four-convex on *I*, then the mappings S_7, S_8 are non-negative and Schur-convex on I^2.

Proof:

We observe that

$$(b-a)\left(\frac{\partial S_8}{\partial b}-\frac{\partial S_8}{\partial a}\right)=\frac{2}{b-a}\int_a^b f(x)dx-\left[f(a)+f(b)\right]$$

$$+\frac{1}{2}[f'(b)-f'(a)](b-a)-\frac{1}{3}f''\left(\frac{a+b}{2}\right)(b-a)^2$$

$$\geq \frac{1}{2}[f'(b)-f'(a)](b-a)-\frac{1}{3}f''\left(\frac{a+b}{2}\right)(b-a)^2 \quad (2.3)$$

$$\geq 0 \quad (2.4)$$

Here inequality (2.3) is due to inequality (2.2), and inequality (2.4) is a consequence of the Hermite-Hadamard inequality for convex function f'', thus by the criteria (1.1) S_8 are Schur-convex on I^2. Hence we get $S_8(a,b) \geq S_8\left(\frac{a+b}{2},\frac{a+b}{2}\right)=0$.

Since S_8 is non-negative, we observe that

$$(b-a)\left(\frac{\partial S_7}{\partial b}-\frac{\partial S_7}{\partial a}\right)=-\frac{2}{b-a}\int_a^b f(x)dx+\left[f(a)+f(b)\right]$$

$$-\frac{1}{3}[f'(b)-f'(a)](b-a)+\frac{1}{12}[f''(a)+f''(b)](b-a)^2 \quad (2.5)$$

$$\geq -\frac{1}{3}[f'(b)-f'(a)](b-a)+\frac{1}{12}[f''(a)+f''(b)+2f''\left(\frac{a+b}{2}\right)](b-a)^2.$$

It is shown in [7] for a convex function *g* that the function

$$S_9(a,b)=\frac{1}{4}\left[g(a)+g(b)\right]+\frac{1}{2}g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \quad (\text{if } b \neq a)$$

is Schur-convex, specially we have $S_9(a,b) \geq 0$. We set $g = f''$, then it is convex, we see that RHS of inequality (2.5) is non-negative, so by the criteria (1.1), S_7 is Schur-convex.

Furthermore, we give a Schur-convexity theorem for the following mapping:

$$S_{10}(a,b)=f\left(\frac{a+b}{2}\right)-\frac{1}{2}\left[f(a)+f(b)\right]+\frac{1}{12}[f'(b)-f'(b)](b-a)+\frac{1}{24}f''\left(\frac{a+b}{2}\right)(b-a)^2.$$

Theorem 2.3. Let $I \subseteq \mathrm{R}, \mathrm{f} \in \mathrm{C}^4(I)$ be four-convex on *I*, then the mappings S_{10} are non-negative and Schur-convex on I^2 .

Proof: We observe that

$$(b-a)\left(\frac{\partial S_{10}}{\partial b}-\frac{\partial S_{10}}{\partial a}\right)=-\frac{1}{3}[f'(b)-f'(a)](b-a)+\frac{1}{12}[f''(a)+f''(b)+2f''\left(\frac{a+b}{2}\right)](b-a)^2.$$

Since $S_9(a,b) \geq 0$ for convex function $g = f''$, as in the above, we can conclude that $S_{10}(a,b)$ are non-negative and Schur-convex.

Remark 2.2. For smooth four-convex functions, we see that both S_8 and S_{10} are non-negative and Schur-convex functions, then the sum of S_8 and S_{10} is also non-negative and Schur-convex function, especially it holds that

$$f\left(\frac{a+b}{2}\right)+\frac{1}{24}f''\left(\frac{a+b}{2}\right)(b-a)^2 \geq \frac{1}{b-a}\int_a^b f(x)\,dx$$

Remark 2.3. For positive real numbers *x*, *y*, we denote the arithmetic mean, geometric mean, and logarithmic mean of *x*, *y* by *A*, *G*, *L*. Applying non-negativity of S_7 and S_8 to function $f(\mathrm{t})=\mathrm{e}^t$, $\mathrm{t}\in[\ln x,\ln y]$ then we have

$$\frac{1}{12}G\cdot\left(\ln\frac{y}{x}\right)^2 \leq A-L \leq \frac{1}{12}L\cdot\left(\ln\frac{y}{x}\right)^2.$$

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