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A Note on Parameterized Preconditioned Method for Singular Saddle Point Problems

Yueyan Lv, Naimin Zhang*

School of Mathematics and Information Science, Wenzhou University, Wenzhou, China
Email: 956910361@qq.com, *nmzhang@wzu.edu.cn

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Abstract

Recently, some authors (Li, Yang and Wu, 2014) studied the parameterized preconditioned HSS (PPHSS) method for solving saddle point problems. In this short note, we further discuss the PPHSS method for solving singular saddle point problems. We prove the semi-convergence of the PPHSS method under some conditions. Numerical experiments are given to illustrate the efficiency of the method with appropriate parameters.

Keywords

Singular Saddle Point Problems, Hermitian and Skew-Hermitian Splitting, Preconditioning, Iteration Methods, Semi-Convergence

1. Introduction

We consider the iterative solution of the following linear system:

$$Ax = \begin{pmatrix} B & E \\ -E^* & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b \tag{1}$$

where $B \in C^{p\times p}$ is Hermitian positive definite, $E \in C^{p\times q}$ is rank-deficient, *i.e.*, $p \geq q$, E^* denotes the conjugate transpose of *E*, $f \in C^p$ and $g \in C^q$. Linear systems of the form (1) are called saddle point problems. They arise in many application areas, including computational fluid dynamics, constrained optimization and weighted least-squares problem, see, e.g., [1] [2].

We review the Hermitian and skew-Hermitian splitting (HSS) [3] of coefficient matrix *A*:

$$A = H + S$$

where

$$H = \frac{1}{2}\left(A + A^*\right) = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \frac{1}{2}(A - A^*) = \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}.$$

The PPHSS Iteration Method ([4]): Denote $n = p + q$. Let $x^{(0)} \in C^n$ be an arbitrary initial guess, com-

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pute $x^{(k+1)}$ for $k = 0,1,2,\ldots$ by the following iteration scheme until $\{x^{(k)}\}$ converges,

$$\begin{cases} (\alpha P + H)x^{(k+1/2)} = (\alpha P - S)x^{(k)} + b \\ (\beta P + S)x^{(k+1)} = (\beta P - H)x^{(k+1/2)} + b \end{cases}. \tag{2}$$

where α, β are given positive constants and

$$P = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \tag{3}$$

matrix C is Hermitian positive definite.

Evidently, the iteration scheme (2) of PPHSS method can be rewritten as

$$x^{(k+1)} = T(\alpha,\beta)x^{(k)} + F(\alpha,\beta)b \tag{4}$$

here, $T(\alpha,\beta)$ is the iteration matrix of the PPHSS method. In fact, Equation (4) may also result from the splitting

$$A = M(\alpha,\beta) - N(\alpha,\beta) \tag{5}$$

with

$$M(\alpha,\beta) = \frac{1}{\alpha+\beta} P^{-1}(\alpha P + H)(\beta P + S) = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta(\alpha+1)B & (\alpha+1)E \\ -\alpha E^* & \alpha\beta C \end{pmatrix} \tag{6}$$

Evidently, the matrix $M(\alpha,\beta)$ can act as a preconditioner for solving the linear system (1), which is called the PPHSS preconditioner. The PPHSS method is a special case of the generalized preconditioned HSS (GHSS) method [5]. When $\beta = \alpha/(\alpha+1)$, we can obtain a special case of the PPHSS (SPPHSS) method. In order to analyze the semi-convergence of the PPHSS iteration, we let

$$\overline{H} = P^{-1/2}HP^{-1/2} = \begin{pmatrix} I_P & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{S} = P^{-1/2}SP^{-1/2} = \begin{pmatrix} 0 & \overline{E} \\ -\overline{E}^* & 0 \end{pmatrix}$$

where I_p is the identity matrix of order p and $\overline{E} = B^{-1/2}EC^{-1/2}$. In the same way, we denote

$$\overline{T}(\alpha,\beta) = P^{1/2}M(\alpha,\beta)^{-1}N(\alpha,\beta)P^{-1/2} \tag{7}$$

Owing to the similarity of the matrices $T(\alpha,\beta)$ and $\overline{T}(\alpha,\beta)$, we only need to study the spectral properties of matrix $\overline{T}(\alpha,\beta)$ in order to analyze the semi-convergence of the PPHSS iteration.

2. The Semi-Convergence of the PPHSS Method

As the coefficient matrix A is singular, then the iteration matrix T has eigenvalue 1, and the spectral radius of matrix T cannot be small than 1. For the iteration matrix T of the singular linear systems, we introduce its pseudo-spectral radius $\nu(T)$ by follows,

$$\nu(T) = \max\{|\lambda| : \lambda \in \sigma(T), \lambda \neq 1\},$$

where $\sigma(T)$ is the set of eigenvalues of T.

For a matrix $K \in R^{n\times n}$, the smallest nonnegative integer i such that $rank(K^i) = rank(K^{i+1})$ is called the index of K, and we denote it by $i = index(K)$. In fact, $index(K)$ is the size of the largest Jordan block corresponding to the zero eigenvalue of K.

Lemma 2.1 ([6]). The iterative method (4) is semi-convergent, if and only if,

$$index(I - T(\alpha,\beta)) = 1 \text{ and } \nu(T(\alpha,\beta)) < 1.$$

Lemma 2.2 ([7]). $index(I - T(\alpha,\beta)) = 1$, if and only if, for any $0 \neq Y \in R(A)$, $Y \notin N(AM^{-1})$.

Theorem 2.3. Assume that B and C be Hermitian positive definite, E be of rank-deficient. Then $index(I - T(\alpha,\beta)) = 1$.

Proof. The proof is similar to the proof of Lemma 2.8 in [8], here is omitted.

Lemma 2.4 ([4]). Let B and C be Hermitian positive definite, E be of rank-deficient. Assume that $\overline{S}' = \beta I_q + \frac{1}{\beta}\overline{E}^*\overline{E}$. Then, we can partition $\overline{T}(\alpha,\beta)$ in Equation (7) as

$$\overline{T} = \begin{pmatrix} \frac{\alpha(\beta-1)}{\beta(\alpha+1)} I_p - \frac{(\alpha+\beta)(\alpha\beta+\beta-\alpha)}{\alpha\beta^2(\alpha+1)}\overline{E}\overline{S}'^{-1}\overline{E}^* & -\frac{\alpha+\beta}{\alpha+1}\overline{E}\overline{S}'^{-1} \\ \frac{(\alpha+\beta)(\alpha\beta+\beta-\alpha)}{\alpha\beta(\alpha+1)}\overline{S}'^{-1}\overline{E}^* & \frac{1-\beta}{\alpha+1} I_q + \frac{\beta(\alpha+\beta)}{\alpha+1}\overline{S}'^{-1} \end{pmatrix}.$$

Let $\overline{E} = U^*\overline{\Sigma}'V$ be the singular value decomposition [9] of E, where $U \in C^{p\times p}$ and $V \in C^{q\times q}$ are unitary matrices, and

$$\overline{\Sigma}' = \begin{pmatrix} \overline{\Sigma} \\ 0 \end{pmatrix}, \quad \overline{\Sigma} = diag(\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_q) \in C^{q\times q},$$

$\overline{\sigma}_i (i = 1, 2, \ldots, q)$ are the singular values of $\overline{E}$.

Lemma 2.5. The eigenvalues of the iteration matrix $T(\alpha,\beta)$ of PPHSS iteration method are $\frac{\alpha(\beta-1)}{\beta(\alpha+1)}$ with multiplicity $p-q$, and the roots of quadratic equation

$$\lambda^2 - \frac{(2\alpha\beta+\beta-\alpha)(\alpha\beta-\overline{\sigma}_k^2)}{\alpha(\alpha+1)(\beta^2+\overline{\sigma}_k^2)}\lambda + \frac{\beta(\beta-1)(\alpha^2+\overline{\sigma}_k^2)}{\alpha(\alpha+1)(\beta^2+\overline{\sigma}_k^2)} = 0, \quad k = 1, 2, \ldots, q \tag{8}$$

Proof. Notice the similarity of matrices $\overline{T}(\alpha,\beta)$ and $T(\alpha,\beta)$. The proof is essentially analogous to the proof of Lemma 2.3 in [4] with only technical modifications. So, it is omitted.

Lemma 2.6. If $\overline{\sigma}_k \neq 0$, then the eigenvalue λ of the iteration matrix $T(\alpha,\beta)$ satisfies $\lambda \neq 1$; if $\overline{\sigma}_k = 0$, then $\lambda = 1$ or $\frac{\alpha(\beta-1)}{\beta(\alpha+1)}$.

Proof. If $\overline{\sigma}_k \neq 0$, we give the proof by contradiction. By Lemma 2.5, obviously, when $\lambda = \frac{\alpha(\beta-1)}{\beta(\alpha+1)}$, it can not be equal to 1. We assume $\lambda_\pm = 1$, by some algebra, it can be reduced to

$$-(2\alpha^2 + 2\alpha\beta + \alpha + \beta)\overline{\sigma}_k^2 - (\alpha^2\beta + \alpha\beta^2) \pm \sqrt{d_k^2 - 4e_k} = 0,$$

here, $d_k = (2\alpha\beta+\beta-\alpha)(\alpha\beta-\overline{\sigma}_k^2)$ and $e_k = \alpha\beta(\alpha+1)(\beta-1)(\alpha^2+\overline{\sigma}_k^2)(\beta^2+\overline{\sigma}_k^2)$. It is equivalent to $\overline{\sigma}_k^2(\beta^2+\overline{\sigma}_k^2) = 0$, so $\overline{\sigma}_k = 0$, which is in contradiction with $\overline{\sigma}_k \neq 0$.

If $\overline{\sigma}_k = 0$, we have $\lambda_+ = 1$ and $\lambda_- = \frac{\alpha(\beta-1)}{\beta(\alpha+1)}$, which finishes the proof.

Lemma 2.7 ([10]). Both roots of the real quadratic equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.

Theorem 2.8. If the iteration parameters α and β

$$\frac{\alpha}{2\alpha+1} < \beta \le \alpha, \quad \alpha > 0 \tag{9}$$

then, the pseudo-spectral radius of the PPHSS method satisfies $\nu(T(\alpha,\beta)) < 1$.

Proof. Using condition (9), it follows that $\frac{\alpha|\beta-1|}{\beta(\alpha+1)} < 1$. According to Lemma 2.5, if $\overline{\sigma}_k \neq 0$, we can obtain that

$$|c_k| = \left|\frac{\beta(\beta-1)(\alpha^2+\overline{\sigma}_k^2)}{\alpha(\alpha+1)(\beta^2+\overline{\sigma}_k^2)}\right| = \frac{\beta|\beta-1|(\alpha^2+\overline{\sigma}_k^2)}{\alpha(\alpha+1)(\beta^2+\overline{\sigma}_k^2)} < 1,$$

and

$$|b_k| < \frac{(2\alpha\beta+\beta-\alpha)(\alpha\beta+\bar{\sigma}_k^2)}{\alpha(\alpha+1)(\beta^2+\bar{\sigma}_k^2)} < \frac{\alpha\beta(2\alpha\beta+\beta-\alpha)+[\alpha(\alpha+1)+\beta(\beta-1)]\bar{\sigma}_k^2}{\alpha(\alpha+1)(\beta^2+\bar{\sigma}_k^2)} = 1+c_k .$$

By Lemma 2.7, for the eigenvalues λ of $T(\alpha,\beta)$, it holds $|\lambda|<1$.

If $\bar{\sigma}_k = 0$, by Lemma 2.6, the eigenvalues of $T(\alpha,\beta)$, except 1 are $\frac{\alpha(\beta-1)}{\beta(\alpha+1)}$. According to the definition of pseudo-spectral, we get $\nu(T(\alpha,\beta))<1$.

Theorem 2.9. Let $\bar{\sigma}_{\min} = \min_{1\le k\le q}\{\bar{\sigma}_k : \bar{\sigma}_k \ne 0\}$ and $\bar{\sigma}_{\max} = \max_{1\le k\le q}\{\bar{\sigma}_k : \bar{\sigma}_k \ne 0\}$. Then, the optimal value of the iteration parameter α for the SPPHSS iteration method is given by

$$\alpha^* \equiv \arg\min_{\alpha>0}\left\{\nu\left(T\left(\alpha,\frac{\alpha}{\alpha+1}\right)\right)\right\} = \frac{1}{2}\left(\bar{\sigma}_{\min}^2 + \bar{\sigma}_{\min}\sqrt{4+\bar{\sigma}_{\min}^2}\right),$$

and correspondingly,

$$\nu\left(T\left(\alpha^*,\frac{\alpha^*}{\alpha^*+1}\right)\right) = \frac{2}{2+\bar{\sigma}_{\min}^2+\bar{\sigma}_{\min}\sqrt{4+\bar{\sigma}_{\min}^2}}. \quad (10)$$

Proof. According to Lemma 2.5 and Lemma 2.6, we know that the eigenvalues of the iteration matrix $T(\alpha,\beta)$ are $-\frac{1}{\alpha+1}$ with multiplicity *p*, and

$$\frac{\alpha^2+\bar{\sigma}_k^2}{\alpha^2+(1+\alpha)^2\bar{\sigma}_k^2}, \quad k=1,2,\ldots,q. \quad (11)$$

If $\bar{\sigma}_k = 0$, the eigenvalues with the form of Equation (11) are 1, which can not affect the value of $\nu(T(\alpha,\beta))$. Therefore, without loss of generality, here we only need to discuss the case $\bar{\sigma}_k \ne 0$. The rest is similar to that of the proof of Theorem 3.1 in [4], here is omitted.

3. Numerical Results

In this section, we use an example to demonstrate the numerical results of the PPHSS method as a solver by comparing its iteration steps (IT), elapsed CPU time in seconds (CPU) and relative residual error (RES) with other methods. The iteration is terminated once the current iterate satisfies $RES \le 10^{-8}$ or the number of the prescribed iteration steps $k = 1,000$ are exceeded. All the computations are implemented in MATLAB on a PC computer with Intel (R) Celeron (R) CPU 1000M @ 1.80 GHz, and 2.00 GB memory.

Example 3.1 ([11]). Consider the saddle point problem (1), with the following block form of coefficient matrix:

$$B = \begin{pmatrix} I\otimes L + L\otimes I & 0 \\ 0 & I\otimes L + L\otimes I \end{pmatrix} \in R^{2l^2\times 2l^2}, \quad E = (\hat{E} \quad b_1 \quad b_2) \in R^{2l^2\times(l^2+2)},$$

where symbol $\otimes$ denotes the Kronecker product, and

$$L = \frac{1}{h^2}\cdot tridiag(-1,2,-1) \in R^{l\times l}, \quad \hat{E} = \begin{pmatrix} I\otimes Q \\ Q\otimes I \end{pmatrix} \in R^{2l^2\otimes l^2}, \quad b_1 = \hat{E}\begin{pmatrix} e_{l^2/2} \\ 0 \end{pmatrix}, \quad b_2 = \hat{E}\begin{pmatrix} 0 \\ e_{l^2/2} \end{pmatrix},$$

$$e_{l^2/2} = (1,1,\ldots,1)^T \in R^{l^2/2}, Q = \frac{1}{h}\cdot tridiag(-1,1,0) \in R^{l\times l}, h = \frac{1}{l+1},$$

the right-hand side vector *b* is chosen by $b = Ae_{p+q}$, where $e_{p+q} = (1,1,\ldots,1)^T \in R^{p+q}$, $p = 2l^2$, $q = l^2+2$.

For the Example 3.1, we choose $C = E^T\hat{B}^{-1}E$ where $\hat{B}$ is the block diagonal matrix of *B*. In **Table 1**, it is clear to see that the pseudo-spectral radius of the PPHSS and the SPPHSS methods are much smaller than of the PHSS method when the optimal parameters are employed. In **Table 2**, we list numerical results with respect to

Table 1. The optimal iteration parameters and pseudo-spectral radius.

Method	l	8	16	24	32
PHSS	α_*	1.6328	2.1999	2.6511	3.0367
	$\nu(T(\alpha_*,\alpha_*))$	0.6756	0.8112	0.8667	0.8969
SPPHSS	α^*	1.0216	1.0059	1.0027	1.0015
	$\nu(T(\alpha^*,\frac{\alpha^*}{\alpha^*+1}))$	0.4947	0.4985	0.4993	0.4996
PPHSS	$\alpha_{\exp}$	1.9815	2.6990	2.5976	2.9953
	$\beta_{\exp}$	0.6853	0.7845	1.0336	1.1234
	$\nu(T(\alpha_{\exp},\beta_{\exp}))$	0.5209	0.5258	0.5340	0.5452

Table 2. IT, CPU and RES for $C = E^T \hat{B}^{-1} E$.

Method	l	8	16	24	32
PHSS	IT	26	37	47	54
	CPU	0.399	1.548	7.286	27.178
	RES (10^{-9})	6.6914	7.2250	7.3711	9.3294
SPPHSS	IT	26	26	26	26
	CPU	1.075	1.583	4.879	8.610
	RES (10^{-9})	5.3781	6.7198	7.0689	7.2124
PPHSS	IT	16	16	16	16
	CPU	0.220	1.008	3.620	12.558
	RES (10^{-9})	7.8783	7.6704	7.7535	8.1446
GMRES	IT	883	2560	5450	10376
	CPU	0.524	3.179	12.572	16.264
	RES (10^{-9})	9.8243	9.9925	9.9903	9.9950
PHSS-GMRES	IT	22	35	44	50
	CPU	0.111	0.467	1.649	3.751
	RES (10^{-9})	9.6710	8.0874	9.3990	9.8588
SPPHSS-GMRES	IT	10	11	13	14
	CPU	0.082	0.372	1.831	3.892
	RES (10^{-9})	4.7178	6.7487	6.2498	4.2212
PPHSS-GMRES	IT	11	13	13	16
	CPU	0.070	0.428	1.564	3.366
	RES (10^{-9})	5.4534	2.3975	9.8449	9.8690

IT, CPU and RES of the texting methods with different problem sizes l. We see that the PPHSS and SPPHSS methods with appropriate parameters always outperforms the PHSS method both as a solver and as a preconditioner for GMRES in iteration steps and CPU times. Notice

$$M(\alpha,\beta)^{-1}=\begin{pmatrix} \frac{\alpha+\beta}{\beta(\alpha+1)}B^{-1}\left(I-\frac{1}{\beta}ES'^{-1}E^{*}B^{-1}\right) & -\frac{\alpha+\beta}{\alpha\beta}B^{-1}ES'^{-1} \\ \frac{\alpha+\beta}{\beta(\alpha+1)}S'^{-1}E^{*}B^{-1} & \frac{\alpha+\beta}{\alpha}S'^{-1} \end{pmatrix}$$

where $S'=\beta C+\frac{1}{\beta}E^{*}B^{-1}E$. To compute the matrix-vector products with $M(\alpha,\beta)^{-1}$, we make incomplete LU factorization of B and S' with drop tolerance 0.001. In the two tables, we use restarted GMRES (18) and preconditioned GMRES (18).

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