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Unconditionally Explicit Stable Difference Schemes for Solving Some Linear and Non-Linear Parabolic Differential Equation

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Abstract

We present the numerical method for solution of some linear and non-linear parabolic equation. Using idea [1], we will present the explicit unconditional stable scheme which has no restriction on the step size ratio k/h^2 where k and h are step sizes for space and time respectively. We will also present numerical results to justify the present scheme.

Keywords

Runge-Kutta Methods, Method of Lines, Difference Equation, Non-Linear PDE

1. Introduction

A number of difference schemes for solving partial difference equations have been proposed by using the idea of methods of lines [2] [3]. The scheme is required the condition of step size ratio $\frac{k}{h^2} \le c_0$ for some constant c_0, where k and h are step sizes for space and time respectively. We [1] [4]-[6] have proposed some explicit scheme and overcome this problem. The problem considered in this paper is linear and nonlinear parabolic problem

$$\frac{\partial u(x,t)}{\partial t} = a\{u(x,t)\}^l D^2(u(x,t)) \quad (l = 0, 2, 4, \cdots, a: \text{positve constant}), \tag{1.1}$$

$$(x,t) \in \Omega = \{(x,t); 0 < x < 1, 0 < t \le t_F\},$$

with the initial Dirichlet boundary condition

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$$u(x,t)=\begin{cases} f(x) & (x,0)\in\partial\Omega\cup\Omega \\ 0, & (0,t),(1,t)\in\partial\Omega\cup\Omega. \end{cases} \tag{1.2}$$

where we set

$$D^l=\frac{\partial^l}{\partial x^l}. \quad (l=1,2,3,\cdots) \tag{1.3}$$

We propose the difference approximation to (1.1) where the step size ratio is defined by

$$c=\frac{k}{h^2}. \ (k\neq 1,\ c \text{ is any positive constant}) \tag{1.4}$$

The outline of this paper is as follows. In §2, by using idea of methods of lines, we present the explicit difference approximation to (1.1). In §3 we study the truncation errors of our scheme. In §4 we study the convergence of the scheme with the condition (1.4) and we will show that our scheme converges to the true solution of (1.1). In §5 we study stability of the scheme, and we will show that our scheme is stable for any step size k and h with the condition (1.4). In §6 we show some numerical examples to justify our methods.

2. Difference Scheme

In the same way as in [1], we will approximate (1.1) by replacing the derivative for space and time in the difference operator

$$D^2(u(x,t))=\frac{\partial^2 u(x,t)}{\partial^2 x}\cong\frac{1}{h^2}(\delta^2 u(x,t)),$$

$$\frac{\partial u(x,t)}{\partial x}\cong\frac{1}{h}\Delta u(x,t), \tag{2.1}$$

where δ is the central difference operator, Δ forward difference operator. We denote the approximation to (1.1) at the mesh point $(x,t)=(jh,nk)$

$$u_j^n\cong u(jh,nk).$$

We set

$$L(g(x,t))=\{g(x,t)\}^{2l}. \tag{2.2}$$

We define the difference approximation to (1.1) by the following scheme.
If $|u_j^n|<1$.
Then we set

$$u_j^{n+1}=u_j^n+\frac{ac}{1+2a\hat{c}}\Phi(u_{j-1}^n,u_j^n,u_{j+1}^n). \tag{2.3}$$

If $|u_j^n|\geq 1$.
Then we set

$$u_j^{n+1}=u_j^n+\frac{ac}{1+2a\hat{c}L(u_j^n)}\Phi(u_{j-1}^n,u_j^n,u_{j+1}^n), \tag{2.4}$$

where

$$\Phi(u_{j-1}^n,u_j^n,u_{j+1}^n)=\{u_j^n\}^l\{u_{j-1}^n-2u_j^n+u_{j+1}^n\}. \tag{2.5}$$

The step size $\hat{c}$ in (2.3), (2.4) is defined by

$$\hat{c}=\frac{k^{1+\rho}}{h^2}. \quad (\rho>0) \tag{2.6}$$

If we set

$$\tilde{L}\left(u_j^n\right)=\begin{cases}1 & \left|u_j^n\right|\le 1\\ L\left(u_j^n\right) & \left|u_j^n\right|>1.\end{cases} \tag{2.7}$$

Then, from (2.3), (2.4), we have

$$u_j^{n+1}=u_j^n+\frac{ac}{1+2a\hat{c}\tilde{L}\left(u_j^n\right)}\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right). \tag{2.8}$$

3. Truncation Error

We define the truncation error $T(jh,nk)$ of (2.8)

$$\begin{aligned}T(jh,nk)=u\left(jh,(n+1)k\right)-\Bigg\{&u(jh,nk)+\frac{ac}{\left(1+2a\hat{c}\tilde{L}\left(u(jh,nk)\right)\right)}\\ &\times\Phi\left(u\left((j-1)h,nk\right),u(jh,nk),u\left((j+1)h,nk\right)\right)\Bigg\},\end{aligned} \tag{3.1}$$

where, from the definition of (2.7), we have

$$\tilde{L}\left(u(j,n)\right)=\begin{cases}1 & \left|u_j^n\right|\le 1\\ L\left(u(j,n)\right) & \left|u_j^n\right|>1.\end{cases} \tag{3.2}$$

By Taylor series expansions of the solution $u(jh,nk)$ of (1.1), we have

$$\begin{aligned}&T(jh,nk)=ku_t(jh,nk)+O\left(k^2\right)\\ &\quad-\frac{ac}{\left(1+2a\hat{c}\tilde{L}\left(u(jh,nk)\right)\right)}\left\{u(jh,nk)\right\}^l\left\{\left(u\left(j-1\right)h,nk\right)-2u(jh,nk)+u\left((j+1)h,nk\right)\right\}\\ &=ak\left\{u(jh,nk)\right\}^l u_{xx}(jh,nk)-\frac{ak}{\left(1+2a\hat{c}\tilde{L}\left(u(jh,nk)\right)\right)}\Bigg\{\left\{u(jh,nk)\right\}^l u_{xx}(jh,nk)\\ &\quad+\frac{k}{12c}\left\{u(jh,nk)\right\}^l u_{xxxx}(jh,nk)+O\left(k^3\right)\Bigg\}.\end{aligned} \tag{3.3}$$

From (3.3), we have

$$\begin{aligned}T(jh,nk)=k&\left\{\frac{2a^2\hat{c}\tilde{L}\left(u(jh,nk)\right)}{\left(1+2a\hat{c}\tilde{L}\left(u(jh,nk)\right)\right)}\right\}\left\{u(jh,nk)\right\}^l u_{xx}(jh,nk)\\ &-\frac{ak^2}{12c\left(1+2a\hat{c}\tilde{L}\left(u(jh,nk)\right)\right)}\left\{\left\{u(jh,nk)\right\}^l u_{xxxx}(jh,nk)+O\left(k^3\right)\right\}.\end{aligned} \tag{3.4}$$

If we set

$$\tilde{\rho}=\min\{\rho,1\}, \tag{3.5}$$

and

$$\tilde{c}=\frac{k^{(1+\tilde{\rho})}}{h^2}. \tag{3.6}$$

Then, from (3.4), we have the following result.

Theorem 1. The truncation error of the difference approximation (2.8) to (1.1) is given by

$$T\left(jh,nk\right)=ak^{(1+\tilde{\rho})}w\left(jh,nk\right), \tag{3.7}$$

where

$$\begin{aligned} w\left(jh,nk\right)=&\frac{2ac\tilde{L}\left(u\left(jh,nk\right)\right)}{\left(1+2a\hat{c}\tilde{L}\left(u\left(jh,nk\right)\right)\right)}\left\{u\left(jh,nk\right)\right\}^{l}u_{xx}\left(jh,nk\right) \\ &-k^{(1-\tilde{\rho})}\frac{1}{12c\left(1+2a\hat{c}\tilde{L}\left(u\left(jh,nk\right)\right)\right)}\left\{u\left(jh,nk\right)\right\}^{l}u_{xxxx}\left(jh,nk\right), \end{aligned} \tag{3.8}$$

where $\tilde{\rho}$ and $\tilde{c}$ are defined by (3.5) and (3.6) respectively.

4. Convergence

In this section, we study the convergence of the scheme (2.8). We set the approximation error by

$$e\left(jh,nk\right)=u\left(jh,nk\right)-u_j^n. \tag{4.1}$$

We use the abbreviation's

$$e_j^n=e\left(jh,nk\right),$$

$$T\left(j,n\right)=T\left(jh,nk\right),$$

$$u\left(j,n\right)=u\left(jh,nk\right).$$

From (2.8), (3.7), (4.1), we have

$$\begin{aligned} e_j^{n+1}=&e_j^n+g_1\left(j,n\right)\Phi\left(u\left(j-1,n\right),u\left(j,n\right),u\left(j+1,n\right)\right) \\ &-g_2\left(j,n\right)\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right)+T\left(j,n+1\right) \\ =&e_j^n+\left(g_1\left(j,n\right)-g_2\left(j,n\right)\right)\Phi\left(u\left(j-1,n\right),u\left(j,n\right),u\left(j+1,n\right)\right) \\ &+g_2\left(j,n\right)\left\{\Phi\left(u\left(j-1,n\right),u\left(j,n\right),u\left(j+1,n\right)\right)-\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right)\right\}+T\left(j,n+1\right), \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} g_1\left(j,n\right)&=\frac{ac}{1+2a\tilde{c}\tilde{L}\left(u\left(j,n\right)\right)}, \\ g_2\left(j,n\right)&=\frac{ac}{1+2a\tilde{c}\tilde{L}\left(u_j^n\right)}. \end{aligned} \tag{4.3}$$

From (2.5), we have

$$\begin{aligned} \Phi\left(u\left(j-1,n\right),u\left(j,n\right),u\left(j+1,n\right)\right)&=\left\{u\left(j,n\right)\right\}^{l}\left\{u\left(j-1,n\right)-2u\left(j,n\right)+u\left(j+1,n\right)\right\}, \\ &=h^2\left\{u\left(j,n\right)\right\}^{l}u_{xx}\left(j,n\right)+O\left(h^4\right), \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\Phi\left(u\left(j-1,n\right),u\left(j,n\right),u\left(j+1,n\right)\right)-\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right) \\ &=u\left(j,n\right)^{l}\left\{u\left(j-1,n\right)-2u\left(j,n\right)+u\left(j+1,n\right)\right\}-\left(u_j^n\right)^{l}\left\{u_j^{n-1}-u_j^n+u_{j+1}^n\right\} \\ &=\left\{u\left(j,n\right)^{l}-\left(u_j^n\right)^{l}\right\}\left\{u\left(j-1,n\right)-2u\left(j,n\right)+u\left(j+1,n\right)\right\} \\ &\quad+\left(u_j^n\right)^{l}\left\{\left(u\left(j-1,n\right)-2u\left(j,n\right)+u\left(j+1,n\right)\right)-\left(u_j^{n-1}-u_j^n+u_{j+1}^n\right)\right\} \\ &=h^2\left\{u\left(j,n\right)^{l}-\left(u_j^n\right)^{l}\right\}u_{xx}\left(j,n\right)+h^2\left\{u_j^n\right\}^{l}e_{xx}\left(j,n\right)+O\left(h^4\right). \end{aligned} \tag{4.5}$$

We set the initial conditions of (4.2)

$$e_j^0 = 0,$$
$$e_j^1 = T(j,1). \quad (0 < j < 1/h) \tag{4.6}$$

Form (4.2), (4.4), (4.5), (4.6), we have

$$\begin{aligned} e_j^{n+1} &= e_j^n + \{u(j,n)\}^l (g_1(j,n) - g_2(j,n)) h^2 u_{xx}(j,n) \\ &\quad + g_2(j,n) h^2 \left\{ \left\{ u(j,n)^l - (u_j^n)^l \right\} u_{xx}(j,n) + \{u_j^n\}^l e_{xx}(j,n) \right\} + T(j,n+1) + O(h^4) \\ &= e_j^n + h^2 \left\{ g_1(j,n) u(j,n)^l - g_2(j,n) (u_j^n)^l \right\} u_{xx}(j,n) \\ &\quad + h^2 g_2(j,n) \{u_j^n\}^l e_{xx}(j,n) + T(j,n+1) + O(h^4). \end{aligned} \tag{4.7}$$

From (4.7), we have

$$\begin{aligned} e_j^n &= h^2 \Sigma_{l=1}^{n-1} p_1(j,l) u_{xx}(j,l) \left(\Sigma_{s=1}^l T(j,s) \right) + h^2 \Sigma_{l=1}^{n-1} p_2(j,l) D^2 \left(\Sigma_{s=1}^l T(j,s) \right) + \Sigma_{l=1}^n T(j,l) + O(h^4) \\ &= h^2 \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} p_1(j,l) u_{xx}(j,l) ak \left(\Sigma_{s=1}^l w(j,s) \right) + h^2 \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} p_2(j,l) akD^2 \left(\Sigma_{s=1}^l w(j,s) \right) \\ &\quad + ak^{(1+\tilde{\rho})} \Sigma_{l=1}^n w(j,l) + O(h^4), \end{aligned} \tag{4.8}$$

with

$$p_1(j,n) = \{u(j,n)\}^l g_1(j,n) - \{u_j^n\}^l g_2(j,n), \tag{4.9}$$

$$p_2(j,n) = \frac{ac\{u_j^n\}^{(l)}}{1 + 2a\tilde{c}\{u_j^n\}^{(2l)}}. \tag{4.10}$$

We study the coefficients of (4.8) to $l \geq 2$.
Firstly we consider the case

$$\left| u_i^j \right| \geq 1. \tag{4.11}$$

We set

$$\tilde{c}_1 = (\tilde{c})^{\frac{1}{2l}}. \tag{4.12}$$

Then from (4.3), (4.12), we have

$$\begin{aligned} k^{\tilde{\rho}} \{u(j,n)\}^l g_1(j,n) &= k^{\tilde{\rho}} \frac{ac\{u(j,n)\}^{(l)}}{1 + 2a\tilde{c}\{u(j,n)\}^{(2l)}} = \{\tilde{c}_1\}^l \frac{a\{\tilde{c}_1\}^l \{u(j,n)\}^{(l)}}{1 + 2a\tilde{c}\{u(j,n)\}^{(2l)}} \\ &= \{\tilde{c}_1\}^l \frac{a\{\tilde{c}_1 u(j,n)\}^l}{1 + 2a\{\tilde{c}_1 u(j,n)\}^{(2l)}}. \end{aligned} \tag{4.13}$$

$$\begin{aligned} k^{\tilde{\rho}} \{u_j^n\}^l g_2(j,n) &= k^{\tilde{\rho}} \frac{ac\{u_j^n\}^{(l)}}{1 + 2a\tilde{c}\{u_j^n\}^{(2l)}} = \{\tilde{c}_1\}^l \frac{a\{\tilde{c}_1\}^l \{u_j^n\}^{(l)}}{1 + 2a\tilde{c}\{u_j^n\}^{(2l)}} \\ &= \{\tilde{c}_1\}^l \frac{a\{\tilde{c}_1 u_j^n\}^l}{1 + 2a\{\tilde{c}_1 u_j^n\}^{(2l)}}. \end{aligned} \tag{4.14}$$

We have the equation

$$\left|\frac{a\{\tilde{c}_1u(j,n)\}^{(l)}}{\left(1+2a\{\tilde{c}_1u(j,n)\}^{(2l)}\right)}\right|\le 1, \tag{4.15}$$

$$\left|\frac{a\{\tilde{c}_1u_j^n\}^{(l)}}{\left(1+2a\{\tilde{c}_1u_j^n\}^{(2l)}\right)}\right|\le 1. \tag{4.16}$$

From (4.13), (4.14), (4.15), (4.16), we have

$$\left|k^{\tilde{\rho}}p_1(j,n)\right|\le\left(\left|\{u(j,n)\}^l g_1(j,n)\right|+\left|\{u_j^n\}^l g_2(j,n)\right|\right)\le 2\{\tilde{c}\}^{(l)}. \tag{4.17}$$

If we assume

$$u(x,t)\in C^\infty(\partial\Omega\cup\Omega). \tag{4.18}$$

Then we have

$$\left|\frac{\partial^{i+j}u(x,t)}{\partial x^i t^j}\right|\le K.\quad (i,j=0,1,2,3,\cdots)(K:\text{constat})\qquad (x,t)\in(\partial\Omega\cup\Omega) \tag{4.19}$$

From (3.7), we have

$$\begin{aligned}
|T(j,n)| &= ak^{(1+\tilde{\rho})}|w(j,n)| \\
&= ak^{(1+\tilde{\rho})}\left|\frac{2ac\tilde{L}(u(j,n))}{\left(1+2a\tilde{c}\tilde{L}(u(j,n))\right)}\{u(j,n)\}^l u_{xx}(j,n)\right. \\
&\quad \left.-k^{(1-\tilde{\rho})}\frac{1}{12c\left(1+2a\tilde{c}\tilde{L}(u(j,n))\right)}\{u(j,n)\}^l u_{xxxx}(j,n)\right| \\
&= ak^{(1+\tilde{\rho})}\left|\frac{2ac\{u(j,n))\}^{2l}}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)}\{u(j,n)\}^l u_{xx}(j,n)\right. \\
&\quad \left.-k^{(1-\tilde{\rho})}\frac{1}{12c\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)}\{u(j,n)\}^l u_{xxxx}(j,n)\right| \\
&\le ak^{(1+\tilde{\rho})}C_1,\quad (jh,nk)\in(\partial\Omega\cup\Omega)
\end{aligned} \tag{4.20}$$

with

$$C_1=\left\{2acK^{(2l)}+\frac{1}{12c}k^{(1-\tilde{\rho})}\right\}K^{(l+1)}. \tag{4.21}$$

From (4.20), we have

$$\Sigma_{l=1}^n|T(j,l)|=|\Sigma_{l=1}^n ak^{(1+\tilde{\rho})}|w(j,l)|<ak^{(1+\tilde{\rho})}nC_1,\qquad (jh,lk)\in(\Omega\cup\partial\Omega) \tag{4.22}$$

where C_1 is defined by (4.21).

We have from the condition (1.1)

$$nk\le t_F. \tag{4.23}$$

From (4.17), (4.20), (4.23), we have

$$
\begin{aligned}
&h^2\Sigma_{l=1}^{n-1}\left|p_1(j,l)u_{xx}(j,l)\right|\left(\Sigma_{s=1}^{l}\left|T(j,s)\right|\right)\\
&\le h^2\Sigma_{l=1}^{n-1}ak^{(1+\tilde{\rho})}nC_1\left|p_1(j,l)u_{xx}(j,l)\right|\le\frac{k}{c}\Sigma_{l=1}^{n-1}2aC_1\{t_F\}K\left\{\tilde{c}_{(1)}\right\}^{(l)}\\
&\le\frac{2a}{c}\{t_F\}^2C_1K\left\{\tilde{c}_{(1)}\right\}^{(l)},\qquad (jh,lk)\in(\Omega\cup\partial\Omega)
\end{aligned}
\tag{4.24}
$$

where C_1 is defined by (4.21).

In the same way to (4.16), from (4.10), we have

$$
\begin{aligned}
\left|k^{\tilde{\rho}}p_2(j,n)\right|&=\left|k^{\tilde{\rho}}\frac{ac\left\{u_j^n\right\}^{(l)}}{1+2a\tilde{c}\tilde{L}\left(u_j^n\right)}\right|=\left|\left\{\tilde{c}_1\right\}^{(l)}\frac{a\left\{\tilde{c}_1u_j^n\right\}^{(l)}}{1+2a\left\{\tilde{c}_1u_j^n\right\}^{(2l)}}\right|\\
&\le\left|\left\{\tilde{c}_1\right\}^{(l)}\right|\left|\frac{a\left\{\tilde{c}_1u_j^n\right\}^{(l)}}{\left|1+2a\left\{\tilde{c}_1u_j^n\right\}^{(2l)}\right|}\right|\le\left\{\tilde{c}_1\right\}^{(l)}.
\end{aligned}
\tag{4.25}
$$

From (3.8), we have

$$
\begin{aligned}
D^2\left[w(j,n)\right]=D^2\Bigg\{&\frac{2ac\{u(j,n)\}^{(2l)}}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)}\Bigg\{\{u(j,n)\}^{(l)}u_{xx}(j,n)\\
&-\frac{k^{(1-\tilde{\rho})}\{u(j,n)\}^{(2l)}}{12c\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)}\{u(j,n)\}^{(l)}u_{xxxx}(j,n)\Bigg\}\Bigg\}.
\end{aligned}
\tag{4.26}
$$

After some complicate computation, we have

$$
\begin{aligned}
&D^2\left\{\frac{2ac\tilde{L}_2(u(j,n))}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)}\{u(j,n)\}^{(l)}u_{xx}(j,n)\right\}\\
&=\frac{q_4(j,n)}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)^4}u_{xx}(j,n)+2\frac{q_5(j,n)}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)^3}u_{xxx}(j,n)\\
&+\frac{q_6(j,n)}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)}u_{xxxx}(j,n),
\end{aligned}
\tag{4.27}
$$

with

$$
\begin{aligned}
q_4(j,n)=2ac\Big\{&3lu_{xx}(j,n)\{u(j,n)\}^{(3l-1)}+3l(3l-1)\{u_x(j,n)\}^2\{u(j,n)\}^{(3l-2)}\\
&+2al\tilde{c}u_{xx}(j,n)\{u(j,n)\}^{(5l-1)}+2l(5l-1)a\tilde{c}\{u_x(j,n)\}^2\{u(j,n)\}^{(5l-2)}\\
&+6a\tilde{c}lu_{xx}(j,n)\{u(j,n)\}^{(5l-1)}+6a\tilde{c}l(3l-1)\{u_x(j,n)\}^2\{u(j,n)\}^{(5l-2)}\\
&+4a^2\{\tilde{c}\}^2lu_{xx}(j,n)\{u(j,n)\}^{(7l-1)}+4a^2\{\tilde{c}\}^2l(5l-1)\{u_x(j,n)\}^2\{u(j,n)\}^{(7l-2)}\\
&-24l^2a\tilde{c}\{u_x(j,n)\}^2\{u(j,n)\}^{(5l-2)}-16l^2a^2\{\tilde{c}\}^2\{u_x(j,n)\}^2\{u(j,n)\}^{(7l-1)}\Big\},
\end{aligned}
$$

$$
q_5(j,n)=2ac\left\{3lu_x(j,n)\{u(j,n)\}^{(3l-1)}+2a\tilde{c}l\{u_x(j,n)\}^2\{u(j,n)\}^{(5l-1)}\right\},
$$

$$q_6(j,n) = 2ac\{u(j,n)\}^{(3l)}.$$

From (4.27), we have

$$\left| D^2 \left\{ \frac{2ac\{u(j,n)\}^{(2l)}}{\left(1+2\tilde{c}\{u(j,n)\}^{(2l)}\right)} \{u(j,n)\}^{(l)} u_{xx}(j,n) \right\} \right| \le C_2, \qquad (jh,nk) \in (\partial\Omega \cup \Omega) \tag{4.28}$$

with

$$\begin{aligned} C_2 &= 2ac\Big\{\left(1+6l+3l|3l-1|\right)K^{(3l+1)} + \left(10a\tilde{c}l + 2al|5l-1|\tilde{c} + 6a\tilde{c}|3l-1| + 24al^2\tilde{c}\right)K^{(5l+1)} \\ &\quad + \left(4a^2l\tilde{c}^2 + 4a^2l\tilde{c}^2 l|5l-1| + 16a^2l^2\tilde{c}^2\right)K^{(7l+1)}\Big\}. \end{aligned} \tag{4.29}$$

From (4.26), we have

$$\begin{aligned} & D^2 \left\{ \frac{\{u(j,n)\}^{(2l)}}{\left(1+2a\hat{c}\{u_j^n\}^{(2l)}\right)} \{u(j,n)\}^{(l)} u_{xxxx}(j,n) \right\} \\ &= \left\{ \frac{q_7(j,n)}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)^4} + \frac{q_8(j,n)}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)^3} \right\} u_{xxxx}(j,n) \\ &\quad + 2\frac{q_9(j,n)}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)^2} u_{xxxxx}(j,n) + \frac{q_{10}(j,n)}{\left(1+2a\tilde{c}\{u(j,n)\}^{(2l)}\right)} u_{xxxxxx}(j,n), \end{aligned} \tag{4.30}$$

with

$$\begin{aligned} q_7(j,n) &= lu_{xx}\{u(j,n)\}^{(l-1)} + l(l-1)\{u_x(j,n)\}^2\{u(j,n)\}^{(l-2)} \\ &\quad - 2al\tilde{c}u_{xx}(j,n)\{u(j,n)\}^{(3l-1)} - 2a\tilde{c}l(3l-1)\{u_x(j,n)\}^2\{u(j,n)\}^{(3l-2)}, \\ q_8(j,n) &= 8a\tilde{c}l^2\{u_x(j,n)\}^2\{u(j,n)\}^{(3l-2)} - 16a^2l^2\tilde{c}^2\{u_x(j,n)\}^2\{u(j,n)\}^{(5l-2)}, \\ q_9(j,n) &= lu_x(j,n)\{u(j,n)\}^{(l-1)} - 2a\tilde{c}lu_x(j,n)\{u(j,n)\}^{(3l-1)}, \\ q_{10}(j,n) &= 2c\{u(j,n)\}^{(l)}. \end{aligned}$$

From (4.30)

$$\left| D^2 \left\{ \frac{\{u(j,n)\}^{(3l)}}{\left(1+2\tilde{c}\{u(j,n)\}^{(2l)}\right)} u_{xxxx}(j,n) \right\} \right| \le C_3, \qquad (jh,nk) \in (\partial\Omega \cup \Omega) \tag{4.31}$$

with

$$C_3 = \left\{1+2l+l|(l-1)|\right\}K^{(l+1)} + \left\{4a\tilde{c}l + 2a\tilde{c}l|3l-1| + 8a\tilde{c}l^2\right\}K^{(3l+1)} + 16a^2l^2\tilde{c}^2K^{(5l+1)}. \tag{4.32}$$

From (4.26), (4.28), (4.31), we have

$$\left| D^2\left[w(j,n)\right] \right| \le \left\{ C_2 + \frac{k^{(1-\tilde{\rho})}}{12c} C_3 \right\}. \qquad (jh,nk) \in (\partial\Omega \cup \Omega) \tag{4.33}$$

From (4.25), we have

$$h^2\Sigma_{m=1}^{n-1}\left\{p_2(j,m)\Sigma_{s=1}^{m}D^2\left[T(j,s)\right]\right\}=h^2ak^{(1+\tilde{\rho})}\Sigma_{m=1}^{n-1}\left\{\frac{ac\left\{u_j^m\right\}^{(l)}}{\left(1+2a\hat{c}\left\{u_j^m\right\}^{(2l)}\right)}\Sigma_{s=1}^{m}D^2\left[w(j,s)\right]\right\}$$
$$=h^2\Sigma_{m=1}^{n-1}\left\{k^{\tilde{\rho}}\frac{ac\left\{u_j^m\right\}^{(l)}}{\left(1+2a\hat{c}\left\{u_j^m\right\}^{(2l)}\right)}ak\Sigma_{s=1}^{m}D^2\left[w(j,s)\right]\right\}. \tag{4.34}$$

From (4.25), (4.33), (4.34), we have

$$h^2\Sigma_{m=1}^{n-1}\left|\left\{p_2(j,m)\Sigma_{s=1}^{m}D^2\left[T(j,s)\right]\right\}\right|$$
$$\le\frac{k}{c}\Sigma_{m=1}^{n-1}\left\{\tilde{c}_1\right\}^l\left|ak\Sigma_{s=1}^{m}D^2\left[w(j,s)\right]\right|\le\frac{k}{c}\left(n\left\{\tilde{c}_1\right\}^l\right)ak\Sigma_{m=1}^{n-1}\left(C_2+\frac{k^{(1-\tilde{\rho})}}{12c}C_3\right) \tag{4.35}$$
$$\le\frac{a}{c}\left\{\tilde{c}_1\right\}^l\left\{t_F\right\}^2\left(C_2+\frac{k^{(1-\tilde{\rho})}}{12c}C_3\right),\qquad (jh,lk)\in(\Omega\cup\partial\Omega)$$

where C_2 and C_3 are defined by (4.29) and (4.32) respectively.

From (4.20), we have

$$\Sigma_{l=1}^{n}\left|T(j,l)\right|=ak^{(1+\tilde{\rho})}\left|\Sigma_{l=1}^{n}w(j,l)\right|\le ak\Sigma_{l=1}^{n}k^{\tilde{\rho}}\left|w(j,l)\right|$$
$$\le ak^{\tilde{\rho}}\left\{t_F\right\}C_1+0\left(k^2\right),\qquad (jh,lk)\in(\partial\Omega\cup\Omega) \tag{4.36}$$

where C_1 is defined by (4.21).

From (4.8), (4.20) (4.24), (4.35), (4.36), we have

$$\left|e_j^n\right|\le h^2\Sigma_{l=1}^{n-1}\left|p_1(j,l)u_{xx}(j,l)\right|\left(\Sigma_{s=1}^{l}\left|T(j,s)\right|\right)$$
$$+h^2\Sigma_{l=1}^{n-1}\left|p_2(j,l)\right|\left\{D^2\left(\Sigma_{s=1}^{l}\left|T(j,s)\right|\right)\right\}+\Sigma_{l=1}^{n}\left|T(j,l)\right|+O\left(h^4\right)$$
$$\le\frac{2a}{c}\left\{t_F\right\}^{(2)}C_1K\left\{\tilde{c}_1\right\}^{(l)}+\frac{a}{c}\left\{t_F\right\}^2\left(C_2+\frac{k^{(1-\tilde{\rho})}}{12c}C_3\right)\left\{\tilde{c}_1\right\}^l \tag{4.37}$$
$$+ak^{\tilde{\rho}}t_FC_1+0\left(k^2\right),\quad (jh,lk)\in(\Omega\cup\partial\Omega)$$

where C_1,C_2 and C_3 are defined by (4.21), (4.29) and (4.32) respectively.

We set the maximum norm of the function e_j^n

$$\left\|E^n\right\|=\max_{1\le j\le 1/k}\left|e_j^n\right|. \tag{4.38}$$

Then, from (4.37), we have

$$\left\|E^n\right\|<\frac{2a}{c}\left\{t_F\right\}^{(2)}C_1K\left\{\tilde{c}_1\right\}^{(l)}+\frac{a}{c}\left\{\tilde{c}_1\right\}^l\left\{t_F\right\}^2\left(C_2+\frac{k^{(1-\tilde{\rho})}}{12c}C_3\right)+ak^{\tilde{\rho}}t_FC_1+0\left(k^2\right). \tag{4.39}$$

From (4.39), we have

$$\lim_{k\to 0}\left\|E^n\right\|=0. \tag{4.40}$$

Finally we assume

$$\left|u_j^n\right|\le 1. \tag{4.41}$$

Then, from (4.3), we have

$$g_1(j,n)=g_2(j,n). \tag{4.42}$$

From (4.9), (4.42), we have

$$\left|p_1(j,n)\right| = \left|\{u(j,n)\}^l g_1(j,n) - \{u_j^n\}^l g_2(j,n)\right| = \left|\frac{ac}{(1+2a\tilde{c})}\left\{\{u(j,n)\}^l - \{u_j^n\}^l\right\}\right| \le ac\{K\}^{(l)}. \quad (jh,lk)\in(\partial\Omega\cup\Omega) \tag{4.43}$$

In the same way to (4.14), we have

$$\left|p_2(j,n)\right| = \left|\frac{ac\{u_j^n\}^{(l)}}{1+2a\tilde{c}}\right| \le ac. \tag{4.44}$$

From (3.8), we have after some computation,

$$\left|D^2\{w_2(j,n)u_{xx}(j,n)\}\right| \le C_4, \quad (jh,nk)\in(\partial\Omega\cup\Omega) \tag{4.45}$$

with

$$C_4 = \frac{2ac}{(1+2a\tilde{c})}\{l|l-1|+3l+1\}\left\{1+\frac{1}{24ac^2}k^{(1-\tilde{\rho})}\right\}K^{(l+1)}. \tag{4.46}$$

From (4.8), (4.20), (4.43), (4.44), (4.45), we have

$$\begin{aligned}\left|e_j^n\right| &\le h^2\Sigma_{l=1}^{n-1}\left|p_1(j,l)u_{xx}(j,l)ak^{(1+\tilde{\rho})}\left(\Sigma_{s=1}^{l}w(j,s)\right)\right| \\ &\quad + h^2\Sigma_{l=1}^{n-1}\left|p_2(j,l)ak^{(1+\tilde{\rho})}D^2\left(\Sigma_{s=1}^{l}w(j,s)\right)\right| + ak^{(1+\tilde{\rho})}\Sigma_{l=1}^{n}\left|w(j,l)\right| + O(h^4) \\ &\le \left(\frac{k}{c}\right)\Sigma_{l=1}^{n-1}ack^{\tilde{\rho}}\{K\}^{(l+1)}a\{t_F\}C_1 + \left(\frac{k}{c}\right)\Sigma_{l=1}^{n-1}acak^{(1+\tilde{\rho})}D^2\left(\Sigma_{s=1}^{n}w(j,s)\right) + ak^{(1+\tilde{\rho})}nC_1 + O(h^4). \\ &< a^2\{t_F\}^{(2)}\{K\}^{(l+1)}C_1k^{\tilde{\rho}} + a^2\{t_F\}^{(2)}C_4k^{\tilde{\rho}} + ak^{\tilde{\rho}}\{t_F\}C_1 + O(h^4). \quad (jh,lk)\in(\partial\Omega\cup\Omega)\end{aligned} \tag{4.47}$$

where C_1 and C_4 are defined by (4.21) and (4.46) respectively.

Then, in the same way to (4.40), from (4.47), we have

$$\lim_{k\to 0}\left\|E^n\right\| = 0. \tag{4.48}$$

We study $l = 0$. In the almost same way to (4.47), we have

$$\left|e_j^n\right| \le a^2\{t_F\}^{(2)}C_4k^{\tilde{\rho}} + ak^{\tilde{\rho}}\{t_F\}C_1 + O(h^4). \qquad (jh,lk)\in(\partial\Omega\cup\Omega) \tag{4.49}$$

where C_1 and C_4 are defined by (4.21) and (4.46) with $l = 0$ respectively.

From (4.49), we have

$$\lim_{k\to 0}\left\|E^n\right\| = 0. \tag{4.50}$$

From (4.40), (4.48), (4.50), we have

Theorem 2. Suppose that for h_i and k_i, there exists positive numbers $j(h_i)$ and $n(k_i)$ such that

$$j(h_i)h_i \to x\in[0,1](i\to\infty) \quad n(k_i)k_i \to t\in[0,t_F].$$

If the solution $u(x,t)$ of (1.1) satisfies conditions (4.18). Then, the approximation u_j^n generated by the scheme (2.8) converges to the solution $u(x,t)$ of the differential Equation (1.1).

5. Stability

In this section, we study the stability of the numerical process (2.8) and define as follows.

Definition: The numerical processes $\{Y^n \in R_n\}$ is stable if there exists a positive constant K_2 such that

$$\left\|Y^n\right\| \le K_2,$$

where $\|.\|$ denotes some norm and the constant K_2 is depends on initial value.

We prove that the scheme (2.8) are stable in mean of the von Neumann.
We set

$$x = jh.$$

Then, from (4.7), we have

$$\begin{aligned} e(x,n) &= h^2 \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} p_1(x,l) u_{xx}(x,l) ak\left(\Sigma_{s=1}^{l} w(x,s)\right) \\ &+ \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} p_2(x,l) akD^2\left(\Sigma_{s=1}^{l} w(x,s)\right) \\ &+ ak^{(1+\tilde{\rho})} \Sigma_{s=1}^{n} w(x,s) + O\left(h^4\right). \end{aligned} \tag{5.1}$$

From (5.1), we have

$$\begin{aligned} & D^{(2)}\{e(x,n)\} = h^2 \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} \left\{ \left(D^{(2)}\left(p_1(x,l)\right)\right) u_{xx}(j,l) k\left(\Sigma_{s=1}^{l} w(x,s)\right) \right. \\ & + p_1(x,l)\left(D^{(2)} u_{xx}(x,l)\right) k\left(\Sigma_{s=1}^{l} w(x,s)\right) + p_1(x,l) u_{xx}(x,l) k\left(\Sigma_{s=1}^{l} D^{(2)}\left(w(x,s)\right)\right) \\ & + 2\left(D\left(p_1(x,l)\right)\right)\left(D\left(u_{xx}(x,l)\right)\right) k\left(\Sigma_{s=1}^{l} w(x,s)\right) + \left(Dp_1(x,l)\right) u_{xx}(x,l) k\left(\Sigma_{s=1}^{l} D\left(w(x,s)\right)\right) \\ & \left. + p_1(x,l)\left(Du_{xx}(x,l)\right) k\left(\Sigma_{s=1}^{l} D\left(w(x,s)\right)\right)\right\} + \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} \left\{ \left(D^{(2)} p_2(x,l)\right) k\left(\Sigma_{s=1}^{l} (D^{(2)}\left(w(x,s)\right)\right) \right. \\ & \left. + 2\left(Dp_2(x,l)\right) k\left(\Sigma_{s=1}^{l} D^{(3)}\left(w(x,s)\right)\right) + p_2(x,l) k\left(\Sigma_{s=1}^{l} D^{(4)}\left(w(x,s)\right)\right) \right\} \\ & + h^{(1+\tilde{\rho})} \Sigma_{l=1}^{n}\left(D^{(2)} w(x,s)\right) + O\left(h^4\right), \end{aligned} \tag{5.2}$$

where $p_1(x,n), p_2(x,n)$ and $w(x,n)$ are defined by (4.9), (4.10) and (3.8) respectively.

If we assume (4.18) on the solution $u(t,x)$ of (1.1), Then,in the same way to (4.31), (4.33), (4.45), we have

$$\begin{aligned} & \left| h^2 \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} \left(D^{(2)} p_1(x,l)\right) u_{xx}(x,l) k\left(\Sigma_{s=1}^{l} w(x,s)\right) \right| \le C_5, \\ & \left| h^2 \Sigma_{l=1}^{n-1} p_1(x,l)\left(D^{(2)} u_{xx}(x,l)\right) k\left(\Sigma_{s=1}^{l} w(x,s)\right) \right| \le C_5, \\ & \left| h^2 \Sigma_{l=1}^{n-1} p_1(x,l) u_{xx}(x,l) k\left(\Sigma_{s=1}^{l}\left(D^{(2)}\left(w(x,s)\right)\right)\right) \right| \le C_5, \\ & \left| h^2 \Sigma_{l=1}^{n-1}\left(D\left(p_1(x,l)\right)\right)\left(D\left(u_{xx}(x,l)\right)\right) k\left(\Sigma_{s=1}^{l} w(x,s)\right) \right| \le C_5, \\ & \left| h^2 \Sigma_{l=1}^{n-1}\left(Dp_1(x,l)\right) u_{xx}(x,l) k\left(\Sigma_{s=1}^{l} D\left(w(x,s)\right)\right) \right| \le C_5, \\ & \left| h^2 \Sigma_{l=1}^{n-1} p_1(x,l)\left(Du_{xx}(x,l)\right) k\left(\Sigma_{s=1}^{l} D\left(w(x,s)\right)\right) \right| \le C_5, \\ & \left| \Sigma_{l=1}^{n-1} k^{\tilde{\rho}}\left(D^{(2)} p_2(x,l)\right) k\left(\Sigma_{s=1}^{l}\left(D^{(2)}(w(x,s)\right)\right) \right| \le C_5, \\ & \left| \Sigma_{l=1}^{n-1} k^{\tilde{\rho}}\left(Dp_2(x,l)\right) k\left(\Sigma_{s=1}^{l}\left(D^{(3)}\left(w(x,s)\right)\right)\right) \right| \le C_5, \\ & \left| \Sigma_{l=1}^{n-1} k^{\tilde{\rho}} p_2(x,l) k\left(\Sigma_{s=1}^{l}\left(D^{(4)}\left(w(x,s)\right)\right)\right) \right| \le C_5, \quad (x,lk) \in (\partial\Omega \cup \Omega) \end{aligned} \tag{5.3}$$

for some constant C_5.

From (5.2), (5.3), we have the following result.

Lemma 1. If we assume the solution $u(x,t)$ of (1.1) satisfies (4.18), Then there exists the constant C_5 such that

$$\left| \frac{\partial^2}{\partial x^2} e(x,j) \right| \le C_6, \qquad (x,jk) \in (\partial\Omega \cup \Omega) \tag{5.4}$$

$$\text{with } C_6 = 6C_5, \tag{5.5}$$

where C_5 is defined by (5.3). From (2.8), we have

$$\left|u_j^{n+1}\right| \le \left|u_j^n + \frac{ac}{1+2ac\tilde{L}\left(u_j^n\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right)\right| + \left|\frac{ac}{1+2a\hat{c}\tilde{L}\left(u\left(jh,nk\right)\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right) - \frac{ac}{1+2ac\tilde{L}\left(u_j^n\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right)\right|. \tag{5.6}$$

We set the maximum norm of the function u_j^n

$$\left\|U^n\right\| = \max_{1\le j\le 1/k}\left|u_j^n\right|. \tag{5.7}$$

We have the inequality

$$\begin{aligned}
&\left|u_j^n + \frac{ac}{1+2ac\tilde{L}\left(u_j^n\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right)\right| \\
&= \left|u_j^n + \frac{ac}{1+2ac\tilde{L}\left(u_j^n\right)}\left\{u_j^n\right\}^l\left\{u_{j-1}^n - 2u_j^n + u_{j+1}^n\right\}\right| \\
&\le \left|\left(1 - 2\frac{ac\left\{u_j^n\right\}^l}{1+2ac\tilde{L}\left(u_j^n\right)}\right)\right|\left|u_j^n\right| + \left|\frac{ac\left\{u_j^n\right\}^l}{1+2ac\tilde{L}\left(u_j^n\right)}u_{j-1}^n + \frac{ac\left\{u_j^n\right\}^l}{1+2ac\tilde{L}\left(u_j^n\right)}u_{j+1}^n\right| \\
&\le \left\|U^n\right\| + 2\frac{ac\left(\left|\left\{u_j^n\right\}^l\right| - \left\{u_j^n\right\}^l\right)}{1+2ac\tilde{L}\left(u_j^n\right)}\left\|U^n\right\|.
\end{aligned} \tag{5.8}$$

From (1.1), we have

$$\left\{u_j^n\right\}^l \ge 0.$$

From (5.8), we have

$$\left|u_j^n + \frac{ac}{1+2ac\tilde{L}\left(u_j^n\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right)\right| \le \left\|U^n\right\|. \tag{5.9}$$

From (2.8), we have

$$\begin{aligned}
&\frac{ac}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right) - \frac{ac}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right) \\
&= ac\left\{\frac{1}{1+2a\tilde{c}\tilde{L}\left(u_j^n\right)} - \frac{1}{1+2ac\tilde{L}\left(u_j^n\right)}\right\}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right). \\
&= \left(a^2\right)\frac{2c\left(c-\tilde{c}\right)\tilde{L}\left(u_j^n\right)}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\left\{u_j^n\right\}^l\left\{u_{j-1}^n - 2u_j^n + u_{j+1}^n\right\},
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
\left\{u_{j-1}^n - 2u_j^n + u_{j+1}^n\right\} &= \left(u\left(j-1,n\right) - e_{j-1}^n\right) - 2\left(u\left(j,n\right) - e_j^n\right) + \left(u\left(j+1,n\right) - e_{j+1}^n\right) \\
&= h^2\left(u_{xx}\left(j,n\right) - e_{xx}\left(j,n\right)\right) + O\left(h^4\right).
\end{aligned} \tag{5.11}$$

From (5.10), (5.11), we have

$$\begin{aligned}
&\left|\frac{ac}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right) - \frac{ac}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right)\right| \\
&= \left(a^2\right)\left|k\frac{2\left(c-\tilde{c}\right)\tilde{L}\left(u_j^n\right)}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\left\{u_j^n\right\}^l\left(u_{xx}\left(j,n\right) - e_{xx}\left(j,n\right)\right) + O\left(h^2\right)\right|.
\end{aligned} \tag{5.12}$$

Firstly we consider

$$l = 0.$$

Then from (5.9) and (5.12), we have

$$\left|u_j^{n+1}\right| \le \left\|U^n\right\| + C_7 k + O\left(k^2\right), \tag{5.13}$$

with

$$C_7 = \left(2a^2\right)\left(c - \tilde{c}\right)\left(K + C_6\right). \tag{5.14}$$

where K, C_6 are defined by (4.19) and (5.5) respectively.

From (5.14), we have

$$\left\|U^n\right\| \le C_7 t_F + O(k). \tag{5.15}$$

Lastly, we consider

$$l \ge 2.$$

From (5.12), we have

$$\left|\frac{ac}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_j^n,u_j^n,u_{j+1}^n\right) - \frac{ac}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_j^n,u_j^n,u_{j+1}^n\right)\right|$$

$$= \left|a\left(c-\tilde{c}\right)\frac{1}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\frac{2ac\tilde{L}\left(u_j^n\right)}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\left\{u_j^n\right\}^l h^{(2)}\left(u_{xx}(j,n) - e_{xx}(j,n)\right) + O\left(h^2\right)\right|.$$

$$\left|a\frac{k}{c}\left(c-\tilde{c}\right)\left\{\tilde{c}_1\right\}^{-(l)}\left(\frac{\left\{\tilde{c}_1\right\}^l\left\{u_j^n\right\}^{(l)}}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\right)\left(\frac{2ac\tilde{L}\left(u_j^n\right)}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\right)\left(u_{xx}(j,n) - e_{xx}(j,n)\right) + O\left(h^2\right)\right|. \tag{5.16}$$

Firstly, we consider the case $\left|u_j^n\right| \ge 1$.

Then from (5.16), we have

$$\left|\frac{ac}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_j^n,u_j^n,u_{j+1}^n\right) - \frac{ac}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_j^n,u_j^n,u_{j+1}^n\right)\right|$$

$$= \left|a\frac{k}{c}\left(c-\tilde{c}\right)\left\{\tilde{c}_1\right\}^{-(l)}\left(\frac{\left\{\tilde{c}_1 u_j^n\right\}^{(l)}}{\left(1+2a\left\{\tilde{c}_{(1)}u_j^n\right\}^{(2l)}\right)}\right)\left(\frac{2ac\left\{u_j^n\right\}^{(2l)}}{\left(1+2ac\left\{u_j^n\right\}^{(2l)}\right)}\right)\left(u_{xx}(j,n) - e_{xx}(j,n) + O\left(h^2\right)\right)\right|, \tag{5.17}$$

$$\le \left|\frac{k}{2c}\left(c-\tilde{c}\right)\left\{c_1\right\}^{-(l)}\left(\frac{2a\left\{\tilde{c}_1 u_j^n\right\}^{(l)}}{\left(1+2a\left\{\tilde{c}_{(1)}u_j^n\right\}^{(2l)}\right)}\right)\left(\frac{2ac\left\{u_j^n\right\}^{(2l)}}{\left(1+2ac\left\{u_j^n\right\}^{(2l)}\right)}\right)\left(u_{xx}(j,n) - e_{xx}(j,n) + O\left(h^2\right)\right)\right|\left|u_j^n\right|.$$

We have

$$\left|\frac{2a\left\{\tilde{c}_1 u_j^n\right\}^{(l)}}{\left(1+2a\left\{\tilde{c}_{(1)}u_j^n\right\}^{(2l)}\right)}\right| \le 1, \qquad \left|\frac{2ac\left\{u_j^n\right\}^{(2l)}}{\left(1+2ac\left\{u_j^n\right\}^{(2l)}\right)}\right| \le 1. \tag{5.18}$$

From (5.10). (5.17), (5.18), we have

$$\left|\frac{ac}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right) - \frac{ac}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right)\right| \le \left(\frac{1}{2}kC_8 + O\left(k^2\right)\right)\left|u_j^n\right|, \tag{5.19}$$

with

$$C_8 = \left|\frac{1}{c}(c-\tilde{c})\{\tilde{c}_1\}^{-(l)}\right|(K+C_6) = \left|\frac{(1-k^\rho)}{\sqrt{c}\sqrt{k}^\rho}\right|(K+C_6), \tag{5.20}$$

where K and C_6 are defined by (4.19) and (5.5) respectively.

If $0<k<1$, Then we set

$$\rho = \log_k\left(1-\left(\sqrt{c}\right)S_1\right). \quad \left(0<S_1<\frac{(1-k)}{\left(\sqrt{c}\right)}\right) \tag{5.21}$$

From (5.21), we have

$$\frac{(1-k^\rho)}{\sqrt{c}\sqrt{k^\rho}} = \frac{S_1}{\sqrt{\left(1-\left(\sqrt{c}\right)S_1\right)}}. \tag{5.22}$$

If $k>1$, Then we set

$$\rho = \log_k\left(1+\left(\sqrt{c}\right)S_2\right). \quad \left(0<S_2<\frac{(k-1)}{\left(\sqrt{c}\right)}\right) \tag{5.23}$$

From (5.23), we have

$$\frac{(k^\rho-1)}{\sqrt{c}\sqrt{k^\rho}} = \frac{S_2}{\sqrt{\left(1+\left(\sqrt{c}\right)S_2\right)}}. \tag{5.24}$$

From (5.22), (5.24), we set

$$C_9 = \begin{cases} \dfrac{S_1}{\sqrt{\left(1-\left(\sqrt{c}\right)S_1\right)}}(K+C_6) & (0<k<1) \\ \dfrac{S_2}{\sqrt{\left(1+\left(\sqrt{c}\right)S_2\right)}}(K+C_6), & (k>1). \end{cases} \tag{5.25}$$

where S_1 and S_2 are satisfy (5.21) and (5.23) respectively.

From (5.6), (5.19) and (5.25), we have

$$\left|u_{n+1}^j\right| \le \left(1+\frac{k}{2}C_9 + O(k^2)\right)\left\|U^n\right\|,$$

and we have the following result

$$\left\|U^{n+1}\right\| \le \left(1+\frac{k}{2}C_9 + O(k^2)\right)\left\|U^n\right\|. \tag{5.26}$$

From (5.26), we have

$$\left\|U^n\right\| \le \mathrm{e}^{\left(\frac{1}{2}C_9 t_F\right)}\left\|U^0\right\|, \tag{5.27}$$

where C_9 is defined by (5.25).

Secondly, in the case $\left|u_j^n\right| \le 1$, from (5.12), we have

$$\left|\frac{ac}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right)-\frac{ac}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right)\right|$$
$$=\left(a^2\right)c\left|\frac{2\left(c-\tilde{c}\right)}{\left(1+2a\tilde{c}\right)\left(1+2ac\right)}h^2\left\{u_j^n\right\}^l\left(u_{xx}\left(j,n\right)-e_{xx}\left(j,n\right)+O\left(h^2\right)\right)\right|. \tag{5.28}$$

From (5.28), we have

$$\left|\frac{ac}{\left(1+2a\tilde{c}\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right)-\frac{ac}{\left(1+2ac\tilde{L}\left(u_j^n\right)\right)}\Phi\left(u_{j-1}^n,u_j^n,u_{j+1}^n\right)\right|\le kC_{10}\left|u_j^n\right| \tag{5.29}$$

with

$$C_{10}=2a^2\left|\left(c-\tilde{c}\right)\right|\left(K+C_6\right), \tag{5.30}$$

where K and C_6 are defined by (4.19)and (5.5) respectively.

In the same way to (5.26), we have

$$\left\|U^n\right\|\le \mathrm{e}^{\left(C_{10}t_F\right)}\left\|U^0\right\|, \tag{5.31}$$

where C_{10} is defined by (5.30).

From (5.15), (5.27), (5.31), we have

Theorem 3.

If the solution $u\left(t,x\right)$ of (1.1) is analytic on the region $\partial\Omega\cup\Omega$ then the difference approximation (2.8) to (1.1) are stable.

6. Numerical Example

Lastly, we study the numerical test in the following non-linear Equation .

$$\frac{\partial u\left(x,t\right)}{\partial t}=\left\{u\left(x,t\right)\right\}^2\frac{\partial^2 u\left(x,t\right)}{\partial^2 x} \tag{6.1}$$

and the initial and boundary problem given by,

$$u\left(x,0\right)=\begin{cases}4x^2 & \left(0\le x\le \frac{1}{2}\right)\\ 4\left(1-x\right)^2, & \left(\frac{1}{2}\le x\le 1\right)\end{cases} \tag{6.2}$$
$$u_t\left(0,t\right)=u\left(1,t\right)=0.$$

Applying the difference Equation (2.8) to (6.1) with (6.2), we have the the numerical results in **Table 1** and **Figure 1**, **Figure 2**.

Table 1. (x = 0/100, 2/100, 20/100, 50/100, 70/100, 98/100), (t = 0, 2/100, 10/100, 20/100, 50/100).

	x	$2h$	$20h$	$50h$	$70h$	$98h$
t	0	$0.4E-3$	$0.336E+0$	$0.1E+1$	$0.384E+0$	$0.4E-3$
t	2	$0.4E-3$	$0.336E+00$	$0.997E+00$	$0.384E+00$	$0.4E-3$
t	10	$0.4E-3$	$0.336E+00$	$0.986E+00$	$0.384E+00$	$0.4E-3$
t	20	$0.4E-3$	$0.336E+00$	$0.981E+00$	$0.384E+00$	$0.4E-3$
t	50	$0.4E-3$	$0.336E+00$	$0.964E+00$	$0.384E+00$	$0.4E-3$

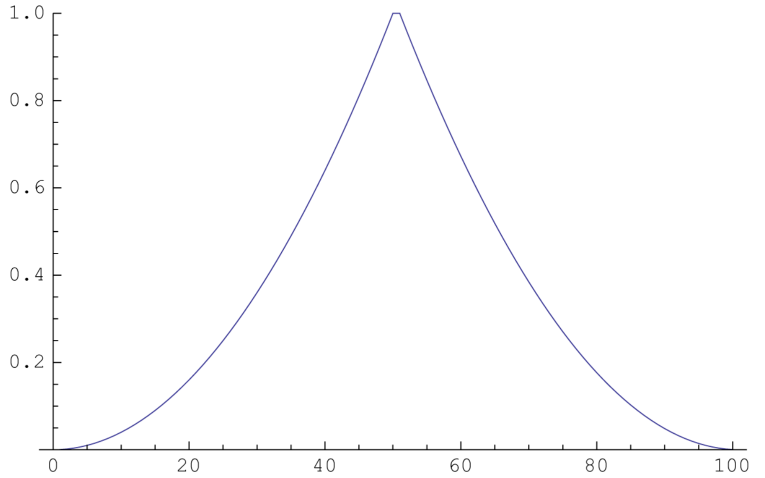

Figure 1. Initial data ($0 \le x \le 1$, $t = 0$).

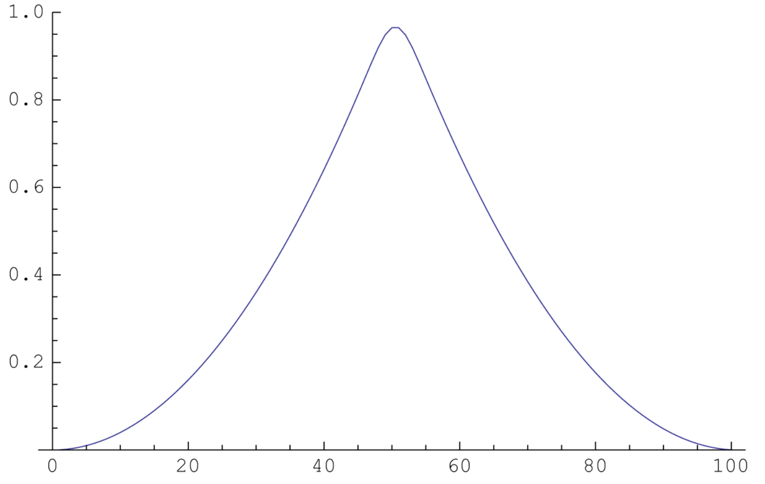

Figure 2. The numerical solution for $0 \le x \le 1$, $t = 50$.

As we see in **Figure 1**, **Figure 2**, the initial data diffuses slowly. Here the interval [0,1] is divided into $M = 100$ with $h = \frac{1}{100}, k = \frac{1}{100}, \rho = \frac{1}{200}$.

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