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Forced Oscillation of Nonlinear Impulsive Hyperbolic Partial Differential Equation with Several Delays

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Abstract

In this paper, we study oscillatory properties of solutions for the nonlinear impulsive hyperbolic equations with several delays. We establish sufficient conditions for oscillation of all solutions.

Keywords

1. Introduction

The theory of partial functional differential equations can be applied to many fields, such as biology, population growth, engineering, control theory, physics and chemistry, see the monograph [1] for basic theory and applications. The oscillation of partial functional differential equations has been studied by many authors see, for example [2]-[7], and the references cited therein.

The theory of impulsive partial differential systems makes its beginning with the paper [8] in 1991. In recent years, the investigation of oscillations of impulsive partial differential systems has attracted more and more attention in the literature see, for example [9]-[13]. Recently, the investigation on the oscillations of impulsive partial differential systems with delays can be found in [14]-[19].

To the best of our knowledge, there is little work reported on the oscillation of second order impulsive partial functional differential equation with delays. Motivated by this observation, in this paper we study the oscillation of nonlinear forced impulsive hyperbolic partial differential equation with several delays of the form

$$\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}(u(x,t))\right]=a(t)\Delta u(x,t)-p(x,t)f(u(x,t))$$

$$-\sum_{j=1}^{n}q_j(x,t)f_j(u(x,t-\sigma_j))+F(x,t),\quad t\neq t_k,\ (x,t)\in\Omega\times\mathbb{R}^+=G$$

$$u(x,t_k^+)=\alpha_k(x,t_k,u(x,t_k)),$$

$$u_t(x,t_k^+)=\beta_k(x,t_k,u_t(x,t_k)),\quad t=t_k,\ k=1,2,\cdots.$$

$$\tag{1}$$

with the boundary conditions

$$\frac{\partial u}{\partial\gamma}+h(x)u=g(x,t),\quad (x,t)\in\partial\Omega\times\mathbb{R}^+ \tag{2}$$

$$u=\varphi(x,t),\quad (x,t)\in\partial\Omega\times\mathbb{R}^+ \tag{3}$$

and the initial condition

$$u(x,t)=\Phi(x,t),\quad (x,t)\in[-\delta,0]\times\Omega. \tag{4}$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and Δ is the Laplacian in the Euclidean N-space $\mathbb{R}^N$, γ is a unit exterior normal vector of $\partial\Omega$, $\delta=\max\{\sigma_j\}$, $\Phi(x,t)\in C^2([-\delta,0]\times\Omega,\mathbb{R})$.

In the sequal, we assume that the following conditions are fulfilled:

(H1) $r(t),a(t)\in PC(\mathbb{R}^+,\mathbb{R}^+)$, σ_j is a positive constant, $p(x,t),q_j(x,t)$ are class of functions which are piece wise continuous in t with discontinuities of first kind only at $t=t_k,k=1,2,\cdots$ and left continuous at $t=t_k,k=1,2,\cdots$

(H2) $f(u),f_j(u)\in C(\mathbb{R}^+,\mathbb{R}^+)$; $\dfrac{f(u)}{u}\geq C$ is a positive constant, $\dfrac{f_j(u)}{u}\geq C_j$ is a positive constant, for $u\neq 0$; $h(x)\in C(\partial\Omega\times\Omega^+,\mathbb{R})$; $F(x,t)\in PC(\mathbb{R}^+\times\overline{\Omega},\mathbb{R})$; $g(t,x)$ and $\varphi(t,x)\in PC(\mathbb{R}^+\times\partial\Omega,\mathbb{R})$; $0<t_1<t_2<\cdots<t_k<\cdots,\lim_{k\to\infty}t_k=\infty$.

(H3) $u(x,t)$ and their derivatives $u_t(x,t)$ are piecewise continuous in t with discontinuities of first kind only at $t=t_k,k=1,2,\cdots$, and left continuous at $t=t_k$, $u(x,t_k)=u(x,t_k^-)$, $u_t(x,t_k)=u_t(x,t_k^-)$, $k=1,2,\cdots$.

(H4) $\alpha_k(x,t_k,u(x,t_k)),\beta_k(x,t_k,u_t(x,t_k))\in PC(\mathbb{R}^+\times\overline{\Omega}\times\mathbb{R},\mathbb{R})$, $k=1,2,\cdots$, and there exist positive constants a_k,a_k^*,b_k,b_k^* and $b_k\leq a_k^*$ such that for $k=1,2,\cdots$,

$$a_k^*\leq\frac{\alpha_k(x,t_k,\xi)}{\xi}\leq a_k$$

$$b_k^*\leq\frac{\beta_k(x,t_k,\xi)}{\xi}\leq b_k.$$

Let us construct the sequence $\{\overline{t_k}\}=\{t_k\}\cup\{t_{k\sigma_j}\}$, where $t_{k\sigma_j}=t_k+\sigma_j$ and $\overline{t_k}<\overline{t_{k+1}}$, $k=1,2,\cdots$.

By a solution of problem (1), (2) ((1),(3)) with initial condition (4), we mean that any function $u(x,t)$ for which the following conditions are valid:

1. If $-\delta\leq t\leq 0$, then $u(x,t)=\Phi(x,t)$.

2. If $0\leq t\leq\overline{t_1}=t_1$, then $u(x,t)$ coincides with the solution of the problem (1) and (2) ((3)) with initial condition.

3. If $\overline{t_k}<t\leq\overline{t_{k+1}},\overline{t_k}\in\{t_k\}\setminus\{t_{k\sigma_j}\}$, then $u(x,t)$ coincides with the solution of the problem (1) and (2) ((3)).

4. If $\overline{t}_k < t \leq \overline{t}_{k+1}, \overline{t}_k \in \{t_{k\sigma_j}\}$, then $u(x,t)$ coincides with the solution of the problem (2) ((3)) and the following equations

$$\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}\left(u(x,t^+)\right)\right] = a(t)\Delta u(x,t^+) - p(x,t)f\left(u(x,t^+)\right)$$

$$-\sum_{j=1}^{n}q_j(x,t)f_j\left(u\left(x,(t-\sigma_j)^+\right)\right) + F(x,t), \quad t \neq t_k, \ (x,t) \in \Omega \times \mathbb{R}^+ = G$$

$$u\left(x,\overline{t}_k^+\right) = u\left(x,\overline{t}_k\right), \quad u_t\left(x,\overline{t}_k^+\right) = u_t\left(x,\overline{t}_k\right), \quad \text{for } \overline{t}_k \in \{t_k\}\setminus\{t_{k\sigma_j}\},$$

or

$$u\left(x,\overline{t}_k^+\right) = \alpha_{k_i}\left(x,\overline{t}_k,u(x,\overline{t}_k)\right), \quad u_t\left(x,\overline{t}_k^+\right) = \beta_{k_i}\left(x,\overline{t}_k,u_t(x,\overline{t}_k)\right), \quad \text{for } \overline{t}_k \in \{t_{k\sigma_j}\}\cap\{t_k\}.$$

Here the number k_i is determined by the equality $\overline{t}_k = t_{k_i}$.

We introduce the notations:

$$\Gamma_k = \{(x,t): t \in (t_k,t_{k+1}), x \in \Omega\}; \quad \Gamma = \bigcup_{k=0}^{\infty}\Gamma_k,$$

$$\overline{\Gamma}_k = \{(x,t): t \in (t_k,t_{k+1}), x \in \overline{\Omega}\}; \quad \overline{\Gamma} = \bigcup_{k=0}^{\infty}\overline{\Gamma}_k,$$

$$p(t) = \min_{x\in\Omega} p(x,t) \quad \text{and} \quad q_j(t) = \min_{x\in\Omega} q_j(x,t).$$

The solution $u \in C^2(\Gamma)\cap C^1(\overline{\Gamma})$ of problem (1), (2) ((1),(3)) is called nonoscillatory in the domain G if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

This paper is organized as follows: Section 2, deals with the oscillatory properties of solutions for the problem (1) and (2). In Section 3, we discuss the oscillatory properties of solutions for the problem (1) and (3). Section 4 presents some examples to illustrate the main results.

2. Oscillation Properties of the Problem (1) and (2)

To prove the main result, we need the following lemmas.

Lemma 2.1. Suppose that λ is the minimum positive eigenvalue of the problem

$$\Delta\eta(x) + \lambda\eta(x) = 0, \quad x \in \Omega,$$

$$\frac{\partial\eta}{\partial\gamma} + h(x)\eta(x) = 0, \quad x \in \partial\Omega,$$

and $\eta(x)$ is the corresponding eigenfunction of λ. Then $\lambda > 0$ and $\eta(x) > 0, x \in \Omega$. **Proof.** The proof of the lemma can be found in [20]. $\square$

Lemma 2.2. Let $u(x,t) \in C^2(\Gamma)\cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1), (2) in *G*. Then the functions

$$v(t) = \int_{\Omega}u(x,t)\eta(x)\,dx \quad \text{and} \quad \int_{\Omega}F(x,t)\eta(x)\,dx \leq 0$$

are satisfies the impulsive differential inequality

$$\left[r(t)v'(t)\right]' + \lambda a(t)v(t) + Cp(t)v(t) + \sum_{j=1}^{n}C_j q_j(t)v(t-\sigma_j) \leq R(t), \quad t \neq t_k \tag{5}$$

$$a_k^* \leq \frac{v(t_k^+)}{v(t_k)} \leq a_k \tag{6}$$

$$b_k^* \leq \frac{v'(t_k^+)}{v'(t_k)} \leq b_k, \quad k = 1, 2, \cdots \tag{7}$$

where

$$R(t) = a(t) \int_{\partial\Omega} \eta(x) g(x,t) \mathrm{d}S.$$

has an eventually positive solution.

Proof. Let $u(x,t)$ be a positive solution of the problem (1), (2) in G. Without loss of generality, we may assume that there exists a $T > 0$, $t_0 > T$ such that $u(x,t) > 0$, $u(x, t - \sigma_j) > 0$, $j = 1, 2, \cdots, n$, for

$$(x,t) \in \Omega \times [t_0, \infty).$$

For $t \geq t_0, t \neq t_k, k = 1, 2, \cdots,$ multiplying Equation (1) with $\eta(x)$, which is the same as that in Lemma 2.1 and then integrating (1) with respect to x over Ω yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(x,t) \eta(x) \mathrm{d}x \right] = a(t) \int_\Omega \Delta u(x,t) \eta(x) \mathrm{d}x - \int_\Omega p(x,t) f(u(x,t)) \eta(x) \mathrm{d}x$$

$$- \sum_{j=1}^n \int_\Omega q_j(x,t) f_j(u(x, t - \sigma_j)) \eta(x) \mathrm{d}x + \int_\Omega F(x,t) \eta(x) \mathrm{d}x.$$

By Green's formula, and the boundary condition we have

$$\int_\Omega \Delta u(x,t) \eta(x) \mathrm{d}x = \int_{\partial\Omega} \left(\eta \frac{\partial u}{\partial \gamma} - u \frac{\partial \eta}{\partial \gamma} \right) \mathrm{d}S + \int_\Omega u \Delta \eta \mathrm{d}x = \int_{\partial\Omega} (\eta(g - hu) - u(-h\eta)) \mathrm{d}S + \int_\Omega u(-\lambda\eta) \mathrm{d}x$$

$$= \int_{\partial\Omega} \eta(x) g(x,t) \mathrm{d}S - \lambda \int_\Omega u(x,t) \eta(x) \mathrm{d}x,$$

where $\mathrm{d}S$ is the surface element on $\partial\Omega$.

Also from condition (H2), and Jenson's inequality we can easily obtain

$$\int_\Omega p(x,t) f(u(x,t)) \eta(x) \mathrm{d}x \geq Cp(t) \int_\Omega u(x,t) \eta(x) \mathrm{d}x$$

$$\int_\Omega q_j(x,t) f_j(u(x, t - \sigma_j)) \eta(x) \mathrm{d}x \geq C_j q_j(t) \int_\Omega u(x, t - \sigma_j) \eta(x) \mathrm{d}x$$

Thus, $v(t) > 0$. Hence we obtain the following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(x,t) \eta(x) \mathrm{d}x \right] + a(t) \lambda \int_\Omega u(x,t) \eta(x) \mathrm{d}x + Cp(t) \int_\Omega u(x,t) \eta(x) \mathrm{d}x$$

$$- \sum_{j=1}^n C_j q_j(t) \int_\Omega u(x, t - \sigma_j) \eta(x) \mathrm{d}x \leq a(t) \int_{\partial\Omega} g(x,t) \eta(x) \mathrm{d}S,$$

$$\left[r(t) v'(t) \right]' + \lambda a(t) v(t) + Cp(t) v(t) + \sum_{j=1}^n C_j q_j(t) v(t - \sigma_j) \leq R(t), t \neq t_k$$

where

$$R(t) = a(t) \int_{\partial\Omega} \eta(x) g(x,t) \mathrm{d}S.$$

For $t \geq t_0$, $t = t_k$, $k = 1, 2, \cdots,$ from (1) and condition (H4), we obtain

$$a_k^* \leq \frac{u(x, t_k^+)}{u(x, t_k)} \leq a_k$$

$$b_k^* \leq \frac{u_t(x, t_k^+)}{u_t(x, t_k)} \leq b_k.$$

According to $v(t) = \int_\Omega u(x,t) \eta(x) \mathrm{d}x,$ we obtain

$$a_k^* \le \frac{v(t_k^+)}{v(t_k)} \le a_k$$

$$b_k^* \le \frac{v'(t_k^+)}{v'(t_k)} \le b_k, \quad k = 1, 2, \cdots.$$

Hence, we obtain that $v(t) > 0$ is a positive solution of impulsive differential inequalities (5)-(7). This completes the proof. $\square$

Lemma 2.3. Let $u(x,t) \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1), (2) in G. If we further assume that $f(-u) = -f(u)$, $u \in (0, +\infty)$ and the impulsive differential inequality (5), and

$$\left[r(t) v'(t) \right]' + \lambda a(t) v(t) + Cp(t) v(t) + \sum_{j=1}^n C_j q_j(t) v(t - \sigma_j) \le -R(t), t \ne t_k \tag{8}$$

$$a_k^* \le \frac{v(t_k^+)}{v(t_k)} \le a_k \tag{9}$$

$$b_k^* \le \frac{v'(t_k^+)}{v'(t_k)} \le b_k, \quad k = 1, 2, \cdots \tag{10}$$

have no eventually positive solution, then each nonzero solution of the problem (1)-(2) is oscillatory in the domain G.

Proof. Let $u(x,t)$ be a positive solution of the problem (1), (2) in G. Without loss of generality, we may assume that there exists a $T > 0$, $t_0 > T$ such that $u(x,t) > 0, u(x, t - \sigma_j) > 0, j = 1, 2, \cdots, n$, for $(x,t) \in \Omega \times [t_0, \infty)$.

From Lemma 2.2, it follows that the function $v(t)$ is an eventually positive solution of the inequality (5) which is a contradictions.

If $u(x,t) < 0$ for $(x,t) \in \Omega \times [t_0, \infty)$, then the function

$$\tilde{u}(x,t) = -u(x,t), \quad (x,t) \in \Omega \times [t_0, \infty),$$

is a positive solution of the following impulsive hyperbolic equation

$$\frac{\partial}{\partial t} \left[r(t) \frac{\partial}{\partial t} (u(x,t)) \right] = a(t) \Delta u(x,t) - p(x,t) f(u(x,t))$$

$$-\sum_{j=1}^n q_j(x,t) f_j(u(x, t - \sigma_j)) + F(x,t), \quad t \ne t_k, \ (x,t) \in \Omega \times \mathbb{R}^+ = G$$

$$u(x, t_k^+) = \alpha_k(x, t_k, u(x, t_k)),$$

$$u_t(x, t_k^+) = \beta_k(x, t_k, u_t(x, t_k)), \quad t = t_k, \quad k = 1, 2, \cdots.$$

$$\frac{\partial u}{\partial \gamma} + h(x) u = -g(x,t), \quad (x,t) \in \partial\Omega \times \mathbb{R}^+.$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[r(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \tilde{u}(x,t) \eta(x) \mathrm{d}x \right] + \lambda a(t) \int_\Omega \tilde{u}(x,t) \eta(x) \mathrm{d}x + Cp(t) \int_\Omega \tilde{u}(x,t) \eta(x) \mathrm{d}x$$

$$-\sum_{j=1}^n C_j q_j(t) \int_\Omega \tilde{u}(x, t - \sigma_j) \eta(x) \mathrm{d}x \le -a(t) \int_{\partial\Omega} g(x,t) \eta(x) \mathrm{d}S,$$

$$\left[r(t) v'(t) \right]' + \lambda a(t) v(t) + Cp(t) v(t) + \sum_{j=1}^n C_j q_j(t) v(t - \sigma_j) \le -R(t), t \ne t_k$$

where

$$R(t) = a(t) \int_{\partial\Omega} \eta(x) g(x,t) \, dS.$$

For $t \geq t_0, t = t_k, k = 1, 2, \cdots,$ from (1) and condition (H4), we obtain

$$a_k^* \leq \frac{\tilde{u}(x,t_k^+)}{\tilde{u}(x,t_k)} \leq a_k$$

$$b_k^* \leq \frac{\tilde{u}_t(x,t_k^+)}{\tilde{u}_t(x,t_k)} \leq b_k.$$

According to $\tilde{v}(t) = \int_{\Omega} \tilde{u}(x,t)\eta(x)\,dx,$ we obtain

$$a_k^* \leq \frac{v(t_k^+)}{v(t_k)} \leq a_k$$

$$b_k^* \leq \frac{v'(t_k^+)}{v'(t_k)} \leq b_k, \quad k = 1, 2, \cdots.$$

Thus, it follows that the function $\tilde{v}(t) = \int_{\Omega} \tilde{u}(x,t)\eta(x)\,dx$ is a positive solution of the inequality (8)-(10) for $t_0 > T$ which is also a contradiction. This completes the proof. $\square$

Now, if we set $g \equiv 0$ in the proof of Lemma 2.3, then we can obtain the following lemma.

Lemma 2.4. Let $u(x,t) \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1), (2) in G. If we further assume that $f(-u) = -f(u), u \in (0, +\infty)$ and the impulsive differential inequality (5), and

$$\left[r(t)v'(t)\right]' + \lambda a(t)v(t) + Cp(t)v(t)$$
$$+ \sum_{j=1}^{n} C_j q_j(t) v(t - \sigma_j) \leq 0, \ t \neq t_k \tag{11}$$

$$a_k^* \leq \frac{v(t_k^+)}{v(t_k)} \leq a_k \tag{12}$$

$$b_k^* \leq \frac{v'(t_k^+)}{v'(t_k)} \leq b_k, \quad k = 1, 2, \cdots \tag{13}$$

has no eventually positive solution, then each nonzero solution of the problem (1), satisfying the boundary condition

$$\frac{\partial u}{\partial \gamma} + h(x)u = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \ t \neq t_k$$

is oscillatory in the domain G.

Proof. Let $u(x,t)$ be a positive solution of the problem (1), (2) in G. Without loss of generality, we may assume that there exists a $T > 0, t_0 > T$ such that $u(x,t) > 0, u(x, t - \sigma_j) > 0, j = 1, 2, \cdots, n,$ for

$$(x,t) \in \Omega \times [t_0, \infty).$$

From Lemma 2.2, it follows that the function $v(t)$ is an eventually positive solution of the inequality (5) which is a contradiction.

If $u(x,t) < 0$ for $(x,t) \in \Omega \times [t_0, \infty),$ then the function $\tilde{u}(x,t) = -u(x,t), (x,t) \in \Omega \times [t_0, \infty),$ is a positive solution of the following impulsive hyperbolic equation

$$\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}\left(u(x,t)\right)\right] = a(t)\Delta u(x,t) - p(x,t) f\left(u(x,t)\right)$$

$$-\sum_{j=1}^{n}q_j(x,t) f_j\left(u(x,t-\sigma_j)\right) + f(x,t), \quad t \neq t_k, \ (x,t) \in \Omega \times \mathbb{R}^+ = G$$

$$u\left(x,t_k^+\right) = \alpha_k\left(x,t_k,u(x,t_k)\right),$$

$$u_t\left(x,t_k^+\right) = \beta_k\left(x,t_k,u_t(x,t_k)\right), \quad t = t_k, \ k = 1,2,\cdots.$$

$$\frac{\partial u}{\partial \gamma} + h(x)u = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+,$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\tilde{u}(x,t)\eta(x)\mathrm{d}x\right] + \lambda a(t)\int_{\Omega}\tilde{u}(x,t)\eta(x)\mathrm{d}x + Cp(t)\int_{\Omega}\tilde{u}(x,t)\eta(x)\mathrm{d}x$$

$$-\sum_{j=1}^{n}C_j q_j(t)\int_{\Omega}\tilde{u}(x,t-\sigma_j)\eta(x)\mathrm{d}x \leq 0,$$

$$\left[r(t)v'(t)\right]' + \lambda a(t)v(t) + Cp(t)v(t) + \sum_{j=1}^{n}C_j q_j(t)v(t-\sigma_j) \leq 0, \ t \neq t_k.$$

For $t \geq t_0, t = t_k, k = 1,2,\cdots,$ from (1) and condition (H4), we obtain

$$a_k^* \leq \frac{\tilde{u}\left(x,t_k^+\right)}{\tilde{u}(x,t_k)} \leq a_k$$

$$b_k^* \leq \frac{\tilde{u}_t\left(x,t_k^+\right)}{\tilde{u}_t(x,t_k)} \leq b_k, \quad k = 1,2,\cdots.$$

According to $\tilde{v}(t) = \int_{\Omega}\tilde{u}(x,t)\eta(x)\mathrm{d}x,$ we obtain

$$a_k^* \leq \frac{v\left(t_k^*\right)}{v(t_k)} \leq a_k$$

$$b_k^* \leq \frac{v'\left(t_k^*\right)}{v'(t_k)} \leq b_k.$$

Thus it follows that the function $\tilde{v}(t) = \int_{\Omega}\tilde{u}(x,t)\eta(x)\mathrm{d}x,$ is a positive solution of the inequality (11)-(13) for $t_0 > T$ which is also a contradiction. This completes the proof. $\square$

Lemma 2.5. Assume that

(A1) the sequence $\{t_k\}$ satisfies $0 < t_0 < t_1 < \cdots, \ \lim_{k\to\infty}t_k = \infty$;

(A2) $m(t) \in PC^1\left[\mathbb{R}^+,\mathbb{R}\right]$ is left continuous at t_k for $k = 1,2,\cdots$;

(A3) for $k = 1,2,\cdots$ and $t \geq t_0,$

$$m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k,$$

$$m\left(t_k^+\right) \leq d_k m(t_k) + e_k,$$

where $p(t), q(t) \in C\left(\mathbb{R}^+,\mathbb{R}\right)$, $d_k \geq 0$ and e_k are constants. PC denote the class of piecewise continuous function from $\mathbb{R}^+$ to $\mathbb{R}$, with discontinuities of the first kind only at $t = t_k, k = 1,2,\cdots$.

Then

$$m(t) \le m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t p(s)\,ds\right) + \int_{t_0}^t \prod_{s < t_k < t} d_k \exp\left(\int_s^t p(r)\,dr\right) q(s)\,ds$$

$$+ \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^t p(s)\,ds\right) e_k.$$

Proof. The proof of the lemma can be found in [21]. □

Lemma 2.6. Let $v(t)$ be an eventually positive (negative) solution of the differential inequality (11)-(13). Assume that there exists $T \ge t_0$ such that $v(t) > 0$ $(v(t) < 0)$ for $t \ge T$. If

$$\lim_{t \to +\infty} \int_{t_0}^t \prod_{t_0 < t_k < s} \frac{b_k^*}{a_k}\,ds = +\infty \tag{14}$$

hold, then $v'(t) \ge 0$ $(v'(t) \le 0)$ for $t \in [T, t_1] \cup \left(\bigcup_{k=l}^{+\infty}(t_k, t_{k+1}]\right)$, where $l = \min\{k : t_k \ge T\}$.

Proof. The proof of the lemma can be found in [22]. □

We begin with the following theorem.

Theorem 2.1. If condition (14), and the following condition

$$\lim_{t \to +\infty} \int_{t_0}^t \prod_{t_0 < t_k < s} \frac{a_k^*}{b_k} r(t_k) F(s)\,ds = +\infty, \tag{15}$$

hold, where

$$F(t) = \frac{\lambda a(t) + C p(t) + \exp(-\delta w(t_1)) \sum_{j=1}^n C_j q_j(t)}{r(t)},$$

then every solution of the problem (1), (2) oscillates in G.

Proof. Let $u(x,t)$ be a nonoscillatory solution of (1), (2). Without loss of generality, we can assume that there exists $T > 0, t_0 \ge T$, such that $u(x,t) > 0$, $u(x,t - \sigma_j) > 0$, $j = 1, 2, \cdots, n$ for $(x,t) \in \Omega \times [t_0, \infty)$.

From Lemma 2.4, we know that $v(t)$ is a positive solution of (11)-(13). Thus from Lemma 2.6, we can find that $v'(t) \ge 0$ for $t \ge t_0$.

For $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \cdots$, define

$$w(t) = r(t) \frac{v'(t)}{v(t)}, \quad t \ge t_0.$$

Then we have $w(t) > 0$, $t \ge t_0$, $r(t)v'(t) - w(t)v(t) = 0$. We may assume that $v(t_0) = 1$, thus we have that for $t \ge t_0$

$$v(t) = \exp\left(\int_{t_0}^t w(s)\,ds\right), \tag{16}$$

$$v'(t) = w(t) \exp\left(\int_{t_0}^t w(s)\,ds\right), \tag{17}$$

$$v''(t) = w^2(t) \exp\left(\int_{t_0}^t w(s)\,ds\right) + w'(t) \exp\left(\int_{t_0}^t w(s)\,ds\right). \tag{18}$$

Substitute (16)-(18) into (11) and then we obtain,

$$r'(t)w(t)\exp\left(\int_{t_0}^t w(s)\,ds\right) + r(t)\left[w^2(t)\exp\left(\int_{t_0}^t w(s)\,ds\right) + w'(t)\exp\left(\int_{t_0}^t w(s)\,ds\right)\right]$$

$$+ \lambda a(t)\exp\left(\int_{t_0}^t w(s)\,ds\right) + C p(t)\exp\left(\int_{t_0}^t w(s)\,ds\right) + \sum_{j=1}^n C_j q_j(t)\exp\left(\int_{t_0}^{t-\sigma_j} w(s)\,ds\right) \le 0.$$

Hence we have

$$r(t)w^2(t) + r(t)w'(t) + \lambda a(t) + Cp(t) + \sum_{j=1}^n C_j q_j(t) \exp\left(-\int_{t-\sigma_j}^t w(s)\,\mathrm{d}s\right) \le 0, \quad t \ne t_k,$$

or

$$r(t)w'(t) + \lambda a(t) + Cp(t) + \sum_{j=1}^n C_j q_j(t) \exp\left(-\int_{t-\sigma_j}^t w(s)\,\mathrm{d}s\right) \le 0, \quad t \ne t_k.$$

From above inequality and condition $b_k \le a_k^*$, it is easy to see that the function $w(t)$ is nonincreasing for $t \ge t_1 \ge \delta + t_0$. Thus $w(t) \le w(t_1)$ for $t \ge t_1$ which implies that

$$r(t)w'(t) + \lambda a(t) + Cp(t) + \exp(-\delta w(t_1)) \sum_{j=1}^n C_j q_j(t) \le 0, \quad t \ne t_k.$$

From (12)-(13), we obtain

$$w(t_k^+) = r(t_k^+) \frac{v'(t_k^+)}{v(t_k^+)} \le r(t_k^+) \frac{b_k v'(t_k)}{a_k^* v(t_k)} = r(t_k) \frac{b_k}{a_k^*} w(t_k),$$

and

$$r(t)w'(t) \le -\lambda a(t) - Cp(t) - \exp(-\delta w(t_1)) \sum_{j=1}^n C_j q_j(t), \quad t \ne t_k.$$

$$w(t_k^+) \le r(t_k) \frac{b_k}{a_k^*} w(t_k), \quad k = 1, 2, \cdots$$

Let

$$-F(t) = \frac{-\lambda a(t) - Cp(t) - \exp(-\delta w(t_1)) \sum_{j=1}^n C_j q_j(t)}{r(t)}.$$

Then according to Lemma 2.5, we have

$$w(t) \le w(t_0) \prod_{t_0 < t_k < t} r(t_k) \frac{b_k}{a_k^*} + \int_{t_0}^t \prod_{s < t_k < t} r(t_k) \frac{b_k}{a_k^*} F(s)\,\mathrm{d}s$$

$$= \prod_{t_0 < t_k < t} \frac{b_k}{a_k^*} \left[w(t_0) r(t_k) - \int_{t_0}^t \prod_{t_0 < t_k < s} r(t_k) \frac{a_k^*}{b_k} F(s)\,\mathrm{d}s \right] < 0.$$

Since $w(t) \ge 0$, the last inequality contradicts condition (15). This completes the proof. $\square$

3. Oscillation Properties of the Problem (1) and (3)

Next we consider the problem (1) and (3). To prove our main result we need the following lemmas.

Lemma 3.1. Suppose that λ_0 is the smallest positive eigen value of the problem

$$\Delta\Psi(x) + \lambda_0 \Psi(x) = 0, \quad x \in \Omega,$$
$$\Psi(x) = 0, \qquad\qquad x \in \partial\Omega,$$

and $\Psi(x)$ is the corresponding eigen function of λ_0. Then $\lambda_0 > 0$ and $\Psi(x) > 0, x \in \Omega$.

Proof. The proof of the lemma can be found in [20]. $\square$

Lemma 3.2. Let $u(x,t) \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1), (3) in G. Then the function

$$\int_\Omega F(x,t)\Psi(x)\,\mathrm{d}x \le 0$$

are satisfies the impulsive differential inequality

$$\left[r(t)v'(t)\right]' + \lambda_0 a(t)v(t) + Cp(t)v(t)$$
$$+ \sum_{j=1}^{n} C_j q_j(t)v(t - \sigma_j) \leq Q(t), \quad t \neq t_k \tag{19}$$

$$a_k^* \leq \frac{v(t_k^+)}{v(t_k)} \leq a_k \tag{20}$$

$$b_k^* \leq \frac{v'(t_k^+)}{v'(t_k)} \leq b_k, \quad k = 1, 2, \cdots \tag{21}$$

where

$$Q(t) = -a(t)\int_{\partial\Omega} \varphi(x,t)\frac{\partial\Psi}{\partial N}\,dS.$$

has the eventually positive solution

$$v(t) = \int_{\Omega} u(x,t)\Psi(x)\,dx.$$

Proof. Let $u(x,t)$ be a positive solution of the problem (1), (3) in G. Without loss of generality, we may assume that there exists a $T > 0, t_0 > T$ such that $u(x,t) > 0, u(x, t - \sigma_j) > 0, j = 1, 2, \cdots, n$, for

$$(x,t) \in \Omega \times [t_0, \infty).$$

For $t \geq t_0, t \neq t_k, k = 1, 2, \cdots$, multiplying equation (1) with $\Psi(x)$, which is the same as that in Lemma 3.1 and then integrating (1) with respect to x over Ω yields

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\int_{\Omega} u(x,t)\Psi(x)\,dx\right] = a(t)\int_{\Omega}\Delta u(x,t)\Psi(x)\,dx - \int_{\Omega} p(x,t)f(u(x,t))\Psi(x)\,dx$$
$$- \sum_{j=1}^{n}\int_{\Omega} q_j(x,t)f_j(u(x,t-\sigma_j))\Psi(x)\,dx + \int_{\Omega} F(x,t)\Psi(x)\,dx.$$

By Green's formula, and the boundary condition we have

$$\int_{\Omega}\Delta u(x,t)\Psi(x)\,dx = \int_{\partial\Omega}\left(\Psi\frac{\partial u}{\partial\gamma} - u\frac{\partial\Psi}{\partial\gamma}\right)dS + \int_{\Omega} u\Delta\Psi\,dx$$
$$= \int_{\partial\Omega} -\varphi(x,t)\frac{\partial\Psi}{\partial\gamma}\,dS + \int_{\Omega} u(-\lambda_0\Psi)\,dx$$
$$= -\int_{\partial\Omega}\varphi(x,t)\frac{\partial\Psi}{\partial\gamma}\,dS - \int_{\Omega} u(x,t)(\lambda_0\Psi(x))\,dx,$$

where dS is the surface element on $\partial\Omega$.

From condition (H2), we can easily obtain

$$\int_{\Omega} p(x,t)f(u(x,t))\Psi(x)\,dx \geq Cp(t)\int_{\Omega} u(x,t)\Psi(x)\,dx$$

$$\int_{\Omega} q_j(x,t)f_j(u(x,t-\sigma_j))\Psi(x)\,dx \geq C_j q_j(t)\int_{\Omega} u(x,t-\sigma_j)\Psi(x)\,dx.$$

The proof is similar to that of Lemma 2.1 and therefore the details are omitted. $\square$

Lemma 3.3. Let $u(x,t) \in C^2(\Gamma)\cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1), (3) in G. If we further assume that $f(-u) = -f(u), \quad u \in (0, +\infty)$ and the impulsive differential inequality (19), and

$$\left[r(t)v'(t)\right]' + \lambda_0 a(t)v(t) + Cp(t)v(t) + \sum_{j=1}^{n} C_j q_j(t)v(t-\sigma_j) \leq -Q(t), \quad t \neq t_k \tag{22}$$

$$a_k^* \le \frac{v(t_k^+)}{v(t_k)} \le a_k \tag{23}$$

$$b_k^* \le \frac{v'(t_k^+)}{v'(t_k)} \le b_k, \quad k = 1, 2, \cdots \tag{24}$$

have no eventually positive solution, then each nonzero solution of the problem (1), (3) is oscillatory in the domain *G*.

Proof. The proof is similar to Lemma 2.3, and hence the details are omitted. □

Futhermore, if we set $\varphi \equiv 0$, then we have the following lemma.

Lemma 3.4. Let $u(x,t) \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1), (3) in *G*. If we further assume that $f(-u) = -f(u)$, $u \in (0, +\infty)$ and the impulsive differential inequality (19), and

$$\left[r(t)v'(t) \right]' + \lambda_0 a(t)v(t) + Cp(t)v(t) + \sum_{j=1}^{n} C_j q_j(t)v(t - \sigma_j) \le 0, \ t \ne t_k \tag{25}$$

$$a_k^* \le \frac{v(t_k^+)}{v(t_k)} \le a_k \tag{26}$$

$$b_k^* \le \frac{v'(t_k^+)}{v'(t_k)} \le b_k, \quad k = 1, 2, \cdots \tag{27}$$

has no eventually positive solution, then each nonzero solution of the problem (1), satisfying the boundary condition

$$u = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}^+, \ t \ne t_k$$

is oscillatory in the domain *G*.

Proof. The proof is similar to Lemma 2.4, and hence the details are omitted. □

Using the above lemmas, we prove the following oscillation result.

Theorem 3.1. If condition (14) and the following condition

$$\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{a_k^*}{b_k} r(t_k) F(s) \, ds = +\infty, \tag{28}$$

hold, where

$$F(t) = \frac{\lambda_0 a(t) + Cp(t)}{r(t)},$$

then every solution of the problem (1), (3) oscillates in G.

Proof. Let $u(x,t)$ be a nonoscillatory solution of (1), (3). Without loss of generality, we can assume that there exists $T > 0$, $t_0 \ge T$, such that $u(x,t) > 0$, $u(x, t - \sigma_j) > 0$, $j = 1, 2, \cdots, n$ for $(x,t) \in \Omega \times [t_0, \infty)$.

From Lemma 3.4, we know that $v(t)$ is a positive solution of (25)-(27). Thus from Lemma 2.6, we can find that $v'(t) \ge 0$ for $t \ge t_0$.

For $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \cdots$, define

$$w(t) = r(t)\frac{v'(t)}{v(t)}, \quad t \ge t_0.$$

Then we have $w(t) > 0$, $t \ge t_0$, $r(t)v'(t) - w(t)v(t) = 0$. We may assume that $v(t_0) = 1$, thus we have that for $t \ge t_0$

$$v(t) = \exp\left(\int_{t_0}^{t} w(s) \, ds \right), \tag{29}$$

$$v'(t) = w(t)\exp\left(\int_{t_0}^t w(s)\,ds\right), \tag{30}$$

$$v''(t) = w^2(t)\exp\left(\int_{t_0}^t w(s)\,ds\right)$$
$$+ w'(t)\exp\left(\int_{t_0}^t w(s)\,ds\right). \tag{31}$$

We substitute (29)-(31) into (25) and can obtain the following inequality,

$$r(t)\left[w^2(t)\exp\left(\int_{t_0}^t w(s)\,ds\right) + w'(t)\exp\left(\int_{t_0}^t w(s)\,ds\right)\right]$$
$$+ \lambda_0 a(t)\exp\left(\int_{t_0}^t w(s)\,ds\right) + Cp(t)\exp\left(\int_{t_0}^t w(s)\,ds\right) \le 0,$$

then we have

$$r(t)w'(t) + \lambda_0 a(t) + Cp(t) \le 0, \ t \ne t_k.$$

From (26)-(27), we can obtain

$$w(t_k^+) = r(t_k^+)\frac{v'(t_k^+)}{v(t_k^+)} \le r(t_k^+)\frac{b_k v'(t_k)}{a_k^* v(t_k)}$$

$$= r(t_k)\frac{b_k}{a_k^*}w(t_k), \ k = 1, 2, \cdots.$$

It follows that

$$r(t)w'(t) \le -\lambda_0 a(t) - Cp(t), \quad t \ne t_k.$$

$$w(t_k^*) \le r(t_k)\frac{b_k}{a_k^*}w(t_k), \ t = t_k.$$

Let

$$-F(t) = \frac{-\lambda_0 a(t) - Cp(t)}{r(t)}.$$

Then according to Lemma 2.5, we have

$$w(t) \le w(t_0)\prod_{t_0 < t_k < t} r(t_k)\frac{b_k}{a_k^*} + \int_{t_0}^t \prod_{s < t_k < t} r(t_k)\frac{b_k}{a_k^*}F(s)\,ds$$

$$= \prod_{t_0 < t_k < t}\frac{b_k}{a_k^*}\left[w(t_0)r(t_k) - \int_{t_0}^t \prod_{t_0 < t_k < s} r(t_k)\frac{a_k^*}{b_k}F(s)\,ds\right] < 0.$$

Since $w(t) \ge 0$, the last inequality contradicts (28). This completes the proof. $\square$

Theorem 3.2. If condition (14) and the following condition

$$\lim_{t \to +\infty}\int_{t_0}^t \prod_{t_0 < t_k < s}\frac{a_k^*}{b_k}r(t_k)C_{j_0}q_{j_0}\,ds = +\infty, \tag{32}$$

hold for some q_{j_0}, then every solution of the problem (1), (3) oscillates in G.

Proof. The proof is obvious and hence the details are omitted. $\square$

4. Examples

In this section, we present some examples to illustrate the main results.

Example 4.1. Consider the impulsive differential equation

$$\frac{\partial}{\partial t}\left((t+\pi)^2\frac{\partial}{\partial t}(u(x,t))\right)=(t+\pi)^2\sin^2(t)\Delta u(x,t)-(t+\pi)^2\cos^2(t)u(x,t)$$

$$-2(t+\pi)\sin^2(t)u\left(x,t-\frac{\pi}{2}\right)-(t+\pi)\sin(2t)\cos(t)\sin(x),\ t>1,t\neq t_k,k=1,2,3,\cdots$$

$$u(x,t_k^+)=\frac{k+1}{k}u(x,t_k)$$

$$u_t(x,t_k^+)=u_t(x,t_k),k=1,2,\cdots$$

(33)

and the boundary condition

$$u(0,t)=u(\pi,t)=0,\ t\geq 0,\ t\neq t_k,\ k=1,2,\cdots.\tag{34}$$

Here $\Omega=(0,\pi)$, $a_k=a_k^*=\dfrac{k+1}{k}$, $b_k=b_k^*=1$, $k=1,2,\cdots$, $r(t)=(t+\pi)^2$, $a(t)=(t+\pi)^2\sin^2(t)$,

$p(t)=(t+\pi)^2\cos^2(t)$, $q_1(t)=2(t+\pi)\sin^2(t)$, $\sigma_1=\dfrac{\pi}{2}$, $f(u)=u$, $f_1(u)=u$, and taking $\{t_k\}_1^{+\infty}=\{2^k\}_1^{+\infty}$.

Moreover

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{b_k^*}{a_k}\mathrm{d}s=\int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}\mathrm{d}s$$

$$=\int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}\mathrm{d}s+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}\mathrm{d}s+\int_{t_2^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}\mathrm{d}s+\int_{t_3^+}^{t_4}\prod_{1<t_k<s}\frac{k}{k+1}\mathrm{d}s+\cdots$$

$$=1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times 2^3+\cdots=\sum_{n=0}^{+\infty}\frac{2^n}{n+1}=+\infty$$

so (14) holds. We take $\lambda_0=1$, $C=C_1=1$, $\delta=\max\{\sigma_1\}=\dfrac{\pi}{2}$, $w(t_1)=\dfrac{1}{t+\pi}$, then

$$F(t)=\frac{(t+\pi)^2+(t+\pi)^2\cos^2(t)}{(t+\pi)^2}=1+\cos^2(t),$$

thus

$$\lim_{t\to+\infty}\int_1^t\prod_{1<t_k<s}\frac{a_k^*}{b_k}r(t_k)F(s)\mathrm{d}s=\lim_{t\to+\infty}\int_1^t\prod_{1<t_k<s}\frac{k+1}{k}(2^k+\pi)^2(1+\cos^2(s))\mathrm{d}s$$

$$\geq\lim_{t\to+\infty}\int_1^t(1+\cos^2 s)\mathrm{d}s=+\infty.$$

Hence (28) holds. Therefore all conditions of Theorem 3.1 are satisfied. Hence every solution of the problem (33), (34) oscillates in $(0,\pi)\times[0,\infty)$. In fact $u(x,t)=\sin x\cos t$ is one such solution of the problem (33) and (34).

Example 4.2. Consider the impulsive differential equation

$$\frac{\partial}{\partial t}\left((t+\pi)^2\frac{\partial}{\partial t}(u(x,t))\right)=(t+\pi)^2\sin^2(t)\Delta u(x,t)-(t+\pi)^2\cos^2(t)u(x,t)$$

$$-2(t+\pi)\sin^2(t)u\left(x,t-\frac{5\pi}{2}\right)+2(t+\pi)\cos^3(t)\cos(x),\ t>1,t\neq t_k,k=1,2,3,\cdots$$

$$u(x,t_k^+)=\frac{k+1}{k}u(x,t_k)$$

$$u_t(x,t_k^+)=u_t(x,t_k),\ k=1,2,\cdots$$

(35)

and the boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, t \geq 0, t \neq t_k, k = 1, 2, \cdots. \tag{36}$$

Here $\Omega = (0,\pi)$, $a_k = a_k^* = \dfrac{k+1}{k}$, $b_k = b_k^* = 1$, $k = 1, 2, \cdots$. $r(t) = (t+\pi)^2$, $a(t) = (t+\pi)^2 \sin^2(t)$,

$p(t) = (t+\pi)^2 \cos^2(t)$, $q_1(t) = 2(t+\pi)\sin^2(t)$, $\sigma_1 = \dfrac{5\pi}{2}$, $h(u) = 0$, $f(u) = u, f_1(u) = u$, and taking

$\{t_k\}_1^{+\infty} = \{2^k\}_1^{+\infty}$. It is easy to check that the conditions of Theorem 2.1 are satisfied. Therefore, every solution of the problem (35), (36) oscillates in $(0,\pi) \times [0,\infty)$. In fact $u(x,t) = \sin t \cos x$ is one such solution of the problem (35) and (36).

Acknowledgements

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