

Hamiltonian Representation of Higher Order Partial Differential Equations with Boundary Energy Flows

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Abstract

This paper presents a system representation that can be applied to the description of the interaction between systems connected through common boundaries. The systems consist of partial differential equations that are first order with respect to time, but spatially higher order. The representation is derived from the instantaneous multisymplectic Hamiltonian formalism; therefore, it possesses the physical consistency with respect to energy. In the interconnection, particular pairs of control inputs and observing outputs, called port variables, defined on the boundaries are used. The port variables are systematically introduced from the representation.

Keywords

Symplectic Structure, Dirac Structure, Hamiltonian Systems, Passivity, Partial Differential Equations, Nonlinear Systems

1. Introduction

Energy is one of the most important concepts for describing physical systems. In analytical mechanics, an energy in the systems can be interpreted as a Hamiltonian. Hamiltonian systems can be characterized by symplectic structures [1] derived from the skew symmetry that cotangent bundles possess. Hamiltonian systems and their symplectic structures have been widely applied not only in physics, but in engineering, particularly, control theory [2] [3]. Specifically, in an electrical circuit, an energy is defined as the time integral of the product of currents and voltages. Energy flows between each circuit also balance if there is no dissipative element. Furthermore, the sum of currents balances between inflows and outflows at any node, and the directed sum of vol-

tages around any closed loop is zero according to Kirchhoff law. Indeed, such properties have been generalized to various physical systems in terms of *port-Hamiltonian systems* [2] originated in *bondgraph theory* [4]. The above particular pairs with the physical dimension of power are called *port variables*, and the energy balances can be regarded as *passivity* [2]. A system is passive if and only if a finite amount of energy can be extracted from the system. In other words, energy changes in interactions can be observed by the port variables, and a supplied energy is less than a stored energy if a system is passive. Passivity-based controls via port-Hamiltonian system representations have been frequently used in control designs [3].

This paper proposes the port-Hamiltonian representation of systems of higher order partial differential equations defined on a domain with a boundary. The representation can be formally formulated from the viewpoint of the multisymplectic formalism [5] [6] under the assumption of first order with respect to time, but possibly higher order with respect to spatial variables. The port representation for Hamiltonian systems with boundary energy flows was initiated by *the distributed port-Hamiltonian system* in [7]. The systems satisfy a power balance defined on the boundary; therefore, it can describe the interaction between the systems connected through common boundaries. Thus, passivity-based controls in this formulation can be enhanced as boundary energy controls. Various aspects of the distributed port-Hamiltonian systems have been studied, e.g., the implicit representation of distributed port-Hamiltonian systems [8], and the relationship between field equations and distributed port-Hamiltonian systems [9]-[11]. The higher order representation of the distributed port-Hamiltonian systems has been proposed in, e.g., [12] [13]; however, they are not related with the multisymplectic formalism.

Thus, we first relate higher order partial differential equations with the implicit Hamiltonian systems [14]. Next, we describe the implicit representation as a Dirac structure defined over the multisymplectic manifold in analogy with the first order formalizations [15]-[17]. Dirac structures [18] [19] are a unified concept of symplectic and Poisson structures. Then, we derive *the Stokes variational differential* from the fact that higher order derivatives yield variations of boundary port variables through integration by parts and Stokes theorem. Finally, we show that the boundary energy balance and the Stokes-Dirac structure [7] [20] that is an extended Dirac structure for distributed port-Hamiltonian systems can be defined in the proposed higher order field port Hamiltonian systems with boundary energy flows.

This paper is organized as follows: In Section 2, we make a brief summary of port-Hamiltonian systems and explain the motivation of this study. Section 3 introduces mathematical preliminaries from some references. Section 4 presents the following three concepts under the assumption of time-spatial splitting: 1) an implicit Hamiltonian representation using the dual structure derived from the multisymplectic instantaneous formalism, 2) Stokes variational differential derived from the integration by parts formula, and 3) the implicit higher order field port Hamiltonian representation with boundary. Section 5 introduces the formal port representation for higher order partial differential equations from the implicit Hamiltonian representation. We call it *higher order field port Hamiltonian systems with boundary energy flows*. Finally, Section 6 illustrates two modeling examples.

2. Summary of Port-Hamiltonian Representations

This section explains the concept of port-Hamiltonian systems by means of a simple example of coupled multi-physical models, and the motivation of this work.

2.1. Port Representation for Lumped Parameter Energy Conserving Physical Systems

Let us consider the following model of the direct current motor consisting of an electrical circuit and an armature:

$$L \frac{di}{dt} + Ri = u - K\omega, \quad J \frac{d\omega}{dt} + B\omega = Ni, \quad (1)$$

where $i = i(t) = dq/dt$ is the current that is the time derivative of the electric charge $q = q(t)$, $\omega = \omega(t) = d\theta/dt$ is the velocity of the angle $\theta = \theta(t)$, and u is the input voltage. In (1), the following constants are defined: the inductance L , the resistance R , the back electromotive force constant K , the inertia moment J , the viscous friction constant B , and the torque constant N . When the dissipative elements and the input are null, *i.e.*, $R = B = 0$ and $u = 0$, the system (1) is energy conserving, and it can be formulated as the following standard Hamiltonian system:

$$\dot{p}'_q = -\frac{\partial \mathcal{H}}{\partial q}, \quad \dot{q} = \frac{\partial \mathcal{H}}{\partial p'_q}, \quad \dot{p}'_\theta = -\frac{\partial \mathcal{H}}{\partial \theta}, \quad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p'_\theta}, \quad (2)$$

where we have defined the Hamiltonian and the momenta as follows:

$$\mathcal{H} = \dot{q}p'_q + \dot{\theta}p'_\theta - \mathcal{L}, \quad \mathcal{L} = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}J\dot{\theta}^2 - \kappa_1\dot{\theta}q + \kappa_2\theta\dot{q}, \quad (3)$$

$$p'_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q} + \kappa_2\theta, \quad p'_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = J\dot{\theta} - \kappa_1q \quad (4)$$

with $K := \kappa_2 - \kappa_1$ and $N := \kappa_1 + \kappa_2$ for certain κ_1 and κ_2 .

We shall augment the Hamiltonian system (2) as the following port-Hamiltonian system with the dissipations $-R\dot{q}$ and $-B\dot{\theta}$ and the input u :

$$\begin{cases} \begin{bmatrix} f_1^p \\ f_1^q \end{bmatrix} = \begin{bmatrix} -R & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_1^p \\ e_1^q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, & y^q = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e_1^p \\ e_1^q \end{bmatrix}, \\ \begin{bmatrix} f_2^p \\ f_2^q \end{bmatrix} = \begin{bmatrix} -B & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_2^p \\ e_2^q \end{bmatrix}, & y^\theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e_2^p \\ e_2^q \end{bmatrix}, \end{cases} \quad (5)$$

where we have defined the following variables called *port variables*:

$$\begin{cases} f_1^p = \frac{dp'_q}{dt}, & f_1^q = \frac{dq}{dt}, & e_1^p = \frac{\partial \mathcal{H}}{\partial p'_q} = \dot{q}, & e_1^q = \frac{\partial \mathcal{H}}{\partial q} = \kappa_1\dot{\theta}, \\ f_2^p = \frac{dp'_\theta}{dt}, & f_2^q = \frac{d\theta}{dt}, & e_2^p = \frac{\partial \mathcal{H}}{\partial p'_\theta} = \dot{\theta}, & e_2^q = \frac{\partial \mathcal{H}}{\partial \theta} = \kappa_2\dot{q}, \end{cases} \quad (6)$$

and, in particular, f_j^i and e_j^i for $i = \{p, q\}$ and $j = \{1, 2\}$ are called *flows* and *efforts*, respectively. Here, we can see that the electrical and mechanical subsystems are coupled by the interconnection of the effort variables: \dot{q} and $\dot{\theta}$.

Then, the summation consisting of the products of the pairs of the port variables is equivalent to the time derivative of the Hamiltonian, *i.e.*, the total energy change of the system. Indeed, we can directly calculate the following power balance:

$$\frac{d\mathcal{H}}{dt} = f_1^p e_1^p + f_1^q e_1^q + f_2^p e_2^p + f_2^q e_2^q = -R\dot{q}^2 - B\dot{\theta}^2 + uy^q, \quad (7)$$

where we have used the relation $y^q = \dot{q} = e_1^p$. The relation (7) means that the dissipations $-R\dot{q}^2$ and $-B\dot{\theta}^2$ stabilize the system by decreasing the energy, and the product uy^q of the input and the output may affect the dissipation rate. Furthermore, the Hamiltonian can be controlled if we can find a suitable input satisfying $\tilde{\mathcal{H}}(t_1) - \tilde{\mathcal{H}}(t_0) = \int_{t_0}^{t_1} uy^q dt$ for the desired Hamiltonian $(\mathcal{H} + \tilde{\mathcal{H}})$ in the time interval $[t_0, t_1]$. A finite amount of energy changes in interactions can be precisely observed by the product of an input-output pair if the system is *passive*. Hence, these controls are called *passivity-based controls*.

2.2. Distributed Port-Hamiltonian Systems

In the case of distributed parameter systems, the representation is called a *distributed port-Hamiltonian system*, and has the following formal structure:

$$\begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix}, \quad \begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} e^p|_{\partial\mathcal{S}} \\ (-1)^p e^q|_{\partial\mathcal{S}} \end{bmatrix} \quad (8)$$

called the *Stokes-Dirac structure* defined on the system domain \mathcal{S} with the boundary $\partial\mathcal{S}$ (see [7] for details), where d is the exterior differential operator, we have defined

$$\begin{cases} f^p = -\frac{\partial \alpha_p}{\partial t} \in \Omega^p(\mathcal{S}), & f^q = -\frac{\partial \alpha_q}{\partial t} \in \Omega^q(\mathcal{S}), & f^b \in \Omega^{n-p}(\partial\mathcal{S}), \\ e^p = \frac{\delta \mathcal{H}}{\delta \alpha_p} \in \Omega^{n-p}(\mathcal{S}), & e^q = \frac{\delta \mathcal{H}}{\delta \alpha_q} \in \Omega^{n-q}(\mathcal{S}), & e^b \in \Omega^{n-q}(\partial\mathcal{S}) \end{cases} \quad (9)$$

for a Hamiltonian functional $\mathcal{H} = \int_{\mathcal{S}} \mathcal{H}$, Ω^k is the space of differential k -forms, $\delta/\delta\alpha_i$ is the variational derivative with respect to the differential form α_i , $r = pq+1$, and $p+q = n+1$. Then, the power balance (7) is extended to the following relation described by differential forms:

$$\frac{d\mathcal{H}}{dt} = \int_{\mathcal{S}} (f_p \wedge e_p + f_q \wedge e_q) = \int_{\partial\mathcal{S}} e_p \wedge e_q. \quad (10)$$

The boundary integral term in (10) is generated from the domain integral term by Stokes theorem: $\int_{\mathcal{S}} d\omega = \int_{\partial\mathcal{S}} \omega$ for an n -form $d\omega$ in the n -dimensional domain \mathcal{S} . Hence, the passivity-based controls can be enhanced as boundary energy controls by regarding e_p and e_q as a input-output pair, *i.e.*, boundary port variables.

2.3. Motivation

As we have seen above, the port-Hamiltonian representations are important for the control of, e.g., nonlinear systems, distributed parameter systems, higher order systems and multi-physical systems from which it is difficult to obtain analytical solutions in a closed form. This paper derives a formal port-Hamiltonian representation of a given partial differential equation including higher order derivatives in terms of the multisymplectic formalism.

In this paper, we assume that a given system of partial differential equations is determined by variational problems. Such a system must be regarded as an energy conserving physical system [15] through Legendre transformations that map Lagrangian systems to Hamiltonian systems. This assumption comes from the fact that any system can be decomposed into a variational subsystem that can be determined by variational calculus and a non-variational subsystem that cannot be introduced from any Lagrangian on a contractible manifold [21]. For example, as we seen in (1), the dissipative terms $-R\dot{q}$ and $-B\dot{\theta}$ cannot be derived from any Lagrangian or Hamiltonian. On the other hand, Lagrangians of the first subsystem can be explicitly calculated by homotopy operators in terms of the exactness of vertical differential forms in variational bi-complex [22] [23]. Hence, we consider only the variational subsystems in this study.

3. Mathematical Preliminary

Mathematical notations used in this paper basically conform to those of the references [22] [24] [25].

3.1. Multi index for Higher Order Derivatives

Let X be an $(m+1)$ -dimensional manifold. Let Q be a fiber manifold on X , and consider the r -th order jet bundle $J^r Q$ over Q . We denote the local coordinates of X , Q , and $J^r Q$, respectively, by x^i , (x^i, q^a) , and (x^i, q_I^a) , where $0 \leq i \leq m$, and $1 \leq a \leq l$. The multi index I describes all variables of the repeated combination of x^i that mean higher order derivatives with respect to the variable, e.g., $q_{x^i x^j}^a = \partial^2 q^a / \partial x^i \partial x^j$. We the order of I by $|I|$, and it is used as $0 \leq |I| \leq r$ that means all derivatives up to the r -th order. Let x^0 be the time coordinate for $t \in \mathbb{R}$ and let x^s be the spatial coordinates for $1 \leq s \leq m$. In some case, we use the abbreviation such as $q_t^a = q_{x^0}^a = q_0^a$.

Example 1. Let $m=1$, $l=1$, and $r=2$. We define the local coordinates of X by $(x^0, x^1) = (t, y)$, and those of Q by $(x^0, x^1; q^1) = (t, y; q)$. Then, the local coordinates of the jet bundle $J^2 Q$ are $(x^0, x^1; q_I^1) = (t, y; q, q_t, q_y, q_{tt}, q_{ty}, q_{yy})$ for all $0 \leq |I| \leq 2$. Note that q is described as a function of t, y ; however, each element of q_I is regarded as independent variables on the bundle. By using the summation convention, for example, we can interpret such as $(\partial \mathcal{L} / \partial q_I) dq_I = (\partial \mathcal{L} / \partial q_t) dq_t + (\partial \mathcal{L} / \partial q_y) dq_y$ for $|I|=1$.

3.2. Time-Spatial Splitting of State Space

Variational problems can be formalized as follows.

Definition 1. Consider the $(m+1,0)$ -form $\mathcal{L} := \mathcal{L}dx \in \Omega^{m+1,0}(J^r Q)$ as a Lagrangian density of a functional $\int_{\mathcal{Y}} \mathcal{L}dx$. We define the variational derivative $d\mathcal{L}$ of the Lagrangian density as the $(m+1,1)$ -form

$$d_v \mathcal{L}dx = \frac{\partial \mathcal{L}}{\partial q_I^a} dq_I^a \wedge dx \in \Omega^{m+1,1}(J^r Q) \tag{11}$$

that determines the stationary condition $d_v \mathcal{L} \equiv 0$ of the variational problem, where $0 \leq |I| \leq r$, and d_v is the vertical differential operator (see Sections B and C). In the following discussions, the variational derivative $d\mathcal{L}$ is lifted on $J^{2r-1}Q$.

Let us consider the following control system as a main objective.

Assumption 1. A given system is defined on a contractible domain \mathcal{S} with a boundary $\partial\mathcal{S}$, and it can be derived from a Lagrangian density in functional forms including derivatives that are first order with respect to the time coordinate and up to $(r-1)$ -th order with respect to spatial coordinates.

Under this assumption, the multi index I up to r -th order used for describing r -th order Lagrangians can be defined as $I = \{K, Kt\}$, where the multi index K is of spatial coordinates for $0 \leq |K| \leq r-1$.

Example 2. In the case of $m=2$, $l=1$ and $r=2$, let $(x^0, x^1, x^2; q^1) = (t, y, z; q)$ be a local coordinates of Q on X . Then, by defining $K = \{\emptyset, y, z\}$ and $I = \{K, Kt\} = \{\emptyset, y, z, t, yt, zt\}$, the local coordinates of J^2Q can be described as $(x^0, x^1, x^2; q_I^1) = (t, y, z; q_t^1)$. Without Assumption 1, I may include tt, yy, zz .

Furthermore, the following second assumption is important when we use the multisymplectic instantaneous formalism (see Section E).

Definition 2. Let us consider the time-spatial split domain $\mathcal{Y} = \mathcal{T} \times \mathcal{S} \subset \mathbb{R} \times \Sigma$ consisting of a time interval $\mathcal{T} \subset \mathbb{R}$ and a spatial domain $\mathcal{S} \subset \Sigma$. Then, a system defined on \mathcal{Y} at an instantaneous time $t \in \mathcal{T}$ is defined on \mathcal{S} .

Assumption 2. A Lagrangian density $\mathcal{L} \in \Omega^{m+1,0}(J^r Q)$ restricted to \mathcal{S} can be described as $\mathcal{L}dx^s \in \Omega^{m,0}(J^r Q)|_{\mathcal{S}}$ at a time $t \in \mathcal{T}$ in $\mathcal{Y} = \mathcal{T} \times \mathcal{S}$, where we denote the spatial (horizontal) volume form $dx^1 \wedge \dots \wedge dx^m$ by dx^s .

Note that, we treat variables including time derivatives, e.g., q_{Kt}^a in a bundle restricted to the spatial domain \mathcal{S} in this setting.

4. Implicit Hamiltonian Representation Induced from Distributions

In this section, we present a symplectic structure for distributions determined by partial differential equations in terms of the implicit Hamiltonian representation [14]. A distribution is a subbundle of a tangent bundle that is defined by a system dynamics, external constraints, and internal constraints generated by degenerate Lagrangians. On the other hand, a field Hamiltonian system is defined by the covariant Hamiltonian in the multisymplectic formalism [6]. However, the covariant Hamiltonian does not correspond to the typical Hamiltonian that are constant with respect to time evolution, e.g., for particle systems, but the instantaneous Hamiltonian derived from the time-spatial splitting.

4.1. Distributions

Definition 3. Consider a system of Pfaff equations $\mathcal{P} = \{\omega_a = 0 | 1 \leq a \leq l\}$ on an n -dimensional manifold M , where ω_a for each a is a differential 1-form. Then, the submanifold N of M is called an integral manifold of \mathcal{P} if $i_{\xi} \omega_a = 0$ for any vector ξ on the tangent space $T_p N$ at each point $p \in N$, where $1 \leq a \leq l$.

Definition 4. Let M be an n -dimensional manifold M . A morphism associating an r -dimensional subspace Δ of the tangent space $T_p M$ with each point $p \in M$ is called an r -th order distribution. The distribution is called regular if the dimension r is invariant.

The integrability of \mathcal{P} can be rephrased by distributions. That is, the r -dimensional distribution Δ is defined by $\Delta_p = \{\xi \in T_p M \mid i_\xi \omega_a = 0, 1 \leq a \leq l\}$ for each point $p \in M$, where $r = n - l$.

4.2. Symplectic Structure Restricted to Distributions

The relationship between Lagrangian and Hamiltonian systems is given by Legendre transformations (see (65) in Section D). For classical field equations, the Legendre transformations (or Lagrangians) are not regular in general, and thus, they are not one-to-one. However, the Legendre transformation can reasonably work under the following weaker condition.

Proposition 1. [5] *Let P^{r-1} be an embedded closed subbundle of Z^{r-1} . Then, the Legendre transformation $\sigma_\mathcal{L} : J^{2r-1}Q \rightarrow Z^{r-1}$ is called almost regular if $\sigma_\mathcal{L}$ is a submersion (i.e., $T\sigma_\mathcal{L}$ is surjective) with respect to the image P^{r-1} . In this case, there exists a vector field ξ on $J^{2r-1}M$ such that $T\sigma_\mathcal{L} \cdot \xi = v$ for any vector field v on P^{r-1} .*

Proposition 2. [5] *The following conditions are equivalent:*

- 1) $\phi \in \Gamma(Q)$ satisfies the Euler-Lagrange equations,
- 2) $j^{2r-1}\phi \in \Gamma(J^{2r-1}Q)$ satisfies the Cartan equations,
- 3) If $\sigma_\mathcal{L}$ is almost regular, $\delta = \sigma_\mathcal{L} \circ j^{2r-1}\phi \in \Gamma(P^{r-1})$ satisfies the Hamilton-De Donder equations,

where Γ is the space of all sections, and $j^{2r-1}\phi$ is the $(2r-1)$ -th jet of the section ϕ .

Remark 1. Note that $\phi = \pi_0^{2r-1} \circ \rho$ is not always the extremum of the original variational problem, i.e., Euler-Lagrange equations even if a certain $\rho \in \Gamma(J^{2r-1}Q)$ is a Cartan equation, where $\pi_0^{2r-1} : J^{2r-1}Q \rightarrow Q$ is the natural projection. This correspondence is valid only if $\sigma_\mathcal{L}$ is regular [5]. Hence, we start from Euler-Lagrange equations in this paper.

Under the assumption of first order Lagrangians with respect to time, the covariant Legendre transformation $\sigma_\mathcal{L} : J^{2r-1}Q \rightarrow Z^{r-1}$ in (65) is restricted the following instantaneous Legendre transformation:

$$\begin{cases} p_a^{Kt} = \frac{\partial \mathcal{L}}{\partial q_{Kt}^a} & \text{if } |Kt| = r, \\ p_a^{Kt} = \frac{\partial \mathcal{L}}{\partial q_{Kt}^a} - D_s p_a^{Kts} & \text{if } 1 \leq |Kt| < r, \end{cases} \quad (12)$$

where $0 \leq |K| \leq r-1$ is the multi index with respect to spatial variables, and $D_s = \sum_{i=1}^m D_i$ is the spatial total divergence. The term $-D_s p_a^{Kts}$ in the second equation of (12) is introduced from $p^{Kta} = 0$ in $-D_i p_a^{Kti}$ of (65), because Lagrangian is first order with respect to time.

From the above preliminaries, we shall relate a distribution Δ_Q of Euler-Lagrange equations with a distribution $\Delta_{Z^{r-1}}$ of Hamiltonian systems on the multisymplectic manifold Z^{r-1} through the relations i) \rightarrow ii) \rightarrow iii) in Proposition 2. Indeed, the following Hamiltonian representation for $\Delta_{Z^{r-1}}$ can be defined.

Definition 5. Consider the $(r-1)$ -th multisymplectic manifold Z^{r-1} with the local coordinates

$(x^i; q_K^a, p_a^{Kt}, p)$ under Assumption 1. Then, the local coordinates of VZ^{r-1} can be written as

$(x^i; q_K^a, p_a^{Kt}, \delta q_K^a, \delta p_a^{Kt}, p)$, because this results from the addition of the local coordinates of Z^{r-1} and the

coordinates $(\delta q_K^a, \delta p_a^{Kt})$ generated by the differential operator δ with respect to q_K^a, p_a^{Kt} , and we have used

the relations: $\delta q_K^a = D_i q_K^a = q_{Kt}^a$, $\delta p_a^{Kt} = D_i p_a^{Kt}$. On the other hand, the local coordinates of V^*Z^{r-1} are

$(x^i; q_K^a, p_a^{Kt}, -\delta p_a^{Kt}, \delta q_K^a, p)$, because it is given by the pairing of $-d\Theta \in V^*Z^{r-1}$ and arbitrary vector

$\xi \in VZ^{r-1}$ that is defined by $i_\xi(-d\Theta)$ between the vertical tangent and cotangent bundles [1]. Note that there is no adjoint variable of p defining an affine structure [5] (p. 214).

Now, we can consider the following induced symplectic structure induced from distributions.

Proposition 3. *Let $\sigma_\mathcal{L} : J^{2r-1}Q \rightarrow Z^{r-1}$ be the almost regular Legendre transformation and let $\Delta_Q \subset VQ$ be a regular distribution on Q that is restricted to \mathcal{S} . We restrict the multisymplectic $(m+2)$ -form Ω over Z^{r-1} (see Section D) to $\Delta_{Z^{r-1}}$, i.e., we define a skew symmetric bilinear form by $\Omega_\Delta = \Omega|_{\Delta_{Z^{r-1}} \times \Delta_{Z^{r-1}}}$. Then, there*

exists the following subbundle $\mathbb{D}_\Delta \subset VZ^{r-1} \oplus V^*Z^{r-1}$ for each $z \in Z^{r-1}$ and a fixed p :

$$\mathbb{D}_\Delta(z) = \left\{ (v_z, \alpha_z) \in V_z Z^{r-1} \times V_z^* Z^{r-1} \mid v_z \in \Delta_{Z^{r-1}}(z), \alpha_z(w_z) = \Omega_\Delta(z)(v_z, w_z), \forall w_z \in \Delta_{Z^{r-1}}(z) \right\}, \quad (13)$$

where we have defined $\Delta_{Z^{r-1}} = V\sigma_{\mathcal{L}} \circ (V\pi_0^{2r-1})^{-1} \Delta_Q \subset VZ^{r-1}$, $V\sigma_{\mathcal{L}}$ is the vertical tangent map of $\sigma_{\mathcal{L}}$, and $(V\pi_0^{2r-1})^{-1} \Delta_Q \subset VJ^{2r-1}Q$ is the distribution lifted along the vertical tangent map $V\pi_0^{2r-1} : VJ^{2r-1}Q \rightarrow VQ$ of the canonical projection $\pi_0^{2r-1} : J^{2r-1}Q \rightarrow Q$.

Proof. According to Proposition 2, there exists a vector field $\xi \in TJ^{2r-1}Q$ for any vector field $v \in \Delta_{Z^{r-1}}$. Thus, there exists also $\zeta = \varpi_{2r-1}\xi$, where $\varpi_{2r-1} : TJ^{2r-1}Q \rightarrow VJ^{2r-1}Q$ is the natural projection. Hence, the relation in (13) is resulted from the nature of the symplectic structure. Indeed, for a given $v_z \in \Delta_{Z^{r-1}}(z) \subset V_z Z^{r-1}$, there always exists the corresponding $(m+1, 1)$ -form $\alpha_z \in V_z^* Z^{r-1}$ that determines a Hamiltonian vector field. \square

Let us introduce Dirac structures on vector spaces from the references [2] [18] [19].

Definition 6. A Dirac structure on a vector space A is a subspace $\mathbb{D} \subset A \oplus A^*$ such that $\mathbb{D} = \mathbb{D}^\perp$, where A^* is the dual space of A , \mathbb{D}^\perp is the orthogonal space of \mathbb{D} with respect to the symmetric pairing $\langle\langle \cdot, \cdot \rangle\rangle$ on $A \oplus A^*$ such that $\langle\langle (v, \alpha), (\bar{v}, \bar{\alpha}) \rangle\rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle$ for $((v, \alpha), (\bar{v}, \bar{\alpha})) \in A \oplus A^*$, and $\langle \cdot, \cdot \rangle$ is the natural pairing between A^* and A .

Corollary 1. \mathbb{D}_Δ in (13) is the (almost) Dirac structure.

Proof. If we fix the coordinate p of Z^{r-1} , i.e., the covariant Hamiltonian, then Equation (13) is the typical form of induced Dirac structures [15]. \square

5. Distirubted Port-Hamiltonian Systems with Higher Order Boundary Energy Flows

In this section, we derive a formal structure of distributed port-Hamiltonian systems with boundary energy flows including higher order derivatives from the previously discussed implicit Hamiltonian representation. The energy flows passing through boundaries of system domains are used for boundary interconnections, or passivity-based boundary controls.

5.1. Boundary Terms Generated by Integration by Parts

In higher order variational problems, the zero boundary condition is usually assumed for simplification or some other reason. Then, boundary terms generated by Stokes theorem after applying integration by parts are eliminated. Actually, these boundary terms are related with the boundary energy flows.

Let us recall such a calculation that yields the boundary term in variational calculus. We first define the following notation for simplification.

Definition 7. From the Legendre transformation $\sigma_{\mathcal{L}}$ in (65), we can derive the following variable:

$$\tilde{p}_a^I := \frac{\partial \mathcal{L}}{\partial q_a^I} = p_a^I + D_I p_a^{II}, \quad (14)$$

where $0 \leq |I| \leq r$, $0 \leq i \leq m$, D_i is the total differential operator (61), and we have set $p_a^{II} = 0$ for $|I| = r$.

Now, we consider the Lagrangian density functional $\int_{\mathcal{Y}} \mathcal{L} dx$. The variational derivative of the Lagrangian density functional can be transformed by the integration by parts formula as follows:

$$\int_{\mathcal{Y}} \frac{\partial \mathcal{L}}{\partial q_a^{Ki}} dq_a^{Ki} \wedge dx = \int_{\mathcal{Y}} \tilde{p}_a^{Ki} dq_a^{Ki} \wedge dx = \int_{\mathcal{Y}} D_i (\tilde{p}_a^{Ki} dq_K^a) \wedge dx - \int_{\mathcal{Y}} D_i \tilde{p}_a^{Ki} dq_K^a \wedge dx, \quad (15)$$

where $0 \leq |K| \leq r-1$. By Stokes theorem [22], for any $0 \leq i \leq m$, the first term in the right side of (15) can be transformed into

$$\int_{\mathcal{Y}} D_i (\tilde{p}_a^{Ki} dq_K^a) \wedge dx = \int_{\mathcal{Y}} \tilde{p}_a^{Ki} dq_K^a \wedge i_{\partial x^i} dx, \quad (16)$$

where D_i is the total divergence that acts as $\sum_{i=0}^m D_i$, and $i_{\partial x^i} dx$ is the volume $(m, 1)$ -form on $\partial \mathcal{Y}$. For the

integrands in (15) and (16), consider the operation of differential forms that separate a coefficient from a vertical basis. For example, $\tilde{p}_a^{Ki} dq_{Ki}^a \wedge dx$ can be decomposed into \tilde{p}_a^{Ki} and dq_{Ki}^a . This operation is defined as follows.

Definition 8. For an $(m+1,1)$ -form $d_v \mathcal{L} dx = \tilde{p}_a^{Ki} dq_{Ki}^a dx \in \Omega^{m+1,1}(J^{2r-1}Q)$, we define the integration by parts operator as the following local expression:

$$\begin{aligned} \mathbb{I}(|Ki|)(d\mathcal{L}) : \Omega^{m+1,1}(J^{2r-1}Q)|_{\mathcal{Y}} &\rightarrow \Omega^{m+1,1}(J^{2r-1}Q)|_{\mathcal{Y} \times \partial \mathcal{Y}}; \left(x^i; \dots, q_{Ki}^a, \dots, \tilde{p}_a^{Ki}, \dots \right) \\ &\mapsto \left\{ \left(x^i; \dots, q_K^a, \dots, -D_i \tilde{p}_a^{Ki}, \dots \right), \left(x^i; \dots, q_K^a, \dots, \tilde{p}_a^{Ki}, \dots \right) \right\}, \end{aligned} \tag{17}$$

where we denoted $\Omega^{m+1,1}(J^{2r-1}Q)|_{\mathcal{Y} \times \partial \mathcal{Y}} = \Omega^{m+1,1}(J^{2r-1}Q)|_{\mathcal{Y}} \times \Omega^{m,1}(J^{2r-1}Q)|_{\partial \mathcal{Y}}$, i.e., the style $\{(A), (B)\}$ means the set of the transformed local coordinates (A) on \mathcal{Y} and (B) on $\partial \mathcal{Y}$, respectively. Here, untransformed coordinates under $|Ki|$ -th order are omitted in (17), and their numbers can be explicitly calculated by $k = |K|$, $0 \leq k \leq r-1$ and $j = r-k$.

Remark 2. For a Lagrangian density $\mathcal{L} dx^s \in \Omega^{m,1}(J^{2r-1}Q)$ restricted to the spatial domain \mathcal{S} in the time-spatial split space, the operation (17) can be also well defined. In this case, the boundary terms generated by Stokes theorem with respect to the time derivative $D_t(\cdot)$ in the total divergence are eliminated, because there is no boundary of a point in the time axis.

The repeated application of the integration by parts operator can yield all variations of boundary terms appeared in variational calculus.

Proposition 4. For some I , where $1 \leq v \leq |I|$, the v -th degree integration by parts operators is defined by

$$\mathbb{I}^v(|I|)(d\mathcal{L}) = \mathbb{I}(|I|-v+1) \circ \dots \circ \mathbb{I}(|I|-1) \circ \mathbb{I}(|I|)(d\mathcal{L}) \tag{18}$$

that can be expressed as the coordinate transformation

$$\begin{cases} p_{\partial a}^{Ki} = 0 & \text{if } |Ki| = r, \\ p_{\partial a}^K = \frac{\partial \mathcal{L}}{\partial q_{Ki}^a} - D_i p_{\partial a}^{Ki} & \text{if } 0 \leq |K| < r. \end{cases} \tag{19}$$

Proof. From the direct calculation of $\mathbb{I}^v(|I|)(d\mathcal{L})$, we obtain the following representation on $\Omega^{m+1,1}(J^{2r-1}Q)|_{\mathcal{Y} \times \partial \mathcal{Y}}$ that is derived from that on $\Omega^{m+1,1}(J^{2r-1}Q)|_{\mathcal{Y}}$:

$$\left\{ \left(x^i; q_j^a, \sum_{|l|=0}^r (-D_i)^{|l|} \tilde{p}_a^l, 0, \dots, 0 \right), \left(x^i; q_j^a, \sum_{|l|=1}^r (-D_i)^{|l|-1} \tilde{p}_a^l, \dots, \sum_{|l|=i}^r (-D_i)^{|l|-i} \tilde{p}_a^l, \dots, \tilde{p}_a^{l-1} - D_i \tilde{p}_a^l, \tilde{p}_a^l, 0 \right) \right\}. \tag{20}$$

The first bracket of (20) includes the nonzero part of the Euler-Lagrange equation; therefore, this operation is equivalent to the variational differential. The last r elements in the second bracket correspond to $p_{\partial a}^K$ in (19) for $0 \leq |K| \leq r$. \square

5.2. Stokes Variational Differential

The symplectic structure induced from distributions does not have any information on boundary energy flows. In this section, we define a variational differential operator with boundary terms generated by integration by parts and Stokes theorem, called the Stokes variational differential. The Stokes variational differential can be used in the induced symplectic structure for relating a given Lagrangian with port-Hamiltonian representations.

Proposition 5. For an r -th order Lagrangian density $\mathcal{L} \in \Omega^{m+1,0}(J^r Q)$ that is first order with respect to time and $(r-1)$ -th order with respect to spatial coordinates, the following variational differential operation can be defined:

$$\partial^r \mathcal{L} = \chi_{J^r Q} \circ \mathbb{I}^r(r) \left(d \left(t_{1,r-1}^r \mathcal{L} \right) \Big|_{\mathcal{S}} \right) : \Omega^{m+1,0}(J^r Q) \Big|_{\mathcal{S}} \rightarrow V^* Z^{r-1} \Big|_{\mathcal{S} \times \partial \mathcal{S}}, \tag{21}$$

where $\chi_{J^{r-1}Q} : V^*J^1J^{r-1}Q \rightarrow V^*Z^{r-1}$ (given as $\Omega_Z^b \circ \kappa_*^{-1}$). We call \mathfrak{d}^r the Stokes variational differential on \mathcal{S} with $\hat{\delta}\mathcal{S}$.

Before the proof of Proposition 5, we should prepare the following bundle maps. We first consider the map between VZ^{r-1} and V^*Z^{r-1} over Z^{r-1} .

Lemma 1. We can define the following bundle map Ω_Z^b :

$$\begin{aligned} \Omega_Z^b : VZ^{r-1} &\rightarrow V^*Z^{r-1}; \\ (x^i; q_K^a, p_a^{Ki}, \delta q_K^a, \delta p_a^{Ki}, p) &\mapsto (x^i; q_K^a, p_a^{Ki}, -\delta p_a^{Ki}, \delta q_K^a, p). \end{aligned} \tag{22}$$

Proof. This can be proven by the direct calculation with respect to the symplectic form (see Definition 5). \square

Lemma 2. By using the Legendre transformation $\sigma_{\mathcal{L}} : J^{2r-1}Q \rightarrow Z^{r-1}$, we can define the following bundle map under the first order assumption:

$$\begin{aligned} \kappa_*^{-1} : V^*J^1J^{r-1}Q &\rightarrow VZ^{r-1}; \\ (x^i; q_K^a, \delta q_K^a, p_a^{Ki}, \delta p_a^{Ki}) &\mapsto (x^i; q_K^a, \delta p_a^{Ki}, \delta q_K^a, p_a^{Ki}, p), \end{aligned} \tag{23}$$

where $1 \leq |Ki| \leq r$, and p is defined by $\sigma_{\mathcal{L}}$.

Proof. There exists the following bundle map $\kappa_{J^{r-1}Q}$ under the assumption:

$$\begin{aligned} \kappa_{J^{r-1}Q} : VV^*J^{r-1}Q &\rightarrow V^*VJ^{r-1}Q; \\ (x^i; q_K^a, p_a^{Ki}, \delta q_K^a, \delta p_a^{Ki}) &\mapsto (x^i; q_K^a, \delta q_K^a, \delta p_a^{Ki}, p_a^{Ki}). \end{aligned} \tag{24}$$

This can be proven in analogy with the diffeomorphism in the lumped parameter case, i.e., $\kappa_Q : TT^*Q \rightarrow T^*TQ$; $(x; q, p, \delta q, \delta p) \mapsto (x; q, \delta q, \delta p, p)$ ([16], p. 140, [26]). The inverse map of (24) can be extended as a map on the affine bundle (23). Indeed, we have the bundle map between the bundle $V^*J^1J^{r-1}Q$ and the original bundle $VJ^{1*}J^{r-1}Q$ of the tangent affine bundle $VZ^{r-1} \cong VJ^{1*}J^{r-1}Q$. Because the local coordinates of $J^{1*}J^{r-1}Q$ are $(x^i; q_K^a, p_a^{Ki})$, in the case of first order with respect to time, the local coordinates of $J^{1*}J^{r-1}Q$ are $(x^i; q_K^a, p_a^{Ki})$, and those of $J^1J^{r-1}Q$ are $(x^i; q_K^a, q_{Ki}^a)$. Thus, under the assumption, we can regard $VJ^{1*}J^{r-1}Q$ and $V^*J^1J^{r-1}Q$ as, respectively, $VV^*J^{r-1}Q$ and $V^*VJ^{r-1}Q$ by identifying $J^1J^{r-1}Q$ and $VJ^{r-1}Q$. \square

Next, we regard the derivative of Lagrangian density $d\mathcal{L}$ as a differential $(m+1,1)$ -form on $J^1J^{r-1}Q$ in the above discussion by using $J^rQ \hookrightarrow J^1J^{r-1}Q$.

Lemma 3. The standard variational derivative

$$\begin{aligned} d_v \circ t_r^{2r-1} : \Omega^{m+1,0}(J^rQ) &\rightarrow \Omega^{m+1,1}(J^{2r-1}Q); \\ \mathcal{L}dx &\mapsto \frac{\partial \mathcal{L}}{\partial q_t^a} dq_t^a \wedge dx \end{aligned} \tag{25}$$

can be regarded as the following derivative under the assumption of first order Lagrangians with respect to time:

$$\begin{aligned} d_v \circ t_{1,r-1}^r : \Omega^{m+1,0}(J^1J^{r-1}Q) &\rightarrow V^*J^1J^{r-1}Q; \\ \mathcal{L}dx &\mapsto (\tilde{p}_a^K dq_K^a + \tilde{p}_a^{Ki} dq_{Ki}^a) \wedge dx, \end{aligned} \tag{26}$$

where we have defined the inclusions $t_{2r-1}^r : J^rQ \hookrightarrow J^{2r-1}Q; (x^i; q_K^a) \mapsto (x^i; q_{KKi}^a)$, and

$$t_{1,r-1}^r : J^rQ \hookrightarrow J^1J^{r-1}Q; (x^i; q_K^a) \mapsto (x^i; q_K^a, q_{Ki}^a)$$

Proof. By $t_{1,2r-2}^{2r-1} : J^{2r-1}Q \hookrightarrow J^1J^{2r-2}Q; (x^i; q_{KKi}^a) \mapsto (x^i; q_{KK}^a, q_{KKi}^a)$, and $t_{1,2r-2}^{1,r-1} : J^1J^{r-1}Q \hookrightarrow J^1J^{2r-2}Q; (x^i; q_K^a, q_{Ki}^a) \mapsto (x^i; q_{KK}^a, q_{KKi}^a)$, we can rewrite (25) as

$$d_v \circ t_{1,2r-2}^{1,r-1} : \Omega^{m+1,0}(J^1J^{r-1}Q) \rightarrow V^*J^1J^{2r-2}Q; \mathcal{L}dx \mapsto (\tilde{p}_a^K dq_K^a + \tilde{p}_a^{Ki} dq_{Ki}^a) \wedge dx, \tag{27}$$

where we have used $\Omega^{m+1,1}(Y) \cong V^*Y$. Moreover, the inclusion $t_{1,2r-2}^{1,r-1}$ yields the dual map

$$\begin{aligned} V^* J^1 J^{2r-2} Q &\rightarrow V^* J^1 J^{r-1} Q; \\ (x^i; q_{KK}^a, \delta q_{KK}^a, p_a^{KKi}, \delta p_a^{KKi}) &\mapsto (x^i; q_K^a, \delta q_K^a, p_a^{Ki}, \delta p_a^{Ki}). \end{aligned} \quad (28)$$

Under the assumption, we can specify by $i = t$ in the above equations. \square

Proof of Proposition 5: From Lemmas 1-3, we can see that the map \mathfrak{d}^r is well-defined. \square

5.3. Local Expression of Induced Symplectic Structures

From the previous preparations, we can derive the relationship between the distribution Δ_Q and an instantaneous Hamiltonian system on Z^{r-1} that is described by the induced symplectic structure \mathbb{D}_Δ using the Stokes variational differential $\mathfrak{d}^r \mathcal{L}$.

Definition 9. Let $\mathcal{L} = \mathcal{L}dx \in \Omega^{m+1,0}(J^r Q)$ be a Lagrangian density that is first order with respect to time, and let $\Delta_Q \subset VQ$ be a regular distribution on Q . Consider the symplectic structure \mathbb{D}_Δ induced on Z^{r-1} in (13), and the Stokes variational differential $\mathfrak{d}^r \mathcal{L} : \Omega^{m+1,0}(J^r Q)|_{\mathcal{S}} \rightarrow V^* Z^{r-1}|_{\mathcal{S} \times \partial \mathcal{S}}$ on \mathcal{S} with $\partial \mathcal{S}$ in (21). Let $P_\Delta = V\sigma_{\mathcal{L}}(\Delta_{J^{2r-1}Q}) \subset VZ^{r-1}$ be the image of Δ_Q , where we have defined $\Delta_{J^{2r-1}Q} = (V\pi_0^{2r-1})^{-1} \Delta_Q$. Then, we define the implicit higher order field Hamiltonian representation on \mathcal{S} with $\partial \mathcal{S}$ as

$$(\mathfrak{X}_{Z^{r-1}}, \mathfrak{d}^r \mathcal{L}) \in \mathbb{D}_\Delta, \quad (29)$$

where $\mathfrak{X}_{Z^{r-1}} \in P_\Delta$.

Theorem 1. *The local expression of the implicit higher order field port Hamiltonian representation on \mathcal{S} with $\partial \mathcal{S}$ determined by $(\mathfrak{X}_{Z^{r-1}}, \mathfrak{d}^r \mathcal{L}) \in \mathbb{D}_\Delta$ is given as follows:*

$$\begin{cases} D_t q_K^a \in \Delta_{Z^{r-1}}, & q_{Kt}^a = D_t q_K^a, & (-1)^{|K|} D_K (D_t \tilde{p}_a^{Kt} - \tilde{p}_a^K) \in \Delta_{Z^{r-1}}^\circ, \\ \tilde{p}_a^{Kt} = \frac{\partial \mathcal{L}}{\partial q_{Kt}^a}, & \tilde{p}_a^K = \frac{\partial \mathcal{L}}{\partial q_K^a}, \end{cases} \quad (30)$$

where \tilde{p}_a^K is the covariant momentum, and we have defined the null space

$\Delta_{Z^{r-1}}^\circ = \left\{ \alpha \in V^* Z^{r-1} \mid \langle \alpha, v \rangle = 0, \forall v \in P_\Delta \subset VZ^{r-1} \right\}$. On the boundary, there is the following $(m,1)$ -form:

$$(-D_s)^{j-1} \frac{1}{w(K)} \tilde{p}_a^{\bar{K}} \cdot D_s^{|\bar{K}|-j} dq_t^a|_{\partial \mathcal{S}}, \quad (31)$$

where $1 \leq j \leq |K|$, K is the multi index with respect to spatial coordinates, \bar{K} is the multi index generated by repeated permutations (see Section B), and we have defined the spatial total divergence $D_s = \sum_{i=1}^m D_i$.

Proof. Let $\mathfrak{X}_{Z^{r-1}} = (x^i; q_K^a, p_a^{Kt}, D_t q_K^a, D_t p_a^{Kt}, p)$ be the vector field on VZ^{r-1} . For a vector field $\mathfrak{X}_{Z^{r-1}}$, we consider the following local expression of the symplectic form Ω

$$\Omega\left((x^i, q_K^a, p_a^{Kt}, u', \alpha', p), (x^i, q_K^a, p_a^{Kt}, u, \alpha, p)\right) = \langle \alpha, u' \rangle - \langle \alpha', u \rangle, \quad (32)$$

where $\langle \alpha, u \rangle : VZ^{r-1} \times V^* Z^{r-1} \rightarrow \mathbb{R}$ is the natural pairing between $u \in \mathfrak{X}_{Z^{r-1}} \subset VZ^{r-1}$ and $\alpha \in V^* Z^{r-1}$. On the other hand, from the local expression of the Stokes variational differential on \mathcal{S}

$$\mathfrak{d}^r \mathcal{L} = (x^i; q_K^a, \tilde{p}_a^{Kt}, -\delta \tilde{p}_a^{Kt}, \delta q_K^a, p), \quad (33)$$

we denote the 1-form fields by $u^* := -\delta \tilde{p}_a^{Kt}$ and $\alpha^* := \delta q_K^a$. From the equivalent condition between the last relation in (13) and (32)

$$\langle u^*, u \rangle + \langle \alpha^*, \alpha \rangle = \langle \alpha, u' \rangle - \langle \alpha', u \rangle, \quad (34)$$

we obtain the condition

$$\langle -\delta \tilde{p}_a^{Kt}, u \rangle + \langle q_{Kt}^a, \alpha \rangle = \langle \alpha, D_t q_K^a \rangle - \langle D_t \tilde{p}_a^{Kt}, u \rangle \tag{35}$$

for any u and α . Then, Equation (35) yields (30), because $\tilde{p}_a^{Kt} = \partial \mathcal{L} / \partial q_{Kt}^a$, and $\delta \tilde{p}_a^{Kt} = \partial \mathcal{L} / \partial q_K^a$ that can be derived from the Euler-Lagrange equation. Indeed, the pairing with respect to u gives the third relation in (30) as follows:

$$\left(D_t \tilde{p}_a^{Kt} - \frac{\partial \mathcal{L}}{\partial q_K^a} \right) \delta q_K^a = (-1)^{|K|} D_K \left(D_t \tilde{p}_a^{Kt} - \frac{\partial \mathcal{L}}{\partial q_K^a} \right) \delta q^a + \Phi, \tag{36}$$

where Φ corresponds to the boundary term in (31). The term (31) is obtained from the calculation in (20). Here, the total divergence $D_t(\cdot)$ with respect to time has been eliminated. Consequently, in (30), the first relation means a given vector field, the second relation is the definition of jet variables, the third relation is the Euler-Lagrange equation before applying integration by parts, and the fourth relation is the definition of the momentum. \square

The above representation can be converted to the following formal form of port representations [7].

Corollary 2. *The implicit higher order field port Hamiltonian system defined on \mathcal{S} with $\partial \mathcal{S}$ can be rewritten as the following port Hamiltonian system:*

$$\begin{bmatrix} f_p^a \\ f_{qK}^a \end{bmatrix} = \begin{bmatrix} 0 & -(-1)^{|K|} D_K \\ D_K^T & 0 \end{bmatrix} \begin{bmatrix} e_p^a \\ e_{qK}^a \end{bmatrix}, \tag{37}$$

where $1 \leq a \leq l$, and we have defined the variables

$$\begin{cases} (f_p^a, e_p^a) = \left((-1)^{|Kt|} D_{Kt} \tilde{p}_a^{Kt}, q_t^a \right), & (f_{qK}^a, e_{qK}^a) = (D_t q_K^a, \tilde{p}_a^K), \\ (f_{\partial|\bar{K}|,j}^a, e_{\partial|\bar{K}|,j}^a) = \left(D_s^{|\bar{K}|-j} e_p^a \Big|_{\partial \mathcal{S}}, -(-D_s)^{j-1} e_{q\bar{K}}^a \Big|_{\partial \mathcal{S}} \right), \\ \tilde{p}_a^{Kt} = \frac{\partial \mathcal{L}}{\partial q_{Kt}^a}, & \tilde{p}_a^K = \frac{\partial \mathcal{L}}{\partial q_K^a} \end{cases} \tag{38}$$

for $1 \leq j \leq |K|$. We call $f_{\partial|\bar{K}|,j}^a$ and $e_{\partial|\bar{K}|,j}^a$ boundary port variables.

Proof. Form the third and second relations in (30), we can obtain the first and second rows in (37), respectively. The product of the pair of the third relation in (38) that is equivalent to the boundary term in (31), where dq_t^a in (31) has been interpreted as an infinitesimal variation of q_t^a . \square

5.4. Power Balance and Passivity

This section derives the power balance of the Hamiltonian representation discussed in the previous section, and define the formal representation of higher order field port Hamiltonian systems with boundary energy flows.

In the time-spatial split space, the instantaneous Hamiltonian (70) on \mathcal{S} is given by

$$\mathcal{H}_{\mathcal{S}} = \int_{\mathcal{S}} \left(\sum_{|K|=0}^{r-1} p_a^{Kt} q_{Kt}^a - \mathcal{L} \right) dx^s, \tag{39}$$

where $1 \leq j \leq |K|$. The following relation corresponds to the power balance of distributed port-Hamiltonian systems.

Proposition 6. *The system (37) satisfies the power balance*

$$\int_{\mathcal{S}} \left(e_p^a f_p^a + e_{qK}^a f_{qK}^a \right) dx^s + \int_{\partial \mathcal{S}} \left(\sum_{j=1}^{|\bar{K}|} e_{\partial|\bar{K}|,j}^a f_{\partial|\bar{K}|,j}^a \right) \mathbf{i}_{\partial \mathcal{S}} dx^s = 0. \tag{40}$$

Proof. The power balance can be derived from the interior product between the derivative of the instantaneous Hamiltonian (39) and arbitrary vector field is zero. Let $\mathcal{H}(x, q_t^a, p_a^{Kt}) = p_a^{Kt} q_{Kt}^a - \mathcal{L}(x, q_t^a)$ be the generalized energy density of (39) defined on $\Omega_v^0(Z^{r-1} \oplus J^{2r-1}M)$, where $0 \leq |I| \leq r$, and $Z^{r-1} \oplus J^{2r-1}Q$ is the Whitney

bundle with the local coordinates $(x^i; q_j^a, p_a^{K_i}, p)$ for $0 \leq |K| \leq r-1$ and $0 \leq |J| \leq 2r-1$. Next, consider the pairing between $d_v \mathcal{H}$ and the vector field $\mu = D_t q_j^a (\partial/\partial q_j^a) + D_t p_a^{K_t} (\partial/\partial p_a^{K_t}) \in V(Z^{r-1} \oplus J^{2r-1}M)$ as follows:

$$\begin{aligned} \frac{d\mathcal{H}_S}{dt} &= \mathbf{i}_\mu d_v \mathcal{H} = \mathbf{i}_\mu d_v (p_a^{K_t} q_{K_t}^a - \mathcal{L}) = \mathbf{i}_\mu \left(p_a^{K_t} dq_{K_t}^a + q_{K_t}^a dp_a^{K_t} - \frac{\partial \mathcal{L}}{\partial q_t^a} dq_t^a \right) \\ &= \mathbf{i}_\mu \left(-D_s p_a^{K_{ts}} dq_{K_t}^a + q_{K_t}^a dp_a^{K_t} - \frac{\partial \mathcal{L}}{\partial q_K^a} dq_K^a \right) \\ &= \mathbf{i}_\mu \left(D_t D_s p_a^{K_{ts}} dq_K^a + q_{K_t}^a dp_a^{K_t} - \frac{\partial \mathcal{L}}{\partial q_K^a} dq_K^a \right) \\ &= \left(D_t D_s p_a^{K_{ts}} + D_t p_a^{K_t} - \frac{\partial \mathcal{L}}{\partial q_K^a} \right) q_{K_t}^a = \left(D_t \tilde{p}_a^{K_t} - \frac{\partial \mathcal{L}}{\partial q_K^a} \right) q_{K_t}^a, \end{aligned} \tag{41}$$

where we have used $\tilde{p}_a^{K_t} = p_a^{K_t} + D_s p_a^{K_{ts}} = \partial \mathcal{L} / \partial q_{K_t}^a$, and the time total divergence $D_t(\cdot)$ is eliminated in the same way of the proof of *Theorem 1*. From the energy conservation $\mathbf{i}_\mu d_v \mathcal{H} = 0$, we obtain $D_t \tilde{p}_a^{K_t} = \partial \mathcal{L} / \partial q_K^a$. On the other hand, we have

$$\begin{aligned} \mathbf{i}_\mu d_v \mathcal{L} &= \mathbf{i}_\mu \left[\frac{\partial \mathcal{L}}{\partial q_{K_t}^a} dq_{K_t}^a + \frac{\partial \mathcal{L}}{\partial q_K^a} dq_K^a \right] \\ &= \mathbf{i}_\mu \left[-D_t \tilde{p}_a^{K_t} dq_K^a + (-1)^{|K|} D_K \frac{\partial \mathcal{L}}{\partial q_K^a} dq^a \sum_{j=1}^{|\bar{K}|} D_s \left((-D_s)^{j-1} \frac{1}{w(K)} \frac{\partial \mathcal{L}}{\partial q_{\bar{K}}^a} \cdot D_s^{|\bar{K}|-j} q_t^a \right) \right] \\ &= -D_t \tilde{p}_a^{K_t} q_{K_t}^a - (-1)^{|K_t|} D_{K_t} \tilde{p}_a^{K_t} q_t^a - D_s \sum_{j=1}^{|\bar{K}|} e_{\partial|\bar{K}|,j}^a f_{\partial|\bar{K}|,j}^a = 0. \end{aligned} \tag{42}$$

By substituting (42) into (41), the integrand of (40) is given as follows:

$$-\mathbf{i}_\mu d_v \mathcal{H} = (-1)^{|K_t|} D_{K_t} \tilde{p}_a^{K_t} q_t^a + \frac{\partial \mathcal{L}}{\partial q_K^a} q_{K_t}^a + D_s \sum_{j=1}^{|\bar{K}|} e_{\partial|\bar{K}|,j}^a f_{\partial|\bar{K}|,j}^a = 0. \tag{43}$$

By applying Stokes theorem to the integral of the third term of the second equation in (43), (40) is given. \square

Proposition 7. *The system (37) is passive.*

Proof. The Hamiltonian (39) and the power balance (40) correspond to the finite constant and the duality product before the time integration, respectively, in the definition of the passivity (see *Section A*). \square

Consequently, we at last reach the final result that means the system (37) is just a higher order representation of distributed port-Hamiltonian systems.

Theorem 2. *The system (37) is the Stokes-Dirac structure.*

Proof. We have already proven that the system (37) is a Dirac structure in *Proposition 1*. On the other hand, the power balance (40) corresponds with the main property of distributed port-Hamiltonian systems described by the Stokes-Dirac structure, and it can be regarded as the higher order version of the structure [12]. \square

6. Examples

This section presents two modeling examples.

6.1. Timoshenko Beam Equation

The 1-dimensional Timoshenko beam equation

$$\begin{cases} A_\rho \epsilon_u - GAD_y (\epsilon_y - \theta) = 0, \\ I_\rho \theta_u - EI \theta_{yy} - GA (\epsilon_y - \theta) = 0 \end{cases} \tag{44}$$

is derived from the Lagrangian density functional on $\Omega^{2,0}(J^3Q)$

$$\mathcal{L} = \int_{\mathcal{Y}} \mathcal{L} dx = \frac{1}{2} \int_{\mathcal{Y}} \left\{ A_{\rho} \epsilon_t^2 + I_{\rho} \theta_t^2 - GA(\epsilon_y - \theta)^2 - EI\theta_y^2 \right\} dx, \tag{45}$$

where $t \in [t_0, t_1]$ is the time coordinate, $y \in \mathcal{S}$ is the spatial coordinate along the longitudinal axis, $\epsilon \in \mathbb{R}$ is the shearing, $\theta \in \mathbb{R}$ is the rotation at each point in y , A_{ρ} is the unit mass, I_{ρ} is the moment of inertia, EI is the elastic stiffness, and GA is the shearing stiffness.

Let $m = 2$, $l = 2$, and $r = 3$. From $(x^0, x^1; q^1, q^2) = (t, y; \epsilon(t, y), \theta(t, y))$ and the maximum of higher order degrees $\max|K| = 2$ in (44), we derive $q_{Kt}^1 = D_{Kt}\epsilon$ for $0 \leq |Kt| \leq 3$. By defining J^3Q with the local coordinate $q_{Kt}^2 = D_{Kt}\theta$, we set $dx = dt \wedge dy$ and $\mathcal{Y} = [t_0, t_1] \times \mathcal{S}$.

In (38), from $k = 1$, $0 \leq |K| \leq 1$, $j = 1$, and $a = 2$, we obtain

$$\left\{ \begin{aligned} (f_1^p, e_1^p) &= \left((-1)^1 D_t \tilde{p}_1^t, \epsilon_t \right) = \left(D_t (A_{\rho} \epsilon_t), \epsilon_t \right), \\ (f_2^p, e_2^p) &= \left((-1)^1 D_t \tilde{p}_2^t, \theta_t \right) = \left(D_t (I_{\rho} \theta_t), \theta_t \right), \\ (f_q^2, e_q^2) &= \left(D_t \theta, \tilde{p}_2 \right) = \left(D_t \theta, GA(\epsilon_y - \theta) \right), \\ (f_{qy}^1, e_{qy}^1) &= \left(D_t \epsilon_y, \frac{1}{w(y)} \tilde{p}_1^{\bar{y}} \right) = \left(D_t \epsilon_y, -GA(\epsilon_y - \theta) \right), \\ (f_{qy}^2, e_{qy}^2) &= \left(D_t \theta_y, \frac{1}{w(y)} \tilde{p}_2^{\bar{y}} \right) = \left(D_t \theta_y, -EI\theta_y \right), \\ (f_{\partial 1,1}^1, e_{\partial 1,1}^1) &= \left(D_s^{1-1} q_t^1, -(-D_s)^{1-1} e_{qy}^1 \right) = \left(\epsilon_t, -e_{qy}^1 \right), \\ (f_{\partial 1,1}^2, e_{\partial 1,1}^2) &= \left(D_s^{1-1} q_t^2, -(-D_s)^{1-1} e_{qy}^2 \right) = \left(\theta_t, -e_{qy}^2 \right), \end{aligned} \right. \tag{46}$$

where $D_s = D_y$ and we have defined

$$\begin{aligned} \tilde{p}_1^t &= \frac{\partial \mathcal{L}}{\partial \epsilon_t}, & \tilde{p}_2^t &= \frac{\partial \mathcal{L}}{\partial \theta_t}, \\ \tilde{p}_2 &= \frac{\partial \mathcal{L}}{\partial \theta}, & \tilde{p}_1^{\bar{y}} &= \frac{\partial \mathcal{L}}{\partial \epsilon_y}, & \tilde{p}_2^{\bar{y}} &= \frac{\partial \mathcal{L}}{\partial \theta_y}. \end{aligned} \tag{47}$$

Hence, we have

$$\left\{ \begin{aligned} \begin{bmatrix} f_1^p \\ f_2^p \\ f_2^q \\ f_{qy}^1 \\ f_{qy}^2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & D_y & 0 & 0 \\ 0 & 0 & 0 & -1 & D_y \\ D_y & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & D_y & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1^p \\ e_2^p \\ e_2^q \\ e_{qy}^1 \\ e_{qy}^2 \end{bmatrix}, \\ \begin{bmatrix} f_{\partial}^1 \\ f_{\partial}^2 \end{bmatrix} &= \begin{bmatrix} f_{\partial 1,1}^1|_{\partial \mathcal{S}} \\ f_{\partial 1,1}^2|_{\partial \mathcal{S}} \end{bmatrix}, & \begin{bmatrix} e_{\partial}^1 \\ e_{\partial}^2 \end{bmatrix} &= \begin{bmatrix} e_{\partial 1,1}^1|_{\partial \mathcal{S}} \\ e_{\partial 1,1}^2|_{\partial \mathcal{S}} \end{bmatrix}, \end{aligned} \right. \tag{48}$$

where two lines form the first equation in (48) is equivalent to (44), and three lines from the bottom are equalities. Moreover, the system (48) satisfies

$$\int_{\mathcal{S}} (e_1^p f_1^p + e_2^p f_2^p + e_q^2 f_q^2 + e_{qy}^1 f_{qy}^1 + e_{qy}^2 f_{qy}^2) dx^s + \int_{\partial \mathcal{S}} (e_{\partial}^1 f_{\partial}^1 + e_{\partial}^2 f_{\partial}^2) \mathbf{i}_{\partial x^s} dx^s = 0, \tag{49}$$

where note that Stokes theorem cannot be applied to (e_q^2, f_q^2) ; therefore the corresponding boundary term has not been defined.

6.2. Potential Boussinesq Equation

The 1-dimensional potential Boussinesq equation ([27], p. 237) that expresses shallow water waves

$$q_{yyt} + \frac{1}{2} D_y^2 (q_{yy}^2) + q_{yyyyy} = 0 \quad (50)$$

is derived from the Lagrangian density functional defined on $\Omega^{2,0}(J^7Q)$

$$\mathcal{L} = \int_y \left(\frac{1}{2} q_{yt}^2 + \frac{1}{6} q_{yy}^3 - \frac{1}{2} q_{yyy}^2 \right) dx, \quad (51)$$

where $t \in [t_0, t_1]$ is the time coordinate, $y \in \mathcal{S}$ is the spatial coordinate along the water surface, and $q \in \mathbb{R}$ is the height of the wave.

Let $m=2$, $l=1$, and $r=7$. We have defined J^7Q with the local coordinate $q_{Kt}^1 = D_{Kt}q$ for $0 \leq |Kt| \leq 7$, $dx = dt \wedge dy$, and $\mathcal{Y} = [t_0, t_1] \times \mathcal{S}$ by $(x^0, x^1; q^1) = (t, y; q(t, y))$ and the maximum order $\max |K| = 6$ in (50).

By substituting $k=3$, $0 \leq |K| \leq 3$, $1 \leq j \leq 3$, and $a=1$ to (38), we get

$$\left\{ \begin{array}{l} (f_1^p, e_1^p) = \left((-1)^2 D_{yt} \tilde{p}_1^{yt}, q_t \right) = (D_{yt} q_{yt}, q_t), \\ (f_{qyy}^1, e_{qyy}^1) = \left(D_t q_{yy}, \frac{1}{w(yy)} \tilde{p}_1^{yy} \right) = \left(D_t q_{yy}, \frac{1}{2} q_{yy}^2 \right), \\ (f_{qyyy}^1, e_{qyyy}^1) = \left(D_t q_{yyy}, \frac{1}{w(yyyy)} \tilde{p}_1^{yyy} \right) = (D_t q_{yyy}, -q_{yyy}), \\ (f_{\partial 2,1}^1, e_{\partial 2,1}^1) = (D_s^{2-1} q_t^1, -(-D_s)^{1-1} e_{qyy}^1) = (D_y q_t, -e_{qyy}^1), \\ (f_{\partial 2,2}^1, e_{\partial 2,2}^1) = (D_s^{2-2} q_t^1, -(-D_s)^{2-1} e_{qyy}^1) = (q_t, D_y e_{qyy}^1), \\ (f_{\partial 3,1}^1, e_{\partial 3,1}^1) = (D_s^{3-1} q_t^1, -(-D_s)^{1-1} e_{qyyy}^1) = (D_y^2 q_t, -e_{qyyy}^1), \\ (f_{\partial 3,2}^1, e_{\partial 3,2}^1) = (D_s^{3-2} q_t^1, -(-D_s)^{2-1} e_{qyyy}^1) = (D_y q_t, D_y e_{qyyy}^1), \\ (f_{\partial 3,3}^1, e_{\partial 3,3}^1) = (D_s^{3-3} q_t^1, -(-D_s)^{3-1} e_{qyyy}^1) = (q_t, -D_y^2 e_{qyyy}^1), \end{array} \right. \quad (52)$$

where $D_s = D_y$ and we have defined

$$\tilde{p}_1^{yt} = \frac{\partial \mathcal{L}}{\partial q_{yt}}, \quad \tilde{p}_1^{yy} = \frac{\partial \mathcal{L}}{\partial q_{yy}}, \quad \tilde{p}_1^{yyy} = \frac{\partial \mathcal{L}}{\partial q_{yyy}} \quad (53)$$

Hence, the following system representation is given:

$$\left\{ \begin{array}{l} \begin{bmatrix} f_p^1 \\ f_{qyy}^1 \\ f_{qyyy}^1 \end{bmatrix} = \begin{bmatrix} 0 & -D_y^2 & D_y^3 \\ D_y^2 & 0 & 0 \\ D_y^3 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_p^1 \\ e_{qyy}^1 \\ e_{qyyy}^1 \end{bmatrix}, \\ f_{\partial}^1 = \begin{bmatrix} f_{\partial 2,1}^1|_{\partial \mathcal{S}} \\ f_{\partial 2,2}^1|_{\partial \mathcal{S}} \\ f_{\partial 3,1}^1|_{\partial \mathcal{S}} \\ f_{\partial 3,2}^1|_{\partial \mathcal{S}} \\ f_{\partial 3,3}^1|_{\partial \mathcal{S}} \end{bmatrix}, \quad e_{\partial}^1 = \begin{bmatrix} e_{\partial 2,1}^1|_{\partial \mathcal{S}} \\ e_{\partial 2,2}^1|_{\partial \mathcal{S}} \\ e_{\partial 3,1}^1|_{\partial \mathcal{S}} \\ e_{\partial 3,2}^1|_{\partial \mathcal{S}} \\ e_{\partial 3,3}^1|_{\partial \mathcal{S}} \end{bmatrix}, \end{array} \right. \quad (54)$$

where the first line of the first equation in (54) is equivalent to (50), and two lines from the bottom are equalities. Moreover, the system (54) satisfies

$$\int_S \left(e_1^p f_1^p + e_{qyy}^1 f_{qyy}^1 + e_{qyyy}^1 f_{qyyy}^1 \right) dx^s + \int_{\partial S} \left\{ \left(e_\delta^1 \right)^T f_\delta^1 \right\} \mathbf{i}_{\partial x^s} dx^s = 0. \quad (55)$$

7. Conclusions

This paper derived the higher order field port Hamiltonian system with boundary energy flows from systems of higher order partial differential equations that are determined by variational problems in terms of the multisymplectic instantaneous formalism. By defining the symplectic structure induced from distributions and the Stokes variational differential including the integration by parts operators, we clarified the implicit Hamiltonian representation of the systems of higher order partial differential equations, and its local expression corresponds to the distributed port-Hamiltonian system.

In this paper, we assumed that Lagrangians are first order with respect to time, but possibly higher order with respect to spatial variables for simplification. This assumption can be generalized. On the other hand, the formal representation including time derivatives up to first order corresponds to the distributed port-Hamiltonian systems.

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Appendix

A. Passivity

Consider the following pairing between $L_e^q(U)$ and $L_e^q(U^*)$:

$$\langle y, u \rangle_T := \int_0^T \langle y(t), u(t) \rangle dt \quad (56)$$

for $u \in L_e^q(U)$ and $y \in L_e^q(U^*)$, where $\langle \cdot, \cdot \rangle$ is the duality product, U is a finite dimensional linear space, U^* is its dual space, and we have defined the extended L_e^q space that is the set of all measurable functions in L^q truncated to a finite time interval. Note that the duality product $\langle y(t), u(t) \rangle$ corresponds to power.

Definition 10. Let $G : L_e^q(U) \rightarrow L_e^q(U^*)$. Then G is passive if there exists a constant H such that

$$\langle Gu, u \rangle_T \geq -H, \quad \forall u \in L^q(U), \quad \forall T \geq 0, \quad (57)$$

where the left-side of (57) is assumed to be well-defined.

Hence, G is passive if and only if a finite amount of energy can be extracted from the system defined by G .

Corollary 3. For a point t in the time axis,

$$\langle Gu(t), u(t) \rangle \geq -\frac{dH}{dt}, \quad \forall u \in L^q(U). \quad (58)$$

B. Differential Forms on Bundles

A differential (i, j) -form $g\eta \wedge \psi$ defined on the r -th order jet bundle $J^r Q$ are defined by a horizontal i -form $\psi = dx^1 \wedge \cdots \wedge dx^i \in \Omega_h^i(J^r Q)$ and a vertical j -form $\eta = dq_{I_1}^{a_1} \wedge \cdots \wedge dq_{I_j}^{a_j} \in \Omega_v^j(J^r Q)$, where $\Omega^j(J^r Q)$ is the space of differential j -forms on $J^r Q$, $g = g(x^i, q_i^a)$ is a smooth function defined on $J^r Q$, and a_k and I_k are different combination selected from a and I for $1 \leq k \leq j$.

Let $\Omega^{i,j}(J^r Q)$ be the space of differential (i, j) -forms defined on $J^r Q$. (i, j) -forms such that $n = i + j$ are called n -forms, and their space is denoted by $\Omega^n(J^r Q)$.

The exterior differential operator $d = d_v + d_h$ for (i, j) -forms is defined by the vertical differential operator

$$d_v : \Omega^{i,j}(J^r Q) \rightarrow \Omega^{i,j+1}(J^r Q); \quad g\eta \wedge \psi \mapsto \sum_{a,I} \frac{\partial g}{\partial q_I^a} dq_I^a \wedge \eta \wedge \psi \quad (59)$$

and the horizontal differential operator

$$d_h : \Omega^{i,j}(J^r Q) \rightarrow \Omega^{i+1,j}(J^r Q); \quad g\eta \wedge \psi \mapsto \sum_{k=1}^m D_k g \eta \wedge dx^k \wedge \psi, \quad (60)$$

where the total differential operator with respect to x^i is defined by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{a=1}^l \sum_{|I|=0}^{\infty} q_{Ii}^a \frac{\partial}{\partial q_I^a} \quad (61)$$

that is equivalent to partial differential, and $D_i dq_I = d(D_i q_I) = dq_{I,i}$. Note that $q_{I,i} = \partial q_I / \partial x^i$, and $q_{Ii} = (w(I)/w(Ii))q_{I,i}$, where $w(I)$ is the weight of the index I [22] [24], and $q_I^a = (1/w(I))q_I^a$ for the index \bar{I} generated by the repeated permutation of the combination in I .

C. Euler-Lagrange Equations

An Euler-Lagrange equation is determined by the stationary condition $d\mathcal{L}dx = 0$ of the variational derivative $d\mathcal{L}$ of a Lagrange density $\mathcal{L} = \mathcal{L}dx \in \Omega^{m,0}(J^r Q)$. If variables on boundaries are zero, the local expression of Euler-Lagrange equations is given by the stationary condition of

$$d_v \mathcal{L}dx = \frac{\partial \mathcal{L}}{\partial q_I^a} dq_I^a \wedge dx = (-1)^{|I|} D_I \frac{\partial \mathcal{L}}{\partial q_I^a} dq^a \wedge dx, \quad (62)$$

where $dx = dx^0 \wedge \cdots \wedge dx^m$ is the $(m+1)$ -volume form, D_I is the total differential operator D_i with respect to all index in I , integration by parts is used in the second equality, and the term $\sum_{i=0}^m D_i(\cdot)$ is eliminated by Stokes theorem under the assumption of the zero boundary condition.

D. Multisymplectic Covariant Formalism

The Hamiltonian representation of lumped parameter systems are determined by the symplectic 2-form $\omega = -d\theta$ on cotangent bundle T^*Q , where $\theta = pdq$ is the canonical 1-form. Then, for a given Hamiltonian $H : T^*Q \rightarrow \mathbb{R}$, Hamiltonian vector field μ is defined by $i_\mu \omega = dH$ ([1], p. 187), where $(q, p) \in T^*Q$, i is

the interior product.

The (covariant) Hamiltonian representation of field equations are determined by the multisymplectic $(m + 2)$ -form $\Omega = -d\Theta$ on the multisymplectic manifold Z^{r-1} ([5], p. 211), where Θ is the canonical $(m + 1)$ -form. Then, Z^{r-1} is defined as the subbundle of $\wedge^{m+1}(J^{r-1}Q)$ over $J^{r-1}Q$ defining $(m + 1)$ -forms

$$Z^{r-1} = \left\{ z \in \wedge^{m+1}(J^{r-1}Q) \mid i_{\xi} i_{\nu} z = 0, \forall \nu, \xi \in VJ^{r-1}Q \right\}, \tag{63}$$

where the space $\Omega^{m+1}(J^{r-1}Q)$ of $(m + 1)$ -forms over $J^{r-1}Q$ is defined by the space of all sections of $(m + 1)$ -th degree exterior power cotangent $(r - 1)$ -th jet bundle [5], and the vertical tangent bundle $V\mathcal{Y}$ has been defined as a vector subbundle $T\mathcal{Y} \supset V\mathcal{Y} = \text{Ker}T\pi$ determined by the tangent map $T\pi : T\mathcal{Y} \rightarrow TX$ for a bundle $\pi : \mathcal{Y} \rightarrow X$. Note that the local coordinates (x^i, \dot{x}^i) of the tangent bundle TX when those of the manifold X are (x^i) , and the local coordinates of TY are $(x^i, y^a, \dot{x}^i, \dot{y}^a)$ when those of the bundle $\pi : Y \rightarrow X$ are (x^i, y^a) , where those of VY are (x^i, y^a, \dot{y}^a) . Let $(x^i; q_K^a, p_a^{Ki}, p)$ for $0 \leq |K| \leq r - 1$ be the local coordinates of Z^{r-1} . Any $z \in Z^{r-1}$ in (63) can be locally written as

$$z = p dx + \sum_{|K|=0}^{r-1} p_a^{Ki} dq_K^a \wedge i_{\partial x^i} dx, \tag{64}$$

where we assume that the multi index Ki satisfies $|Ki| = |K| + 1$. Then, $\Theta = z$ is defined by using (64).

On the other hand, the covariant Lagrangian system can be defined as $-d\Theta_{\mathcal{L}}$ on $J^{2r-1}Q$ by the Cartan form $\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^{2r-1}Q)$, where $\sigma_{\mathcal{L}}^* \Theta = \Theta_{\mathcal{L}}$, and $\sigma_{\mathcal{L}} : J^{2r-1}Q \rightarrow Z^{r-1}$ is the covariant Legendre transformation on $J^{r-1}Q$

$$\begin{cases} p_a^K = \frac{\partial \mathcal{L}}{\partial q_K^a} + c_a^K & \text{if } |K| = r, \\ p_a^K = \frac{\partial \mathcal{L}}{\partial q_K^a} - D_i p_a^{Ki} + c_a^K & \text{if } 1 \leq |K| < r, \\ c_a^K = 0 & \text{if } |K| = 1, \\ p = \mathcal{L} - \sum_{|K|=0}^{r-1} p_a^{Ki} q_{Ki}^a, \end{cases} \tag{65}$$

where $\sigma_{\mathcal{L}}^*$ is the pull-back, and $(x^i; q_I^a)$ are the local coordinates of J^rQ for $0 \leq |I| \leq r$. The functions c_a^{Ki} and c_a^K in (65) give arbitrariness in the global expression of $\Theta_{\mathcal{L}}$; therefore, this is not used in the local expression, i.e., $c_a^{Ki} = c_a^K = 0$.

In the above, the covariant Lagrangian system determined by the variational problem of the r -th order Lagrangian density $\mathcal{L} dx \in \Omega^{m,0}(J^rQ)$ on J^rQ is defined by the Cartan form $\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^{2r-1}Q)$ on $J^{2r-1}Q$ ([5], p. 210). Note that the covariant Hamiltonian p determines the affine structure of Z^{r-1} that is the essential of the symplectic structure.

E. Multisymplectic Instantaneous Formalism

The instantaneous formalism is the covariant representation with time-spatial splitting. The time-spatial splitting is equivalent to choosing an infinitesimal supersurface parametrized by time in the configuration space Q . Bundles with time-spatial splitting consist of the Cauchy surface $\Sigma \subset X$ and time-spatial vector fields ζ on Q . Then, ζ means the direction of the time evolution of the system, and ζ on $Q|_{\Sigma}$ transversally intersects to $Q|_{\Sigma}$ everywhere, where $Q|_{\Sigma}$ means the restricted Q to Σ .

The instantaneous representation is defined on the space $\Gamma(J^{2r-1}Q|_{\Sigma})$ that consists of all sections γ of

$J^{2r-1}Q$ restricted to Σ . For a given global section $\gamma \in \Gamma(J^{2r-1}Q|_{\Sigma})$, the local coordinates of $\Gamma(J^{2r-1}Q|_{\Sigma})$ is given as $\gamma_J^a = q_J^a \circ \gamma$ ([6], p. 379), where q_J^a is the local coordinates of $J^{2r-1}Q$ for $0 \leq |J| \leq 2r-1$. Then, the subbundle $j^{2r-1}Q|_{\Sigma}$ of $\Gamma(J^{2r-1}Q|_{\Sigma})$ that can be identified with the tangent bundle $T^{2r-1}Q|_{\Sigma}$ by choosing the direction of the time evolution of ζ , where the local coordinates of $j^{2r-1}Q|_{\Sigma}$ are obtained from γ_U^a by using the multi index U with respect to time for $0 \leq |U| \leq 2r-1$. Hence, by restricting the system to $j^{2r-1}Q|_{\Sigma}$, the spatial derivatives in γ_J^a are eliminated.

$T^{2r-1}Q|_{\Sigma}$ and $T^*(T^{r-1}Q|_{\Sigma})$ are the vector bundle over $T^{r-1}Q|_{\Sigma}$ with $2r$ numbers of local coordinates, respectively, (γ_L^a) for $0 \leq |L| \leq 2r-1$ and (γ_K^a, π_a^K) for $0 \leq |K| \leq r-1$. On the dual bundle of $T^{2r-1}Q|_{\Sigma}$, i.e., $T^*(T^{r-1}Q|_{\Sigma})$, the instantaneous Hamiltonian systems are derived from the canonical form

$$\mathcal{G}_{\Sigma} = \int_{\Sigma} \left(\sum_{|L|=0}^{r-1} \pi_a^L dq_L^a \right) dx^s, \tag{66}$$

where the instantaneous momentum

$$\pi_a^L = \sum_{|K|=0}^{r-1} (-1)^{|K|} w(L, K) D_K p_a^{KL} \tag{67}$$

is calculated by integration by parts, and the weight

$$w(L, K) = (|L| + |K|)! / (|L|! |K|!) \tag{68}$$

based on the combination have been defined.

Instantaneous Lagrange systems are determined on $T^{2r-1}Q|_{\Sigma}$ by $\mathcal{G}_{\mathcal{L}, \Sigma} = \sigma_{\mathcal{L}, \Sigma}^* \mathcal{G}_{\Sigma}$ [6, pp. 382], where the instantaneous Legendre transformation

$$\sigma_{\mathcal{L}, \Sigma} : T^{2r-1}Q|_{\Sigma} \rightarrow T^*(T^{r-1}Q|_{\Sigma}) \tag{69}$$

is given by $\gamma_U^a \mapsto (\gamma_L^a, \pi_a^L)$ for $0 \leq |U| \leq 2r-1$ and $0 \leq |L| \leq r-1$ that are multi indexes with respect to time. The image in $T^*(T^{r-1}Q|_{\Sigma})$ of the bundle map $\sigma_{\mathcal{L}, \Sigma}$ is the instantaneous primal constraint set P_{Σ}^{r-1} . The instantaneous Hamiltonian $\mathcal{H}_{\Sigma} : P_{\Sigma}^{r-1} \rightarrow \mathbb{R}$ on P_{Σ}^{r-1} is defined as follows ([6], p. 384):

$$\mathcal{H}_{\Sigma} = \int_{\Sigma} \left(\sum_{|L|=0}^{r-1} \pi_a^L q_{L_t}^a - \mathcal{L} \right) dx^s. \tag{70}$$

Because P_{Σ}^{r-1} possesses the (pre-)symplectic structure ω_{Σ} of $T^*(T^{r-1}Q|_{\Sigma})$, instantaneous Hamiltonian systems on P_{Σ}^{r-1} are given by evolutonal vector fields ν such that $i_{\nu} \omega_{\Sigma} = d\mathcal{H}_{\Sigma}$.