Journal of Applied Mathematics and Physics, 2015, 3, 1391-1405
Published Online November 2015 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2015.311167

New Exact Solutions of the (2 + 1)-Dimensional AKNS Equation

Yepeng Sun

School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan, China
Email: yepsun@163.com

Received 11 October 2015; accepted 17 November 2015; published 20 November 2015

Abstract

N-soliton solutions and the bilinear form of the (2 + 1)-dimensional AKNS equation are obtained by using the Hirota method. Moreover, the double Wronskian solution and generalized double Wronskian solution are constructed through the Wronskian technique. Furthermore, rational solutions, Matveev solutions and complexitons of the (2 + 1)-dimensional AKNS equation are given through a matrix method for constructing double Wronskian entries. The three solutions are new.

Keywords

(2 + 1)-Dimensional AKNS Equation, Rational Solutions, Matveev Solutions, Complexitons

1. Introduction

It is one of the most important topics to search for exact solutions of nonlinear evolution equations in soliton theory. Moreover, various methods have been developed, such as the inverse scattering transformation [1], the Darboux transformation [2], the Hirota method [3], the Wronskian technique [4] [5], source generation procedure [6] [7] and so on. In 1971, Hirota first proposed the formal perturbation technique to obtain N-soliton solution of the KdV equation. Satsuma gave the Wronskian representation of the N-soliton solution to the KdV equation [8]. Then the Wronskian technique was developed by Freeman and Nimmo [4] [5]. In 1992, Matveev introduced the generalized Wronskian to obtain another kind of exact solutions called Positons for the KdV equation [9]. Recently, Ma first introduced a new kind of exact solution called complexitons [10]. By using these methods, exact solutions of many nonlinear soliton equations are obtained [11]-[16].

The AKNS (Ablowitz-Kaup-Newell-Segur) equation is one of the most important physical models [17]-[19]. In 1997, Lou and Hu have obtained the (2 + 1)-dimensional AKNS equation from the inner parameter dependent symmetry constraints of the KP equation [20]. Moreover, Lou *et al*. have studied Painlev e' integrability of the

How to cite this paper: Sun, Y.P. (2015) New Exact Solutions of the (2 + 1)-Dimensional AKNS Equation. *Journal of Applied Mathematics and Physics*, **3**, 1391-1405. http://dx.doi.org/10.4236/jamp.2015.311167

(2 + 1)-dimensional AKNS equation [21]. In this paper, we will apply the Hirota method and the Wronskian technique to obtain new exact solutions of the (2 + 1)-dimensional AKNS equation.

This paper is organized as follows. In Section 2, the bilinear form of the (2 + 1)-dimensional AKNS equation and its N-soliton solutions are obtained through the Hirota method. In Section 3, the double Wronskian solution and generalized double Wronskian solution are constructed by using the Wronskian technique. In Sections 4 and 5, rational solutions and Matveev solutions are given. In Section 6, complexitons of the (2 + 1)-dimensional AKNS equation are provided. Finally, we give some conclusions.

2. N-Soliton Solutions of the (2 + 1)-Dimensional AKNS Equation

We consider the following (2 + 1)-dimensional AKNS equation [21]

$$p_t + p_{xx} + pu_x = 0,\quad q_t - q_{xx} - qu_x = 0,\quad u_y + 2pq = 0. \tag{2.1}$$

Through the dependent variable transformation

$$p = \frac{g}{f},\ q = \frac{h}{f},\ u = 2\frac{f_x}{f}, \tag{2.2}$$

Equation (2.1) is transformed into the following bilinear form

$$\left(D_t + D_x^2\right) g \cdot f = 0, \tag{2.3a}$$

$$\left(D_t - D_x^2\right) h \cdot f = 0, \tag{2.3b}$$

$$D_x D_y f \cdot f + 2gh = 0, \tag{2.3c}$$

where D is the well-known Hirota bilinear operator defined by

$$D_t^m D_x^n f \cdot g = \left(\partial_t - \partial_{t'}\right)^m \left(\partial_x - \partial_{x'}\right)^n f(t,x) g(t',x')\Big|_{t'=t,\,x'=x}.$$

Expanding f, g and h as the series

$$f(t,x,y) = 1 + f^{(2)}\varepsilon^2 + f^{(4)}\varepsilon^4 + \cdots + f^{(2j)}\varepsilon^{2j} + \cdots, \tag{2.4a}$$

$$g(t,x,y) = g^{(1)}\varepsilon + g^{(3)}\varepsilon^3 + \cdots + g^{(2j+1)}\varepsilon^{2j+1} + \cdots, \tag{2.4b}$$

$$h(t,x,y) = h^{(1)}\varepsilon + h^{(3)}\varepsilon^3 + \cdots + h^{(2j+1)}\varepsilon^{2j+1} + \cdots, \tag{2.4c}$$

substituting Equation (2.4) into (2.3) and comparing the coefficients of the same power of ε yields

$$g_t^{(1)} + g_{xx}^{(1)} = 0,\quad g_t^{(3)} + g_{xx}^{(3)} = -\left(D_t + D_x^2\right) g^{(1)} \cdot f^{(2)},$$

$$g_t^{(5)} + g_{xx}^{(5)} = -\left(D_t + D_x^2\right)\left(g^{(1)} \cdot f^{(4)} + g^{(3)} \cdot f^{(2)}\right), \cdots,$$

$$h_t^{(1)} - h_{xx}^{(1)} = 0,\quad h_t^{(3)} - h_{xx}^{(3)} = -\left(D_t - D_x^2\right) h^{(1)} \cdot f^{(2)},$$

$$h_t^{(5)} - h_{xx}^{(5)} = -\left(D_t - D_x^2\right)\left(h^{(1)} \cdot f^{(4)} + h^{(3)} \cdot f^{(2)}\right), \cdots,$$

$$f_{xy}^{(2)} = -g^{(1)}h^{(1)},\quad 2f_{xy}^{(4)} = -D_x D_y f^{(2)} \cdot f^{(2)} - 2\left(g^{(1)}h^{(3)} + g^{(3)}h^{(1)}\right),$$

$$f_{xy}^{(6)} = -D_x D_y f^{(2)} \cdot f^{(4)} - \left(g^{(1)}h^{(5)} + g^{(3)}h^{(3)} + g^{(5)}h^{(1)}\right), \cdots.$$

Taking

$$g^{(1)} = \mathrm{e}^{\xi_1},\quad \xi_1 = \omega_1 t + k_1 x + l_1 y + \xi_1^{(0)}, \tag{2.5a}$$

$$h^{(1)} = \mathrm{e}^{\eta_1},\quad \eta_1 = \alpha_1 t + \beta_1 x + \gamma_1 y + \eta_1^{(0)}. \tag{2.5b}$$

we can obtain

$$f^{(2)} = -\frac{1}{(k_1+\beta_1)(l_1+\gamma_1)}e^{\xi_1+\eta_1}, \quad f^{(m)} = 0, \quad m = 4,6,\cdots,$$

$$g^{(n)} = 0, \ h^{(n)} = 0, \ n = 3,5,\cdots, \ \omega_1 = -k_1^2, \ \alpha_1 = \beta_1^2.$$

Letting $\varepsilon = 1$, then $g_1 = g^{(1)}$, $h_1 = h^{(1)}$, $f_1 = 1 + f^{(2)}$. Thus, the one-soliton solution is given as follows.

$$p = \frac{e^{\xi_1}}{1+e^{\xi_1+\eta_1+\theta_{13}}}, \quad q = \frac{e^{\eta_1}}{1+e^{\xi_1+\eta_1+\theta_{13}}}, \quad u = \frac{2(k_1+\beta_1)e^{\xi_1+\eta_1+\theta_{13}}}{1+e^{\xi_1+\eta_1+\theta_{13}}}, \tag{2.6}$$

where $e^{\theta_{13}} = -\frac{1}{(k_1+\beta_1)(l_1+\gamma_1)}$.

In the same way, we can obtain the following N-soliton solutions of Equation (2.3).

$$f_n = \sum_{\mu=0,1} A_1(\mu)\exp\left[\sum_{j=1}^{2n}\mu_j\xi_j + \sum_{1\le j<\rho}^{2n}\mu_j\mu_\rho\theta_{j\rho}\right], \tag{2.7a}$$

$$g_n = \sum_{\mu=0,1} A_2(\mu)\exp\left[\sum_{j=1}^{2n}\mu_j\xi_j + \sum_{1\le j<\rho}^{2n}\mu_j\mu_\rho\theta_{j\rho}\right], \tag{2.7b}$$

$$h_n = \sum_{\mu=0,1} A_3(\mu)\exp\left[\sum_{j=1}^{2n}\mu_j\xi_j + \sum_{1\le j<\rho}^{2n}\mu_j\mu_\rho\theta_{j\rho}\right], \tag{2.7c}$$

where

$$\omega_j = -k_j^2, \ \alpha_j = \beta_j^2, \ \xi_{n+j} = \eta_j, \ (j = 1,\cdots,n), \tag{2.8a}$$

$$e^{\theta_{j\rho}} = 2(k_j - k_\rho)(l_j - l_\rho), \ (j < \rho = 2,3,\cdots,n), \tag{2.8b}$$

$$e^{\theta_{(n+j)(n+\rho)}} = 2(\beta_j - \beta_\rho)(\gamma_j - \gamma_\rho), \ (j < \rho = 2,3,\cdots,n), \tag{2.8c}$$

$$e^{\theta_{j,(n+\rho)}} = -\frac{1}{(k_j+\beta_\rho)(l_j+\gamma_\rho)}, \ (j,\rho = 1,2,\cdots,n), \tag{2.8d}$$

$A_1(\mu)$, $A_2(\mu)$ and $A_3(\mu)$ take over all possible combinations of $\mu_j = 0,1(j = 1,2,\cdots,2n)$ and satisfy the following condition

$$\sum_{j=1}^{n}\mu_j = \sum_{j=1}^{n}\mu_{n+j}, \quad \sum_{j=1}^{n}\mu_j = 1 + \sum_{j=1}^{n}\mu_{n+j}, \quad 1 + \sum_{j=1}^{n}\mu_j = \sum_{j=1}^{n}\mu_{n+j}.$$

3. The Double Wronskian Solution and Generalized Double Wronskian Solution

Let us first specify some properties of the Wronskian determinant. As is well known, the double Wronskian determinant is

$$W^{N,M}(\varphi;\psi) = det\left(\varphi, \partial_x\varphi, \cdots, \partial_x^{N-1}\varphi; \psi, \partial_x\psi, \cdots, \partial_x^{M-1}\psi\right),$$

where $\varphi = (\varphi_1(x), \varphi_2(x), \cdots, \varphi_{N+M}(x))^{\mathrm{T}}$ and $\psi = (\psi_1(x), \psi_2(x), \cdots, \psi_{N+M}(x))^{\mathrm{T}}$. The following two determinantal identities were often used [4] [5]. The one is

$$|D,a,b||D,c,d| - |D,a,c||D,b,d| + |D,a,d||D,b,c| = 0, \tag{3.1}$$

where D is a $N \times (N-2)$ matrix and a,b,c and d represent N column vectors. The other is

$$\sum_{j=1}^{N}|\alpha_1,\cdots,\alpha_{j-1},b\alpha_j,\alpha_{j+1},\cdots,\alpha_N| = \left(\sum_{j=1}^{N}b_j\right)|\alpha_1,\cdots,\alpha_N|, \tag{3.2}$$

where $\alpha_j\left(1\le j\le N\right)$ are N column vectors and $b\alpha_j$ denotes $\left(b_1\alpha_{1j},b_2\alpha_{2j},\cdots,b_N\alpha_{Nj}\right)^{\mathrm{T}}$.

Employing the Wronskian technique, we have the following result.

Theorem 1. *The* (2 + 1)-*dimensional AKNS Equation* (2.3) *has the double Wronskian solution*

$$g=2W^{N+2,M}\left(\varphi;\psi\right),\ f=W^{N+1,M+1}\left(\varphi;\psi\right),\ h=-2W^{N,M+2}\left(\varphi;\psi\right), \tag{3.3}$$

where φ_j and ψ_j satisfy the following conditions

$$\varphi_{j,x}=-\frac{1}{2}\lambda_j\varphi_j,\ \varphi_{j,t}=-2\varphi_{j,xx},\ \varphi_{j,y}=\varphi_{j,x},\ \left(j=1,2,\cdots,N+M+2\right), \tag{3.4a}$$

$$\psi_{l,x}=\frac{1}{2}\lambda_l\psi_l,\ \psi_{l,t}=2\psi_{l,xx},\ \psi_{l,y}=\psi_{l,x},\ \left(l=1,2,\cdots,N+M+2\right). \tag{3.4b}$$

Proof. In the following, we use the abbreviated notation of Freeman and Nimmo for the Wronskian and its derivatives [4] [5], then Equation (3.3) becomes

$$g=2\left|\widehat{N+1};\widehat{M-1}\right|,\ f=\left|\widehat{N};\widehat{M}\right|,\ h=-2\left|\widehat{N-1};\widehat{M+1}\right|. \tag{3.5}$$

First, we calculate various derivatives of *g* and *f* with respect to *x* and *t*.

$$f_x=\left|\widehat{N-1},N+1;\widehat{M}\right|+\left|\widehat{N};\widehat{M-1},M+1\right|,$$

$$\begin{aligned}f_{xx}=&\left|\widehat{N-2},N,N+1;\widehat{M}\right|+\left|\widehat{N-1},N+2;\widehat{M}\right|+2\left|\widehat{N-1},N+1;\widehat{M-1},M+1\right|\\&+\left|\widehat{N};\widehat{M-2},M,M+1\right|+\left|\widehat{N};\widehat{M-1},M+2\right|,\end{aligned}$$

$$g_x=2\left|\widehat{N},N+2;\widehat{M-1}\right|+2\left|\widehat{N+1};\widehat{M-2},M\right|,$$

$$\begin{aligned}g_{xx}=&2\left|\widehat{N-1},N+1,N+2;\widehat{M-1}\right|+2\left|\widehat{N},N+3;\widehat{M-1}\right|+4\left|\widehat{N},N+2;\widehat{M-2},M\right|\\&+2\left|\widehat{N+1};\widehat{M-3},M-1,M\right|+2\left|\widehat{N+1};\widehat{M-2},M+1\right|,\end{aligned}$$

$$f_t=-2\left(\left|\widehat{N-2},N+1,N;\widehat{M}\right|+\left|\widehat{N-1},N+2;\widehat{M}\right|\right)+2\left(\left|\widehat{N};\widehat{M-2},M+1,M\right|+\left|\widehat{N};\widehat{M-1},M+2\right|\right),$$

$$\begin{aligned}g_t=&-4\left(\left|\widehat{N-1},N+2,N+1;\widehat{M-1}\right|+\left|\widehat{N},N+3;\widehat{M-1}\right|\right)\\&+4\left(\left|\widehat{N+1};\widehat{M-3},M,M-1\right|+\left|\widehat{N+1};\widehat{M-2},M+1\right|\right).\end{aligned}$$

Then a direct calculation gives

$$\begin{aligned}g_tf-f_tg+g_{xx}f-2g_xf_x+gf_{xx}=&6\left|\widehat{N};\widehat{M}\right|\left|\widehat{N-1},N+1,N+2;\widehat{M-1}\right|\\&-2\left|\widehat{N};\widehat{M}\right|\left(\left|\widehat{N},N+3;\widehat{M-1}\right|-2\left|\widehat{N},N+2;\widehat{M-2},M\right|+\left|\widehat{N+1};\widehat{M-3},M-1,M\right|\right)\\&+6\left|\widehat{N};\widehat{M}\right|\left|\widehat{N+1};\widehat{M-2},M+1\right|-2\left|\widehat{N-2},N,N+1;\widehat{M}\right|\left|\widehat{N+1};\widehat{M-1}\right|\\&+2\left|\widehat{N+1};\widehat{M-1}\right|\left(3\left|\widehat{N-1},N+2;\widehat{M}\right|+2\left|\widehat{N-1},N+1;\widehat{M-1},M+1\right|\right)\\&+2\left|\widehat{N+1};\widehat{M-1}\right|\left(3\left|\widehat{N};\widehat{M-2},M,M+1\right|-\left|\widehat{N};\widehat{M-1},M+2\right|\right)\\&-4\left|\widehat{N},N+2;\widehat{M-1}\right|\left(\left|\widehat{N-1},N+1;\widehat{M}\right|+\left|\widehat{N};\widehat{M-1},M+1\right|\right)\\&-4\left|\widehat{N+1};\widehat{M-2},M\right|\left(\left|\widehat{N-1},N+1;\widehat{M}\right|+\left|\widehat{N};\widehat{M-1},M+1\right|\right).\end{aligned} \tag{3.6}$$

Utilizing Equation (3.2) and Equation (3.4), we get

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N},N+2;\widehat{M-1}\right|=-\left|\widehat{N-1},N+1,N+2;\widehat{M-1}\right|-\left|\widehat{N},N+3;\widehat{M-1}\right|+\left|\widehat{N},N+2;\widehat{M-2},M\right|, \tag{3.7a}$$

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N+1};\widehat{M-2},M\right|=-\left|\widehat{N},N+2;\widehat{M-2},M\right|+\left|\widehat{N+1};\widehat{M-3},M-1,M\right|+\left|\widehat{N+1};\widehat{M-2},M+1\right|, \tag{3.7b}$$

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N-1},N+1;\widehat{M}\right|=-\left|\widehat{N-2},N,N+1;\widehat{M}\right|-\left|\widehat{N-1},N+2;\widehat{M}\right|+\left|\widehat{N-1},N+1;\widehat{M-1},M+1\right|, \tag{3.7c}$$

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N};\widehat{M-1},M+1\right|=-\left|\widehat{N-1},N+1;\widehat{M-1},M+1\right|+\left|\widehat{N};\widehat{M-2},M,M+1\right|+\left|\widehat{N};\widehat{M-1},M+2\right|. \tag{3.7d}$$

Noting

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N},N+2;\widehat{M-1}\right|\left|\widehat{N};\widehat{M}\right|=\left|\widehat{N},N+2;\widehat{M-1}\right|\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N};\widehat{M}\right|, \tag{3.8a}$$

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N+1};\widehat{M-2},M\right|\left|\widehat{N};\widehat{M}\right|=\left|\widehat{N+1};\widehat{M-2},M\right|\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N};\widehat{M}\right|, \tag{3.8b}$$

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N-1},N+1;\widehat{M}\right|\left|\widehat{N+1};\widehat{M-1}\right|=\left|\widehat{N-1},N+1;\widehat{M}\right|\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N+1};\widehat{M-1}\right|, \tag{3.8c}$$

$$\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N};\widehat{M-1},M+1\right|\left|\widehat{N+1};\widehat{M-1}\right|=\left|\widehat{N};\widehat{M-1},M+1\right|\left(\sum\frac{1}{2}\lambda_j\right)\left|\widehat{N+1};\widehat{M-1}\right|. \tag{3.8d}$$

Using Equation (3.7) and Equation (3.8), then Equation (3.6) becomes

$$\begin{aligned}&g_t f-f_t g+g_{xx}f-2g_x f_x+gf_{xx}\\&=8\left|\widehat{N-1},N+1,N+2;\widehat{M-1}\right|\left|\widehat{N};\widehat{M}\right|+8\left|\widehat{N};\widehat{M-2},M,M+1\right|\left|\widehat{N+1};\widehat{M-1}\right|\\&-8\left|\widehat{N},N+2;\widehat{M-1}\right|\left|\widehat{N-1},N+1;\widehat{M}\right|+8\left|\widehat{N+1};\widehat{M-2},M+1\right|\left|\widehat{N};\widehat{M}\right|\\&+8\left|\widehat{N-1},N+2;\widehat{M}\right|\left|\widehat{N+1};\widehat{M-1}\right|-8\left|\widehat{N+1};\widehat{M-2},M\right|\left|\widehat{N};\widehat{M-1},M+1\right|.\end{aligned} \tag{3.9}$$

According to (3.1), it is easy to see that Equation (3.9) is equal to zero. So, the proof of Equation (2.3a) is completed. Similarly Equations (2.3 b) and (2.3 c) can also be proved.

In the following, we give some exact solutions. From Equation (3.4), we deduce that

$$\varphi_j=\mathrm{e}^{-\xi_j}c_j,\ \psi_j=\mathrm{e}^{\xi_j}d_j,\ \xi_j=\frac{\lambda_j}{2}x+\frac{\lambda_j}{2}y+\frac{\lambda_j^2}{2}t, \tag{3.10}$$

where c_j and d_j $(j=1,2,\cdots,N+M+2)$ are arbitrary real constants.

Taking $c_j=d_j=1$, the double Wronskian solution of Equation (2.3) is obtained as follows:

$$f=\left|\mathrm{e}^{-\xi_j},\partial_x\mathrm{e}^{-\xi_j},\cdots,\partial_x^N\mathrm{e}^{-\xi_j};\mathrm{e}^{\xi_j},\partial_x\mathrm{e}^{\xi_j},\cdots,\partial_x^M\mathrm{e}^{\xi_j}\right|,$$

$$g=2\left|\mathrm{e}^{-\xi_j},\partial_x\mathrm{e}^{-\xi_j},\cdots,\partial_x^{N+1}\mathrm{e}^{-\xi_j};\mathrm{e}^{\xi_j},\partial_x\mathrm{e}^{\xi_j},\cdots,\partial_x^{M-1}\mathrm{e}^{\xi_j}\right|,$$

$$h=-2\left|\mathrm{e}^{-\xi_j},\partial_x\mathrm{e}^{-\xi_j},\cdots,\partial_x^{N-1}\mathrm{e}^{-\xi_j};\mathrm{e}^{\xi_j},\partial_x\mathrm{e}^{\xi_j},\cdots,\partial_x^{M+1}\mathrm{e}^{\xi_j}\right|.$$

Letting $N=0$ and $M=0$ gives

$$f=\mathrm{e}^{\xi_2-\xi_1}-\mathrm{e}^{\xi_1-\xi_2},\ g=(\lambda_1-\lambda_2)\mathrm{e}^{-\xi_1-\xi_2},\ h=(\lambda_1-\lambda_2)\mathrm{e}^{\xi_1+\xi_2},$$

then one-soliton solution of Equation (2.1) is

$$p=\frac{1}{2}(\lambda_1-\lambda_2)\frac{e^{-\xi_1-\xi_2}}{\sinh(\xi_2-\xi_1)},\ q=\frac{1}{2}(\lambda_1-\lambda_2)\frac{e^{\xi_1+\xi_2}}{\sinh(\xi_2-\xi_1)},$$

$$u=\frac{1}{2}(\lambda_2-\lambda_1)\frac{e^{\xi_2-\xi_1}}{\sinh(\xi_2-\xi_1)}-\frac{1}{2}(\lambda_1-\lambda_2)\frac{e^{\xi_1-\xi_2}}{\sinh(\xi_2-\xi_1)}.$$

Choosing $N=1$ and $M=0$ yields

$$f=\frac{1}{2}(\lambda_1-\lambda_2)e^{-\xi_1-\xi_2+\xi_3}+\frac{1}{2}(\lambda_3-\lambda_1)e^{-\xi_1+\xi_2-\xi_3}+\frac{1}{2}(\lambda_2-\lambda_3)e^{\xi_1-\xi_2-\xi_3},$$

$$g=\frac{1}{4}(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)e^{-\xi_1-\xi_2-\xi_3},$$

$$h=(\lambda_1-\lambda_2)e^{\xi_1+\xi_2-\xi_3}+(\lambda_2-\lambda_3)e^{-\xi_1+\xi_2+\xi_3}+(\lambda_3-\lambda_1)e^{\xi_1-\xi_2+\xi_3}.$$

So, we have

$$p=\frac{1}{2}\frac{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)e^{-\xi_1-\xi_2-\xi_3}}{(\lambda_1-\lambda_2)e^{-\xi_1-\xi_2+\xi_3}+(\lambda_3-\lambda_1)e^{-\xi_1+\xi_2-\xi_3}+(\lambda_2-\lambda_3)e^{\xi_1-\xi_2-\xi_3}},$$

$$q=2\frac{(\lambda_1-\lambda_2)e^{\xi_1+\xi_2-\xi_3}+(\lambda_2-\lambda_3)e^{-\xi_1+\xi_2+\xi_3}+(\lambda_3-\lambda_1)e^{\xi_1-\xi_2+\xi_3}}{(\lambda_1-\lambda_2)e^{-\xi_1-\xi_2+\xi_3}+(\lambda_3-\lambda_1)e^{-\xi_1+\xi_2-\xi_3}+(\lambda_2-\lambda_3)e^{\xi_1-\xi_2-\xi_3}},$$

$$u=\frac{1}{2}\left[\frac{(\lambda_1-\lambda_2)(\lambda_3-\lambda_1-\lambda_2)e^{-\xi_1-\xi_2+\xi_3}+(\lambda_3-\lambda_1)(\lambda_2-\lambda_1-\lambda_3)e^{-\xi_1+\xi_2-\xi_3}}{(\lambda_1-\lambda_2)e^{-\xi_1-\xi_2+\xi_3}+(\lambda_3-\lambda_1)e^{-\xi_1+\xi_2-\xi_3}+(\lambda_2-\lambda_3)e^{\xi_1-\xi_2-\xi_3}}\right.$$
$$\left.+\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_2-\lambda_3)e^{\xi_1-\xi_2-\xi_3}}{(\lambda_1-\lambda_2)e^{-\xi_1-\xi_2+\xi_3}+(\lambda_3-\lambda_1)e^{-\xi_1+\xi_2-\xi_3}+(\lambda_2-\lambda_3)e^{\xi_1-\xi_2-\xi_3}}\right].$$

Similarly, when $N=0$ and $M=1$, we get

$$p=-2\frac{(\lambda_1-\lambda_2)e^{-\xi_1-\xi_2+\xi_3}+(\lambda_3-\lambda_1)e^{-\xi_1+\xi_2-\xi_3}+(\lambda_2-\lambda_3)e^{\xi_1-\xi_2-\xi_3}}{(\lambda_1-\lambda_2)e^{\xi_1+\xi_2-\xi_3}+(\lambda_3-\lambda_1)e^{\xi_1-\xi_2+\xi_3}+(\lambda_2-\lambda_3)e^{-\xi_1+\xi_2+\xi_3}},$$

$$q=\frac{1}{2}\frac{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)e^{\xi_1+\xi_2+\xi_3}}{(\lambda_1-\lambda_2)e^{\xi_1+\xi_2-\xi_3}+(\lambda_3-\lambda_1)e^{\xi_1-\xi_2+\xi_3}+(\lambda_2-\lambda_3)e^{-\xi_1+\xi_2+\xi_3}},$$

$$u=\frac{1}{2}\left[\frac{(\lambda_1-\lambda_2)(\lambda_1+\lambda_2-\lambda_3)e^{\xi_1+\xi_2-\xi_3}+(\lambda_3-\lambda_1)(\lambda_1-\lambda_2+\lambda_3)e^{\xi_1-\xi_2+\xi_3}}{(\lambda_1-\lambda_2)e^{\xi_1+\xi_2-\xi_3}+(\lambda_3-\lambda_1)e^{\xi_1-\xi_2+\xi_3}+(\lambda_2-\lambda_3)e^{-\xi_1+\xi_2+\xi_3}}\right.$$
$$\left.+\frac{(\lambda_2-\lambda_3)(\lambda_2+\lambda_3-\lambda_1)e^{-\xi_1+\xi_2+\xi_3}}{(\lambda_1-\lambda_2)e^{\xi_1+\xi_2-\xi_3}+(\lambda_3-\lambda_1)e^{\xi_1-\xi_2+\xi_3}+(\lambda_2-\lambda_3)e^{-\xi_1+\xi_2+\xi_3}}\right].$$

In the following, we will prove that Equation (2.3) has the generalized double Wronskian solution. First, we give the following lemma [19].

Lemma 1. *Assume that* $P=(p_{ij})$ *is an* $l\times l$ *operator matrix and its entries* p_{ij} *are differential operators.* $B=(b_{ij})$ *is an* $l\times l$ *function matrix with column vector set* b_i *and row vector set* b'_j $(i=1,2,\cdots,l;\ j=1,2,\cdots,l)$, *then*

$$\sum_{i=1}^{l}|b_1,\cdots,p_ib_i,\cdots,b_l|=\sum_{j=1}^{l}\begin{vmatrix}b'_1\\ \vdots\\ p'_jb'_j\\ \vdots\\ b'_l\end{vmatrix},\tag{3.11}$$

where $p_i b_i = \left(p_{1i}b_{1i}, p_{2i}b_{2i}, \cdots, p_{li}b_{li}\right)^{\mathrm{T}}$, $p'_j b'_j = \left(p'_{j1}b'_{j1}, p'_{j2}b'_{j2}, \cdots, p'_{jl}b'_{jl}\right)$.

Using the Lemma 1 and the Wronskian technique, we construct the following result.

Theorem 2. *The* (2 + 1)-*dimensional AKNS Equation* (2.3) *has the generalized double Wronskian solution*

$$g = 2W^{N+2,M}(\varphi;\psi),\ f = W^{N+1,M+1}(\varphi;\psi),\ h = -2W^{N,M+2}(\varphi;\psi), \tag{3.12}$$

where φ_j and ψ_j satisfy the following conditions

$$\varphi_{j,x} = -A\varphi_j,\ \varphi_{j,t} = -2\varphi_{j,xx},\ \varphi_{j,y} = \varphi_{j,x},\ \left(j = 1,2,\cdots,N+M+2\right), \tag{3.13a}$$

$$\psi_{l,x} = A\psi_l,\quad \psi_{l,t} = 2\psi_{l,xx},\quad \psi_{l,y} = \psi_{l,x},\quad \left(l = 1,2,\cdots,N+M+2\right), \tag{3.13b}$$

$A = \left(a_{ij}\right)$ is an $\left(N+M+2\right)\times\left(N+M+2\right)$ arbitrary real matrix independent of *x* and *t*.

In fact, similar the proof of Theorem 1, we only need to verify that identities (3.7) hold.

(1) If $trA \neq 0$, setting

$$p_{ij} = \begin{cases} -\partial_x & 1 \le i \le N+M+2; 1 \le j \le N+1; \\ \partial_x & 1 \le i \le N+M+2;\ N+2 \le j \le N+M+2, \end{cases}$$

from Lemma 1, we can get

$$\begin{aligned} &\sum_{j=1}^{N+M+2} \begin{vmatrix} \varphi_1 & \cdots & \partial_x^N \varphi_1 & \psi_1 & \partial_x \psi_1 & \cdots & \partial_x^M \psi_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\partial_x \varphi_j & \cdots & -\partial_x\left(\partial_x^N \varphi_j\right) & \partial_x \psi_j & \partial_x\left(\partial_x \psi_j\right) & \cdots & \partial_x\left(\partial_x^M \psi_j\right) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{N+M+2} & \cdots & \partial_x^N \varphi_{N+M+2} & \psi_{N+M+2} & \partial_x \psi_{N+M+2} & \cdots & \partial_x^M \psi_{N+M+2} \end{vmatrix} \\ &= \left|\widehat{N}; \widehat{M-1}, M+1\right| - \left|\widehat{N-1}, N+1; \widehat{M}\right|. \end{aligned} \tag{3.14}$$

Using Equation (3.13), the left-hand side of (3.14) is equal to

$$\sum_{j=1}^{N+M+2} \begin{vmatrix} \varphi_1 & \cdots & \partial_x^N \varphi_1 & \psi_1 & \cdots & \partial_x^M \psi_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{l=1}^{N+M+2} a_{jl}\varphi_l & \cdots & \sum_{l=1}^{N+M+2} a_{jl}\partial_x^N \varphi_l & \sum_{l=1}^{N+M+2} a_{jl}\psi_l & \cdots & \sum_{l=1}^{N+M+2} a_{jl}\partial_x^M \psi_l \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{N+M+2} & \cdots & \partial_x^N \varphi_{N+M+2} & \psi_{N+M+2} & \cdots & \partial_x^M \psi_{N+M+2} \end{vmatrix} = \sum_{j=1}^{N+M+2} a_{jj} \left|\widehat{N}; \widehat{M}\right|.$$

Therefore,

$$trA\left|\widehat{N}; \widehat{M}\right| = \left|\widehat{N}; \widehat{M-1}, M+1\right| - \left|\widehat{N-1}, N+1; \widehat{M}\right|. \tag{3.15}$$

From (3.15), we derive further

$$trA\left|\widehat{N}, N+2; \widehat{M-1}\right| = -\left|\widehat{N-1}, N+1, N+2; \widehat{M-1}\right| - \left|\widehat{N}, N+3; \widehat{M-1}\right| + \left|\widehat{N}, N+2; \widehat{M-2}, M\right|, \tag{3.16a}$$

$$trA\left|\widehat{N+1}; \widehat{M-2}, M\right| = -\left|\widehat{N}, N+2; \widehat{M-2}, M\right| + \left|\widehat{N+1}; \widehat{M-3}, M-1, M\right| + \left|\widehat{N+1}; \widehat{M-2}, M+1\right|, \tag{3.16b}$$

$$trA\left|\widehat{N-1}, N+1; \widehat{M}\right| = -\left|\widehat{N-2}, N, N+1; \widehat{M}\right| - \left|\widehat{N-1}, N+2; \widehat{M}\right| + \left|\widehat{N-1}, N+1; \widehat{M-1}, M+1\right|, \tag{3.16c}$$

$$trA\left|\widehat{N}; \widehat{M-1}, M+1\right| = -\left|\widehat{N-1}, N+1; \widehat{M-1}, M+1\right| + \left|\widehat{N}; \widehat{M-2}, M, M+1\right| + \left|\widehat{N}; \widehat{M-1}, M+2\right|, \tag{3.16d}$$

$$\begin{aligned} \left(trA\right)^2\left|\widehat{N}; \widehat{M}\right| &= \left|\widehat{N-2}, N, N+1; \widehat{M}\right| + \left|\widehat{N-1}, N+2; \widehat{M}\right| - 2\left|\widehat{N-1}, N+1; \widehat{M-1}, M+1\right| \\ &+ \left|\widehat{N}; \widehat{M-2}, M, M+1\right| + \left|\widehat{N}; \widehat{M-1}, M+2\right|. \end{aligned} \tag{3.17}$$

It is obvious that (3.7) hold.

(2) If $trA=0$, we can consider this as a limit case where trA tends to zero. Then (3.15)-(3.17) become

$$\left|\widehat{N};\widehat{M-1},M+1\right|=\left|\widehat{N-1},N+1;\widehat{M}\right|, \tag{3.18a}$$

$$\left|\widehat{N},N+2;\widehat{M-2},M\right|=\left|\widehat{N-1},N+1,N+2;\widehat{M-1}\right|+\left|\widehat{N},N+3;\widehat{M-1}\right|, \tag{3.18b}$$

$$\left|\widehat{N+1};\widehat{M-2},M+1\right|=\left|\widehat{N},N+2;\widehat{M-2},M\right|-\left|\widehat{N+1};\widehat{M-3},M-1,M\right|, \tag{3.18c}$$

$$\left|\widehat{N-1},N+1;\widehat{M-1},M+1\right|=\left|\widehat{N+1},N,N+1;\widehat{M}\right|+\left|\widehat{N-1},N+2;\widehat{M}\right|, \tag{3.18d}$$

$$\left|\widehat{N};\widehat{M-1},M+2\right|=\left|\widehat{N-1},N+1;\widehat{M-1},M+1\right|-\left|\widehat{N};\widehat{M-2},M,M+1\right|, \tag{3.18e}$$

$$\begin{aligned}&\left|\widehat{N-2},N,N+1;\widehat{M}\right|+\left|\widehat{N-1},N+2;\widehat{M}\right|-2\left|\widehat{N-1},N+1;\widehat{M-1},M+1\right|\\&=-\left|\widehat{N};\widehat{M-2},M,M+1\right|-\left|\widehat{N};\widehat{M-1},M+2\right|.\end{aligned} \tag{3.18f}$$

Using (3.18), Equation (3.12) still satisfies Equation (2.3).

From Equation (3.13), we can get the general solution

$$\varphi=\mathrm{e}^{-2A^2t-Ax-Ay}C,\ \psi=\mathrm{e}^{-2A^2t-Ax-Ay}D, \tag{3.19}$$

where $C=\left(c_1,c_2,\cdots,c_{N+M+2}\right)^{\mathrm{T}}$ and $D=\left(d_1,d_2,\cdots,d_{N+M+2}\right)^{\mathrm{T}}$ are real constant vectors. Thus, we have the following result.

Theorem 3. *$A=\left(a_{ij}\right)$ is an $\left(N+M+2\right)\times\left(N+M+2\right)$ arbitrary real matrix independent of x and t. Equation* (2.3) *has double Wronskian solution* (3.12), *where φ and ψ are constructed by* (3.19). *The corresponding solution of Equation* (2.1) *can be expressed as*

$$p=2\frac{W^{N+2,M}\left(\varphi;\psi\right)}{W^{N+1,M+1}\left(\varphi;\psi\right)},\ q=-2\frac{W^{N,M+2}\left(\varphi;\psi\right)}{W^{N+1,M+1}\left(\varphi;\psi\right)},\ u=2\left[\ln W^{N+1,M+1}\left(\varphi;\psi\right)\right]_x. \tag{3.20}$$

4. Rational Solutions

In the section, we will give rational solutions of the (2 + 1)-dimensional AKNS Equation (2.1).

Expanding (3.19) leads to

$$\varphi=\mathrm{e}^{-2A^2t}\mathrm{e}^{-Ax-Ay}C=\sum_{s=0}^{\infty}\left[\sum_{l=0}^{\left[\frac{s}{2}\right]}\frac{(-1)^{s-l}\,2^l}{l!\left(s-2l\right)!}t^l\left(x+y\right)^{s-2l}\right]A^sC, \tag{4.1a}$$

$$\psi=\mathrm{e}^{2A^2t}\mathrm{e}^{Ax+Ay}D=\sum_{s=0}^{\infty}\left[\sum_{l=0}^{\left[\frac{s}{2}\right]}\frac{2^l}{l!\left(s-2l\right)!}t^l\left(x+y\right)^{s-2l}\right]A^sD. \tag{4.1b}$$

If

$$A=\begin{pmatrix}k_1 & & & 0\\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_{N+M+2}\end{pmatrix},\ k_i\neq k_j\left(i\neq j\right), \tag{4.2}$$

we can obtain solution solutions of Equation (2.3), where

$$\varphi_j=c_j\mathrm{e}^{-2k_j^2t-k_jx-k_jy},\ \psi_j=d_j\mathrm{e}^{2k_j^2t+k_jx+k_jy}\left(j=1,2,\cdots,N+M+2\right). \tag{4.3}$$

If

$$A=\begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}_{(N+M+2)\times(N+M+2)}, \tag{4.4}$$

it is obvious to know that $A^{N+M+2}=0$. Thus (4.1) can be truncated as

$$\varphi=\sum_{s=0}^{N+M+1}\left[\sum_{l=0}^{\left[\frac{s}{2}\right]}\frac{(-1)^{s-l}2^l}{l!(s-2l)!}t^l(x+y)^{s-2l}\right]A^sC, \tag{4.5a}$$

$$\psi=\sum_{s=0}^{N+M+1}\left[\sum_{l=0}^{\left[\frac{s}{2}\right]}\frac{2^l}{l!(s-2l)!}t^l(x+y)^{s-2l}\right]A^sD. \tag{4.5b}$$

The components of φ and ψ are

$$\varphi_j=c_j-c_{j-1}(x+y)+c_{j-2}\left(-2t+\frac{(x+y)^2}{2}\right)+\cdots+c_1\sum_{l=0}^{[\frac{j-1}{2}]}\frac{(-1)^{j-l-1}2^l}{l!(j-1-2l)!}t^l(x+y)^{j-1-2l}, \tag{4.6a}$$

$$\psi_j=d_j+d_{j-1}(x+y)+d_{j-2}\left(2t+\frac{(x+y)^2}{2}\right)+\cdots+d_1\sum_{l=0}^{[\frac{j-1}{2}]}\frac{2^l}{l!(j-1-2l)!}t^l(x+y)^{j-1-2l}, \tag{4.6b}$$

$$(j=1,2,\cdots,N+M+2).$$

In (4.6), taking $c_1=d_1=1$, $c_k=d_k=0\,(k=2,3,\cdots,N+M+2)$, then (4.6) becomes

$$\varphi_j=\sum_{l=0}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^{j-l-1}2^l}{l!(j-1-2l)!}t^l(x+y)^{j-1-2l},\quad \psi_j=\sum_{l=0}^{\left[\frac{j-1}{2}\right]}\frac{2^l}{l!(j-1-2l)!}t^l(x+y)^{j-1-2l}. \tag{4.7}$$

Thus, we can calculate some rational solutions of Equation (2.1).

$$p=-\frac{1}{x+y},\quad q=-\frac{1}{x+y},\quad u=\frac{2}{x+y}, \tag{4.8}$$

$$p=\frac{1}{(x+y)^2+2t},\quad q=2\frac{(x+y)^2-2t}{(x+y)^2+2t},\quad u=4\frac{x+y}{(x+y)^2+2t}, \tag{4.9}$$

$$p=2\frac{2t+(x+y)^2}{2t-(x+y)^2},\quad q=\frac{1}{2t-(x+y)^2},\quad u=4\frac{x+y}{2t-(x+y)^2}. \tag{4.10}$$

5. Matveev Solutions

In the following, we will discuss Matveev solutions of the (2 + 1)-dimensional AKNS equation.

Let A be a Jordan matrix

$$A=\begin{pmatrix} J(k_1) & & & 0 \\ & J(k_2) & & \\ & & \ddots & \\ 0 & & & J(k_s) \end{pmatrix}_{(N+M+2)\times(N+M+2)}. \tag{5.1}$$

Without loss of generality, we observe the following Jordan block (dropping the subscript of *k*)

$$J(k)=\begin{pmatrix} k & & & 0 \\ 1 & k & & \\ & \ddots & \ddots & \\ 0 & & 1 & k \end{pmatrix}_{l_i\times l_i}=kI_{l_i}+E_{l_i},\ E_{l_i}=\begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}_{l_i\times l_i}, \tag{5.2}$$

where I_{l_i} is an $l_i\times l_i$ unite matrix. We have

$$J^s(k)=\left(kI_{l_i}+E_{l_i}\right)^s=\left(I_{l_i}+E_{l_i}\partial_k+\frac{1}{2!}E_{l_i}^2\partial_k^2+\cdots+\frac{1}{j!}E_{l_i}^j\partial_k^j+\cdots+\frac{1}{s!}E_{l_i}^s\partial_k^s\right)k^s, \tag{5.3a}$$

i.e.,

$$J^s(k)=T_k k^s,\ T_k=\begin{pmatrix} 1 & & & & & 0 \\ \partial_k & 1 & & & & \\ \frac{1}{2}\partial_k^2 & \partial_k & 1 & & & \\ \frac{1}{6}\partial_k^3 & \frac{1}{2}\partial_k^2 & \partial_k & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ \frac{1}{(l_i-1)!}\partial_k^{l_i-1} & \cdots & \frac{1}{6}\partial_k^3 & \frac{1}{2}\partial_k^2 & \partial_k & 1 \end{pmatrix}. \tag{5.3b}$$

Substituting (5.2) into (4.1), we get

$$\varphi(k)=T_k\mathrm{e}^{-2k^2t-kx-ky}C,\ \psi(k)=T_k\mathrm{e}^{2k^2t+kx+ky}D. \tag{5.4}$$

The components of $\varphi(k)$ and $\psi(k)$ are

$$\varphi_j(k)=\left(c_1\frac{1}{(j-1)!}\partial_k^{j-1}+\cdots+c_{j-1}\partial_k+c_j\right)\mathrm{e}^{-2k^2t-kx-ky},\ \left(j=1,2,\cdots,l_i\right), \tag{5.5a}$$

$$\psi_j(k)=\left(d_1\frac{1}{(j-1)!}\partial_k^{j-1}+\cdots+d_{j-1}\partial_k+d_j\right)\mathrm{e}^{2k^2t+kx+ky},\ \left(j=1,2,\cdots,l_i\right). \tag{5.5b}$$

Specially, taking $c_1=d_1=1,\ \ c_j=d_j=0\left(j=2,3,\cdots,l_i\right),$ then (5.5) becomes

$$\varphi_j(k)=\frac{1}{(j-1)!}\partial_k^{j-1}\mathrm{e}^{-2k^2t-kx-ky},\ \psi_j(k)=\frac{1}{(j-1)!}\partial_k^{j-1}\mathrm{e}^{2k^2t+kx+ky}. \tag{5.6}$$

Thus, Matveev solutions of Equation (2.1) can be obtained, where

$$\varphi=\left(\varphi_1(k_1),\cdots,\varphi_{l_1}(k_1);\varphi_1(k_2),\cdots,\varphi_{l_2}(k_2);\cdots,\varphi_1(k_s),\cdots,\varphi_{l_s}(k_s)\right)^{\mathrm{T}}, \tag{5.7a}$$

$$\psi=\left(\psi_1(k_1),\cdots,\psi_{l_1}(k_1);\psi_1(k_2),\cdots,\psi_{l_2}(k_2);\cdots,\psi_1(k_s),\cdots,\psi_{l_s}(k_s)\right)^{\mathrm{T}}, \tag{5.7b}$$

$$\left(l_1+l_2+\cdots+l_s=N+M+2\right).$$

In (5.7), taking

$$\varphi=\left(\varphi_1(k),\varphi_2(k)\right)^{\mathrm{T}},\ \psi=\left(\psi_1(k),\psi_2(k)\right)^{\mathrm{T}}, \tag{5.8}$$

where $\varphi_j(k)$ and $\psi_j(k)$ are generated from (5.6), we can obtain the Matveev solution of Equation (2.1).

$$p=-\frac{1}{4kt+x+y}\mathrm{e}^{-4k^2t-2kx-2ky},\ q=-\frac{1}{4kt+x+y}\mathrm{e}^{4k^2t+2kx+2ky},\ u=\frac{2}{4kt+x+y}. \tag{5.9}$$

Similarly, choosing

$$\varphi=\left(\varphi_1(k),\varphi_2(k),\varphi_3(k)\right)^{\mathrm{T}},\ \psi=\left(\psi_1(k),\psi_2(k),\psi_3(k)\right)^{\mathrm{T}},\tag{5.10}$$

and $(N,M)=(1,0)$, we get

$$p=\frac{1}{2t+(4kt+x+y)^2}\mathrm{e}^{-4k^2t-2kx-2ky},\tag{5.11a}$$

$$q=\frac{-2t+(4kt+x+y)^2}{2t+(4kt+x+y)^2}\mathrm{e}^{4k^2t+2kx+2ky},\tag{5.11b}$$

$$u=\frac{4(3kt+x+y)}{2t+(4kt+x+y)^2}-\frac{2k(4kt+x+y)^2}{2t+(4kt+x+y)^2}.\tag{5.11c}$$

When $(N,M)=(0,1)$, we have

$$p=\frac{2t+(4kt+x+y)^2}{2t-(4kt+x+y)^2}\mathrm{e}^{-4k^2t-2kx-2ky},\tag{5.12a}$$

$$q=\frac{1}{2t-(4kt+x+y)^2}\mathrm{e}^{4k^2t+2kx+2ky},\tag{5.12b}$$

$$u=\frac{4(3kt+x+y)}{2t-(4kt+x+y)^2}+\frac{2k(4kt+x+y)^2}{2t-(4kt+x+y)^2}.\tag{5.12c}$$

Assume that

$$\varphi=\left(\varphi_1(k_1),\varphi_2(k_1),\varphi_1(k_2)\right)^{\mathrm{T}},\ \psi=\left(\psi_1(k_1),\psi_2(k_1),\psi_1(k_2)\right)^{\mathrm{T}},\tag{5.13}$$

letting $(N,M)=(1,0)$ gives

$$p=-2\frac{(k_1-k_2)^2}{\left[1+2(k_2-k_1)(4k_1t+x+y)\right]\mathrm{e}^{2\xi_1}-\mathrm{e}^{2\xi_2}},\tag{5.14a}$$

$$q=-2\frac{\left[1+2(k_2-k_1)(4k_1t+x+y)\right]\mathrm{e}^{2\xi_2}-\mathrm{e}^{2\xi_1}}{1+2(k_2-k_1)(4k_1t+x+y)-\mathrm{e}^{2\xi_2-2\xi_1}},\tag{5.14b}$$

$$u=2\frac{\left[k_2-2k_1-2k_2(k_2-k_1)(4k_1t+x+y)\right]\mathrm{e}^{2\xi_1}+(2k_1-k_2)\mathrm{e}^{2\xi_2}}{\left[1+2(k_2-k_1)(4k_1t+x+y)\right]\mathrm{e}^{2\xi_1}-\mathrm{e}^{2\xi_2}}.\tag{5.14c}$$

Similarly, taking $(N,M)=(0,1)$ yields

$$p=2\frac{\left[1+2(k_1-k_2)(4k_1t+x+y)\right]\mathrm{e}^{-2\xi_2}-\mathrm{e}^{-2\xi_1}}{1+2(k_1-k_2)(4k_1t+x+y)-\mathrm{e}^{2\xi_1-2\xi_2}},\tag{5.15a}$$

$$q=-2\frac{(k_1-k_2)^2}{\left[1+2(k_1-k_2)(4k_1t+x+y)\right]\mathrm{e}^{-2\xi_1}-\mathrm{e}^{-2\xi_2}},\tag{5.15b}$$

$$u=2\frac{\left[k_2-2k_1-2k_2((k_1-k_2)(4k_1t+x+y)\right]\mathrm{e}^{-2\xi_1}+(2k_1-k_2)\mathrm{e}^{-2\xi_2}}{-\left[1+2(k_1-k_2)(4k_1t+x+y)\right]\mathrm{e}^{-2\xi_1}+\mathrm{e}^{-2\xi_2}}.\tag{5.15c}$$

6. Complexitions of the (2 + 1)-Dimensional AKNS Equation

In the following, we would like to consider that A is a real Jordan matrix.

$$A=\begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_h \end{pmatrix}, \tag{6.1}$$

where

$$J_i=\begin{pmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{pmatrix}, \quad A_i=\begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix},$$

and $\alpha_i, \beta_i \ (i=1,2,\cdots,h)$ are real constants. Then, from (4.1), complexitons can be obtained.

In order to prove that, we first observe the simplest case when

$$A=\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}=\alpha I_2+\beta\sigma_2, \quad \sigma_2=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.2}$$

Substituting (6.2) into (4.1a) yields

$$\varphi=\mathrm{e}^{-\left[2\left(\alpha^2-\beta^2\right)t+\alpha(x+y)\right]I_2}\cdot\mathrm{e}^{-\left[4\alpha\beta t+\beta(x+y)\right]\sigma_2}C. \tag{6.3}$$

Expanding the above φ and taking advantage of $\sigma_2^2=-I_2$, we have

$$\varphi=\mathrm{e}^{-2\left(\alpha^2-\beta^2\right)t-\alpha(x+y)}\left[\cos\left(4\alpha\beta t+\beta(x+y)\right)I_2-\sin\left(4\alpha\beta t+\beta(x+y)\right)\sigma_2\right]C. \tag{6.4a}$$

Similarly,

$$\psi=\mathrm{e}^{2\left(\alpha^2-\beta^2\right)t+\alpha(x+y)}\left[\cos\left(4\alpha\beta t+\beta(x+y)\right)I_2+\sin\left(4\alpha\beta t+\beta(x+y)\right)\sigma_2\right]D. \tag{6.4b}$$

Further, we consider the matrix A as a Jordan block J_i

$$A=J_i=A'+E', \tag{6.5}$$

$$A'=I_{l_i}\otimes A_i=\begin{pmatrix} A_i & & & 0 \\ & A_i & & \\ & & \ddots & \\ 0 & & & A_i \end{pmatrix}, \quad E'=E_{l_i}\otimes I_i=\begin{pmatrix} 0 & & & 0 \\ I_2 & 0 & & \\ & \ddots & \ddots & \\ 0 & & I_2 & 0 \end{pmatrix}_{2l_i\times 2l_i}, \tag{6.5b}$$

where the symbol $\otimes$ denotes tensor product of matrices. Noting that $A'E'=E'A'$, we get

$$A^s=\left(A'+E'\right)^s=\left(I_{2l_i}+E'\partial_{\alpha_i}+\cdots+\frac{1}{j!}E'^j\partial_{\alpha_i}^j+\cdots+\frac{1}{s!}E'^s\partial_{\alpha_i}^s\right)A'^s. \tag{6.6}$$

Employing the following formula

$$\alpha_i A_i^p=\alpha_i\left(\alpha_i I_2+\beta_i\sigma_2\right)^p=p\left(\alpha_i I_2+\beta_i\sigma_2\right)^{p-1},\ \left(p=1,2,3,\cdots\right), \tag{6.7}$$

then (6.6) can be written as

$$A^s=\begin{pmatrix} I_2 & & & & 0 \\ I_2\partial_{\alpha_i} & I_2 & & & \\ \frac{1}{2}I_2\partial_{\alpha_i}^2 & I_2\partial_{\alpha_i} & I_2 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{1}{(l_i-1)!}I_2\partial_{\alpha_i}^{l_i-1} & \cdots & \frac{1}{2}I_2\partial_{\alpha_i}^2 & I_2\partial_{\alpha_i} & I_2 \end{pmatrix}A'^s=T\left(\partial_{\alpha_i}\right)A'^s. \tag{6.8}$$

Substituting (6.8) into (4.1) yields

$$\varphi_j(\alpha_i)=T(\partial_{\alpha_i})\mathrm{e}^{-2A'^2t-A'(x+y)}C=T(\partial_{\alpha_i})\left(I_{l_i}\otimes\mathrm{e}^{-2A_i^2t-A_i(x+y)}\right)C, \tag{6.9a}$$

$$\psi_j(\alpha_i)=T(\partial_{\alpha_i})\mathrm{e}^{2A'^2t+A'(x+y)}D=T(\partial_{\alpha_i})\left(I_{l_i}\otimes\mathrm{e}^{2A_i^2t+A_i(x+y)}\right)D, \tag{6.9b}$$

or

$$\varphi_j(\alpha_i)=\frac{1}{(j-1)!}\partial_{\alpha_i}^{j-1}\mathrm{e}^{-2A_i^2t-A_i(x+y)}c_1+\cdots+\partial_{\alpha_i}\mathrm{e}^{-2A_i^2t-A_i(x+y)}c_{j-1}+\mathrm{e}^{-2A_i^2t-A_i(x+y)}c_j, \tag{6.10a}$$

$$\psi_j(\alpha_i)=\frac{1}{(j-1)!}\partial_{\alpha_i}^{j-1}\mathrm{e}^{2A_i^2t+A_i(x+y)}d_1+\cdots+\partial_{\alpha_i}\mathrm{e}^{2A_i^2t+A_i(x+y)}d_{j-1}+\mathrm{e}^{2A_i^2t+A_i(x+y)}d_j, \tag{6.10b}$$

where

$$\varphi_j(\alpha_i)=(\varphi_{j1}(\alpha_i),\varphi_{j2}(\alpha_i))^{\mathrm{T}},\quad \varphi(\alpha_i)=\left(\varphi_1(\alpha_i)^{\mathrm{T}},\varphi_2(\alpha_i)^{\mathrm{T}},\cdots,\varphi_{l_i}(\alpha_i)^{\mathrm{T}}\right)^{\mathrm{T}},$$

$$\psi_j(\alpha_i)=(\psi_{j1}(\alpha_i),\psi_{j2}(\alpha_i))^{\mathrm{T}},\quad \psi(\alpha_i)=\left(\psi_1(\alpha_i)^{\mathrm{T}},\psi_2(\alpha_i)^{\mathrm{T}},\cdots,\psi_{l_i}(\alpha_i)^{\mathrm{T}}\right)^{\mathrm{T}},$$

$$c_j=(c_{j1},c_{j2})^{\mathrm{T}},C=\left(c_1^{\mathrm{T}},c_2^{\mathrm{T}},\cdots,c_{l_i}^{\mathrm{T}}\right)^{\mathrm{T}},d_j=(d_{j1},d_{j2})^{\mathrm{T}},D=\left(d_1^{\mathrm{T}},d_2^{\mathrm{T}},\cdots,d_{l_i}^{\mathrm{T}}\right)^{\mathrm{T}}.$$

According to (6.4), Equation (6.10) can be expressed as the following explicit form:

$$\begin{aligned}\varphi_j(\alpha_i)=\begin{pmatrix}\varphi_{j1}(\alpha_i)\\ \varphi_{j2}(\alpha_i)\end{pmatrix}=\sum_{s=1}^{j}\frac{1}{(j-s)!}\partial_{\alpha_i}^{j-s}\Bigg[&\mathrm{e}^{-2(\alpha_i^2-\beta_i^2)t-\alpha_i(x+y)}\\ &\cdot\begin{pmatrix}c_{s1}\cos(4\alpha_i\beta_it+\beta_i(x+y))+c_{s2}\sin(4\alpha_i\beta_it+\beta_i(x+y))\\ -c_{s1}\sin(4\alpha_i\beta_it+\beta_i(x+y))+c_{s2}\cos(4\alpha_i\beta_it+\beta_i(x+y))\end{pmatrix}\Bigg],\end{aligned} \tag{6.11a}$$

$$\begin{aligned}\psi_j(\alpha_i)=\begin{pmatrix}\psi_{j1}(\alpha_i)\\ \psi_{j2}(\alpha_i)\end{pmatrix}=\sum_{s=1}^{j}\frac{1}{(j-s)!}\partial_{\alpha_i}^{j-s}\quad&\mathrm{e}^{2(\alpha_i^2-\beta_i^2)t+\alpha_i(x+y)}\\ &\cdot\begin{pmatrix}d_{s1}\cos(4\alpha_i\beta_it+\beta_i(x+y))-d_{s2}\sin(4\alpha_i\beta_it+\beta_i(x+y))\\ d_{s1}\sin(4\alpha_i\beta_it+\beta_i(x+y))+d_{s2}\cos(4\alpha_i\beta_it+\beta_i(x+y))\end{pmatrix}\Bigg].\end{aligned} \tag{6.11b}$$

Thus, the double Wronskian (3.12) is the complextion of Equation (2.3), where

$$\varphi=\left(\varphi_1(\alpha_1)^{\mathrm{T}},\cdots,\varphi_{l_1}(\alpha_1)^{\mathrm{T}};\varphi_1((\alpha_2)^{\mathrm{T}},\cdots,\varphi_{l_2}(\alpha_2)^{\mathrm{T}};\cdots;\varphi_1(\alpha_h)^{\mathrm{T}},\cdots,\varphi_{l_h}(\alpha_h)^{\mathrm{T}}\right)^{\mathrm{T}},$$

$$\psi=\left(\psi_1(\alpha_1)^{\mathrm{T}},\cdots,\psi_{l_1}(\alpha_1)^{\mathrm{T}};\psi_1(\alpha_2)^{\mathrm{T}},\cdots,\psi_{l_2}(\alpha_2)^{\mathrm{T}};\cdots;\psi_1(\alpha_h)^{\mathrm{T}},\cdots,\psi_{l_h}(\alpha_h)^{\mathrm{T}}\right)^{\mathrm{T}},$$

$$(l_1+l_2+\cdots+l_h=N+M+2).$$

On the other hand, for $\partial_{\alpha_i}A_i^N=-\sigma_2\partial_{\beta_i}A_i^N$, the partial derivative with respect to α_i can be replaced by the partial derivative with respect to β_i in (6.10) and (6.11).

For example, taking $N=M=0$, $\xi=2(\alpha^2-\beta^2)t+\alpha(x+y)$, $\eta=4\alpha\beta t+\beta(x+y)$ (dropping the subscript) and $\varphi=\left(\mathrm{e}^{-\xi}\cos\eta,-\mathrm{e}^{-\xi}\sin\eta\right)^{\mathrm{T}}$, $\psi=\left(\mathrm{e}^{\xi}\cos\eta,\mathrm{e}^{\xi}\sin\eta\right)^{\mathrm{T}}$, we have

$$p=-2\beta\frac{\mathrm{e}^{-4(\alpha^2-\beta^2)t-2\alpha(x+y)}}{\sin 2(4\alpha\beta t+\beta(x+y))}, \tag{6.12a}$$

$$q=-2\beta\frac{e^{4\left(\alpha^2-\beta^2\right)t+2\alpha(x+y)}}{\sin 2\left(4\alpha\beta t+\beta(x+y)\right)}, \tag{6.12b}$$

$$u=4\beta\cot 2\left(4\alpha\beta t+\beta(x+y)\right). \tag{6.12c}$$

7. Conclusion

In this paper, we have obtained N-solution solutions and the generalized double Wronskian solution of the (2 + 1)-dimensional AKNS equation through the Hirota method and the Wronskian technique, respectively. Moreover, we have given rational solutions, Matveev solutions and complexitons of the (2 + 1)-dimensional AKNS equation. According to our knowledge, the three solutions are novel.

Acknowledgements

The author would like to express his thanks to the Editor and the referee for their comments. This work is supported by the Natural Science Foundation of Shandong Province of China (Grant No. ZR2014AM001), and the youth teacher development program of Shandong Province of China.

References

[1] Gardner, C.S., Greene, J.M., Kruskal, M.D. and Miura, R.M. (1967) Method for Solving the Korteweg de Vries Equation. *Physical Review Letters*, **19**, 1095-1097. http://dx.doi.org/10.1103/PhysRevLett.19.1095

[2] Matveev, V.B. and Salle, M.A. (1991) Darboux Transformations and Solitons. Springer-Verlag, Berlin. http://dx.doi.org/10.1007/978-3-662-00922-2

[3] Hirota, R. (1971) Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Physical Review Letters*, **27**, 1192-1194. http://dx.doi.org/10.1103/PhysRevLett.27.1192

[4] Freeman, N.C. and Nimmo, J.J.C. (1983) Soliton Solutions of the KdV and KP Equations: The Wronskian Technique. *Physics Letters A*, **95**, 1-3. http://dx.doi.org/10.1016/0375-9601(83)90764-8

[5] Nimmo, J.J.C. and Freeman, N.C. (1983) A Method of Obtaining the N-Soliton Solution of the Boussinesq Equation in Terms of a Wronskian. *Physics Letters A*, **95**, 4-6. http://dx.doi.org/10.1016/0375-9601(83)90765-X

[6] Hu, X.B. and Wang, H.Y. (2006) Construction of dKP and BKP Equations with Self-Consistent Sources. *Inverse Problems*, **22**, 1903-1920. http://dx.doi.org/10.1088/0266-5611/22/5/022

[7] Hu, X.B. and Wang, H.Y. (2007) New Type of Kadomtsev-Petviashvili Equation with Self-Consistent Soureces and Its Blinear Bäcklund Transformation. *Inverse Problems*, **23**, 1433-1444. http://dx.doi.org/10.1088/0266-5611/23/4/005

[8] Satsuma, J. (1979) A Wronskian Representation of N-Soliton Solutions of Nonlinear Evolution Equations. *Journal of the Physical Society of Japan*, **46**, 359-360. http://dx.doi.org/10.1143/JPSJ.46.359

[9] Matveev, V.B. (1992) Generalized Wronskian Formula for Solutions of the KdV Equation: First Applications. *Physics Letters A*, **166**, 205-208. http://dx.doi.org/10.1016/0375-9601(92)90362-P

[10] Ma, W.X. (2002) Complexiton Solutions to the Korteweg-de Vries Equation. *Physics Letters A*, **301**, 35-44. http://dx.doi.org/10.1016/S0375-9601(02)00971-4

[11] Ma, W.X. (2004) Wronskians, Generalized Wronskians and Solutions to the Korteweg-de Vries Equation. *Chaos, Solitons & Fractals*, **19**, 163-170. http://dx.doi.org/10.1016/S0960-0779(03)00087-0

[12] Ma, W.X. and Maruno, K. (2004) Complexiton Solutions of the Toda Lattice Equation. *Physica A*, **343**, 219-237. http://dx.doi.org/10.1016/j.physa.2004.06.072

[13] Ma, W.X. and You, Y. (2005) Solving the Korteweg-de Vries Equation by Its Bilinear Form: Wronskian Solutions. *Transactions of the American Mathematical Society*, **357**, 1753-1778. http://dx.doi.org/10.1090/S0002-9947-04-03726-2

[14] Ma, W.X. (2005) Complexiton Solutions to Integrable Equations. *Nonlinear Analysis: Theory, Methods & Applications*, **63**, 2461-2471. http://dx.doi.org/10.1016/j.na.2005.01.068

[15] Ma, W.X. (2005) Complexiton Solutions of the Korteweg-de Vries Equation with Self-Consistent Sources. *Chaos, Solitons & Fractals*, **26**, 1453-1458. http://dx.doi.org/10.1016/j.chaos.2005.03.030

[16] Ma, W.X., Li, C.X. and He, J.S. (2009) A Second Wronskian Formulation of the Boussinesq Equation. *Nonlinear Analysis: Theory, Methods & Applications*, **70**, 4245-4258. http://dx.doi.org/10.1016/j.na.2008.09.010

[17] Ablowitz, M.J., Kaup, D.J., Newell, A.C. and Segur, H. (1974) The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems. *Studies in Applied Mathematics*, **53**, 249-315. http://dx.doi.org/10.1002/sapm1974534249

[18] Liu, Q.M. (1990) Double Wronskian Solution of the AKNS and Classical Boussinesq Hierarchies. *Journal of the Physical Society of Japan*, **59**, 3520-3527. http://dx.doi.org/10.1143/JPSJ.59.3520

[19] Chen, D.Y., Zhang, D.J. and Bo, J.B. (2008) New Double Wronskian Solutions of the AKNS Equation. *Science in China Series A*: *Mathematics*, **51**, 55-69. http://dx.doi.org/10.1007/s11425-007-0165-6

[20] Lou, S.Y. and Hu, X.B. (1997) Infinitely Many Lax Pair and Symmetry Constraints of the KP Equation. *Journal of Mathematical Physics*, **38**, 6407-6427. http://dx.doi.org/10.1063/1.532219

[21] Lou, S.Y., Chen, C.L. and Tang, X.Y. (2002) (2 + 1)-Dimensional (M + N)-Component AKNS System: Painlevé Integrability, Infinitely Many Symmetries, Similarity Reductions and Exact Solutions. *Journal of Mathematical Physics*, **43**, 4078-4109. http://dx.doi.org/10.1063/1.1490407