

Boundedness and Oscillation of Third Order Neutral Differential Equations with Deviating Arguments

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Abstract

we consider the third-order neutral functional differential equations with deviating arguments. A new theorem is presented that improves a number of results reported in the literature. Examples are included to illustrate new results.

Keywords

Oscillation, Third Order, Neutral Delay Differential Equations

1. Introduction

In this paper we consider third order neutral differential equations of the form

$$\left(r_2(t)(r_1(t)z'(t))' \right)' + \int_{\alpha}^{\beta} q(t, \xi) f(x(g(t, \xi))) d\sigma(\xi) = 0, t \geq t_0, \quad (1)$$

where $z(t) = x(t) \pm p(t)x(\tau(t))$ and the following conditions are satisfied

(A₁) $p \in C(I, \mathbb{R}), 0 < p(t) \leq p < 1$ and $I = [t_0, \infty)$,

(A₂) $\tau \in C(I, \mathbb{R}), \tau(t) < t$, τ is strictly increasing, $\lim_{t \rightarrow \infty} \tau(t) = \infty$ and we define

$\tau^0(t) = t, \tau^j(t) = \tau(\tau^{j-1}(t)), j = 1, 2, \dots$,

(A₃) $r_i \in C(I, (0, \infty)), \int_{t_0}^{\infty} r_i^{-1}(t) dt = \infty, i = 1, 2$,

(A₄) $f \in C(\mathbb{R}, \mathbb{R}), f$ is non-decreasing and $uf(u) > 0$ for $u \neq 0$,

(A₅) $q \in C(I \times [\alpha, \beta], [0, \infty])$, and $q(t, \xi)$ is not zero on any half line $[t_\mu, \infty) \times [\alpha, \beta], t_\mu \geq t_0$,

(A₆) $g \in C(I \times [\alpha, \beta], \mathbb{R})$, $g(t, \xi) \leq t$ for $t \geq t_0$ and $\xi \in [\alpha, \beta]$, $g(t, \xi)$ is continuous, has positive partial derivative on $I \times [\alpha, \beta]$ with respect to t , nondecreasing with respect to ξ and $\lim_{t \rightarrow \infty} g(t, \xi) = \infty$,

(A₇) $\sigma \in C([\alpha, \beta], \mathbb{R})$, σ is nondecreasing and the integral of Equation (1) is in the sense Riemann-stieltjes.

We mean by a solution of Equation (1) a function $x(t): [t_x, \infty) \rightarrow \mathbb{R}$, $t_x \geq t_0$ such that $z(t)$, $r_1(t)z'(t)$, $r_2(t)(r_1(t)z'(t))'$ and $(r_2(t)(r_1(t)z'(t)))'$ exist and are continuous on $[t_x, \infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory.

Asymptotic properties of solutions of differential equations of the second and third order have been subject of intensive studying in the literature. This problem for neutral differential equations has received considerable attention in recent years (see [1]-[11]).

Recently, in [12] by using Riccati technique, have established some general oscillation criteria for third-order neutral differential equation

$$\left(r_2(t) \left(r_1(t) (x(t) + px(t-\tau))' \right)' \right)' + q(t)x(t-\sigma) = 0.$$

In [3], Candan presented several oscillation criteria for third order neutral delay differential equation

$$\left(r_2(t) \left(r_1(t) (x(t) + px(t-\tau))' \right)' \right)' + \int_{\alpha}^{\beta} q(t, \xi) x(g(t, \xi)) d\sigma(\xi) = 0.$$

[9] and [13] obtained some oscillation criteria for study third order nonlinear neutral differential equations

$$\left(r_2(t) \left(r_1(t) (x(t) + px(t-\tau))' \right)' \right)' + q(t)f(x(t-\sigma)) = 0$$

and

$$\left(r_2(t) \left(r_1(t) (x(t) - px(\tau(t)))' \right)' \right)' + q(t)f(x(g(t))) = 0.$$

In this paper, we establish some oscillation criteria for Equation (1), which complement and extend the results in [3] [13].

We begin with analyzing of the asymptotic behavior of possible non-oscillatory solutions of the Equation (1) in the case when $z(t) = x(t) - px(\tau(t))$. Let $x(t)$ be a non-oscillatory solution of (1) on $[t_0, \infty)$. From (1) it follows that the function $z(t)$ has to be eventually of constant sign, so either

(a) $x(t)z(t) > 0$

or

(b) $x(t)z(t) < 0$

for all sufficiently large t . Denote by N^+ [or N^-] the set of all non-oscillatory solutions $x(t)$ of the Equation (1) such that (a) [or (b)] is satisfied. We begin with some useful lemmas.

Lemma 1.1 Let $z(t) = x(t) - px(\tau(t))$. Assume that (A₁) and (A₂) hold and x be continuous non-oscillatory solution of the functional inequality (a). Then

$$\lim_{t \rightarrow \infty} x(t) = 0 \text{ and } \lim_{t \rightarrow \infty} z(t) = 0.$$

Lemma 1.2 Let $z(t) = x(t) - px(\tau(t))$. Assume that (A₁) and (A₂) hold and x be continuous non-oscillatory solution of the functional inequality (b). If $\lim_{t \rightarrow \infty} z(t) = 0$, then

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

These lemmas are modifications of the Lemma 1 in the paper [14] and the Lemma 2 in the paper [13].

2. Main Results

In this part, for the sake of convenience, we introduce the following notation:

$$\Theta(t) = \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) \text{ and } R(t) = \int_{t_0}^t r_2^{-1}(s) ds.$$

2.1. Oscillation Criteria If $z(t) = x(t) - px(\tau(t))$

In this section, we will establish some oscillation criteria for Equation (1) in the case when $z(t) = x(t) - px(\tau(t))$ and $p(t) = p$.

Lemma 2.1 *Let x be a bounded positive solution of Equation (1) on the interval I . Then there exists a $T_0 \geq t_0$ such that $z(t)$ has the following properties:*

$$z'(t) < 0, (r_1(t)z'(t))' > 0 \text{ and } \left(r_2(t)(r_1(t)z'(t))' \right)' \leq 0 \text{ for } t \geq T_0 \tag{2}$$

Proof. Let x be a bounded positive solution of Equation (1) on the interval I . From (A₁), (A₂) and (A₆), there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t, \xi)) > 0$ for $t \geq t_1$. Then $z(t)$ is bounded and non-oscillatory. Thus, Equation (1) implies that

$$\left(r_2(t)(r_1(t)z'(t))' \right)' = - \int_{\alpha}^{\beta} q(t, \xi) f(x(g(t, \xi))) d\sigma(\xi) \leq 0.$$

Hence, the function $r_2(t)(r_1(t)z'(t))'$ is a non-increasing and of one sign. We claim that

$r_2(t)(r_1(t)z'(t))' > 0$ for $t \geq t_1$. Suppose that $r_2(t)(r_1(t)z'(t))' < 0$ for $t \geq t_2 \geq t_1$. Then there exists a $t_3 \geq t_2$ and constant $K_1 > 0$ such that

$$r_2(t)(r_1(t)z'(t))' < -K_1 \text{ for } t \geq t_3.$$

By integrating the last inequality from t_3 to t , we get

$$r_1(t)z'(t) < r_1(t_3)z'(t_3) - K_1 \int_{t_3}^t \frac{1}{r_2(s)} ds.$$

Letting $t \rightarrow \infty$, from (A₃), we have $\lim_{t \rightarrow \infty} r_1(t)z'(t) = -\infty$. Then there exists a $t_4 \geq t_3$ and constant $K_2 > 0$ such that

$$r_1(t)z'(t) < -K_2 \text{ for } t \geq t_4$$

By integrating this inequality from t_4 to t and using (A₃), we get $\lim_{t \rightarrow \infty} z(t) = -\infty$. This yields that $x \in N^-$ and this contradicts the Lemma 1.1. Now we have $r_2(t)(r_1(t)z'(t))' > 0$ for $t \geq t_2$. Hence $r_1(t)z'(t)$ is increasing function and we have two possible cases for $r_1(t)z'(t)$ either $r_1(t)z'(t) > 0$ eventually or $r_1(t)z'(t) < 0$ eventually for $t \geq t_3 \geq t_2$. If $r_1(t)z'(t) > 0$ for $t \geq t_3$, then there exist a $t_4 \geq t_3$ and a constant $L > 0$ such that

$$r_1(t)z'(t) > L \text{ for } t \geq t_4$$

By integrating this inequality from t_4 to t and using (A₃), we get $\lim_{t \rightarrow \infty} z(t) = \infty$. This means that $z(t) > 0$ and we get $x(t) \geq z(t)$ for all sufficiently large t . Then $\lim_{t \rightarrow \infty} z(t) = \infty$, which contradicts the boundedness of $x(t)$. Hence, $z'(t) < 0$ for $t \geq t_3$. \square

Theorem 2.1 if

$$\limsup_{t \rightarrow \infty} \int_{g(t,\beta)}^t \Theta(s) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du ds > 0. \tag{3}$$

Then every bounded solution $x(t)$ of Equation (1) is either oscillatory or tends to zero.

Proof. Let x be a bounded non-oscillatory solution of Equation (1) on the interval I . Without loss of generality we may assume that $x(t) > 0$. From Lemma 2.1, we get that (2) holds. Now, we have

$$x(t) = z(t) + px(\tau(t)) = z(t) + pz(\tau(t)) + p^2x(\tau^2(t)),$$

for all sufficiently large t . Repeating this procedure and the monotonicity of z , we obtain that there exists an integer $n \geq 0$ such that $\tau^{n+1}(g(t, \xi)) \geq T_0$ and

$$x(g(t, \xi)) = \sum_{r=0}^n p^r z(\tau^r(g(t, \xi))) + p^{n+1}x(\tau^{n+1}(g(t, \xi))) \geq \lambda z(g(t, \xi)),$$

where $\lambda = \sum_{r=0}^n p^r = \frac{1-p^{n+1}}{1-p} > 1$. Hence, we get

$$f(x(g(t, \xi))) \geq f(\lambda z(g(t, \xi))) \geq f(z(g(t, \xi))) \geq f(z(g(t, \beta))). \tag{4}$$

Thus, from Equation (1), we obtain

$$\left(r_2(t)(r_1(t)z'(t))' \right)' \leq -f(z(g(t, \beta)))\Theta(t). \tag{5}$$

Now, since $z(t)$ is bounded decreasing function, then there exist γ such that

$$\lim_{t \rightarrow \infty} z(t) = \gamma.$$

If $z(t) < 0$ for $t \geq T_1 \geq T_0$, then $\gamma < 0$ and which contradicts the Lemma 1.1. Therefore $z(t) > 0$ for $t \geq T_1$ and $\gamma \geq 0$. We shall prove that $\gamma = 0$. Let $\gamma > 0$. For $t > u > T \geq T_1$, we obtain

$$\int_u^{g(t,\beta)} \frac{1}{r_2(v)} \left[r_2(v)(r_1(v)z'(v))' \right] dv = r_1(g(t, \beta))z'(g(t, \beta)) - r_1(u)z'(u).$$

Thus, from Lemma 2.1, we get

$$\begin{aligned} r_1(u)z'(u) &\leq -\int_u^{g(t,\beta)} \frac{1}{r_2(v)} \left[r_2(v)(r_1(v)z'(v))' \right] dv \\ &\leq -\left[r_2(g(t, \beta))(r_1(g(t, \beta))z'(g(t, \beta)))' \right] \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv. \end{aligned} \tag{6}$$

So, for $t > u > s > T$, we have

$$z(g(t, \beta)) - z(g(s, \beta)) = \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} [r_1(u)z'(u)] du.$$

Hence, from (6), we get

$$z(g(t, \beta)) - z(g(s, \beta)) \leq -G(g(t, \beta)) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du \tag{7}$$

where $G(y) = r_2(y)(r_1(y)z'(y))'$. Let us define function

$$F(s, t) = (G(s) - G(g(t, \beta))) \int_s^t \frac{z'(g(u, \beta))g'(u, \beta)}{f(z(g(u, \beta)))} du, \quad t \geq s \geq T$$

We note that $F(t, t) = 0 = F(g(t, \beta), t)$. Deriving $F(s, t)$ partially with respect to s and using Lemma 2.1, (A₄) and (A₆), we get

$$F'_s(s, t) \geq G'(s) \int_s^t \frac{z'(g(u, \beta))g'(u, \beta)}{f(z(g(u, \beta)))} du + G(g(t, \beta)) \frac{z'(g(s, \beta))g'(s, \beta)}{f(z(g(s, \beta)))}$$

From (5), we have $G'(s) \leq -f(z(g(s, \beta)))\Theta(s)$. Hence, we obtain

$$F'_s(s, t) \geq -\Theta(s) f(z(g(s, \beta))) \int_s^t \frac{z'(g(u, \beta))g'(u, \beta)}{f(z(g(u, \beta)))} du + G(g(t, \beta)) \frac{z'(g(s, \beta))g'(s, \beta)}{f(z(g(s, \beta)))} \tag{8}$$

By (A₄) and (A₆), we get

$$\int_s^t \frac{z'(g(u, \beta))g'(u, \beta)}{f(z(g(u, \beta)))} du \leq \frac{1}{f(z(g(s, \beta)))} \int_s^t dz(g(u, \beta)) = \frac{1}{f(z(g(s, \beta)))} (z(g(t, \beta)) - z(g(s, \beta))) \leq 0$$

Thus, from (7), we have

$$\int_s^t \frac{z'(g(u, \beta))g'(u, \beta)}{f(z(g(u, \beta)))} du \leq \frac{-G(g(t, \beta))}{f(z(g(s, \beta)))} \int_{g(s, \beta)}^{g(t, \beta)} \frac{1}{r_1(u)} \int_u^{g(t, \beta)} \frac{1}{r_2(v)} dv du. \tag{9}$$

Then, substituting (8) in (9), it follows that

$$F'_s(s, t) \geq G(g(t, \beta)) \left[\Theta(s) \int_{g(s, \beta)}^{g(t, \beta)} \frac{1}{r_1(u)} \int_u^{g(t, \beta)} \frac{1}{r_2(v)} dv du + \frac{z'(g(s, \beta))g'(s, \beta)}{f(z(g(s, \beta)))} \right].$$

By integrating this inequality from $g(t, \beta)$ to t with respect to s , we obtain

$$0 \geq \int_{g(t, \beta)}^t \Theta(s) \int_{g(s, \beta)}^{g(t, \beta)} \frac{1}{r_1(u)} \int_u^{g(t, \beta)} \frac{1}{r_2(v)} dv du ds + \int_{z(g(g(t, \beta), \beta))}^{z(g(t, \beta))} \frac{d\omega}{f(\omega)}. \tag{10}$$

where $G(g(t, \beta)) > 0$. Since $\lim_{t \rightarrow \infty} z(t) = \gamma > 0$, we get

$$\lim_{t \rightarrow \infty} \int_{z(g(g(t, \beta), \beta))}^{z(g(t, \beta))} \frac{d\omega}{f(\omega)} = 0$$

Hence, from (10), we have

$$\limsup_{t \rightarrow \infty} \int_{g(t, \beta)}^t \Theta(s) \int_{g(s, \beta)}^{g(t, \beta)} \frac{1}{r_1(u)} \int_u^{g(t, \beta)} \frac{1}{r_2(v)} dv du ds \leq 0,$$

which contradicts (3). Therefore, $\lim_{t \rightarrow \infty} z(t) = 0$ and according to the Lemma 1.2 we have that $\lim_{t \rightarrow \infty} x(t) = 0$. \square

In the following Theorem, we establish some sufficient conditions for boundedness and oscillation of Equation (1) under the condition

$$\liminf_{u \rightarrow 0} \frac{f(u)}{u} > 0 \tag{11}$$

Theorem 2.2 Let (11) holds. If there exist an integer $n \geq 0$ such that

$$\limsup_{t \rightarrow \infty} \int_{g(t,\beta)}^t \Theta(s) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du ds > \frac{1-p}{1-p^{n+1}} \limsup_{u \rightarrow 0} \frac{u}{f(u)}, \tag{12}$$

then every bounded solution $x(t)$ of Equation (1) is oscillatory.

Proof. Let x be a bounded non-oscillatory solution of Equation (1) on the interval I . Without loss of generality we may assume that $x(t) > 0$. We can proceed exactly as in the proof of Theorem 2.1 and we use the fact that (12) implies (3). Hence, we get a non-oscillatory solution with the properties $x(t) > 0$, $z(t) > 0$, $z'(t) < 0$ and $(r_1(t)z'(t))' > 0$ for $t \geq T_0$, $\lim_{t \rightarrow \infty} z(t) = 0$ and $\lim_{t \rightarrow \infty} x(t) = 0$. Now, from (4), there exists $T_1 \geq T_0$ such that

$$f(x(g(t,\xi))) \geq f(\lambda z(g(t,\xi))) \geq f(\lambda z(g(t,\beta))) \text{ for } t \geq T_1.$$

Thus, Equation (1) implies that

$$\left(r_2(t)(r_1(t)z'(t))' \right)' \leq -f(\lambda z(g(t,\beta)))\Theta(t).$$

By integrating this inequality from $g(t,\beta)$ to t , we get

$$G(t) \leq G(g(t,\beta)) - \int_{g(t,\beta)}^t f(\lambda z(g(s,\beta)))\Theta(s) ds.$$

where $G(y) = r_2(y)(r_1(y)z'(y))' > 0$. Thus, we obtain

$$0 \leq G(g(t,\beta)) - M \lambda \int_{g(t,\beta)}^t z(g(s,\beta))\Theta(s) ds. \tag{13}$$

where $M = \inf \left\{ \frac{f(u)}{u} : 0 \leq |u| \leq |\lambda z(g(t,\beta))| \right\}$. Since $z(t) > 0$, from the Inequality (7), we get

$$-z(g(s,\beta)) \leq -G(g(t,\beta)) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du. \tag{14}$$

Combining (13) and (14), we have

$$0 < G(g(t,\beta)) \left(1 - M \lambda \int_{g(t,\beta)}^t \Theta(s) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du ds \right).$$

Hence, we get

$$\int_{g(t,\beta)}^t \Theta(s) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du ds < \frac{1}{M \lambda},$$

for $t \geq s \geq T_1$ and this contradicts the condition (12).

Corollary 2.1 Let (11) holds. If

$$\limsup_{t \rightarrow \infty} \int_{g(t,\beta)}^t \Theta(s) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du ds > (1-p) \limsup_{u \rightarrow 0} \frac{u}{f(u)} \tag{15}$$

then every bounded solution $x(t)$ of Equation (1) is oscillatory.

Example 2.1 Consider the differential equation

$$\left(\sqrt{t} \left(\sqrt{t} \left(x(t) - \frac{1}{2} x \left((\sqrt{t}-1)^2 \right) \right) \right)' \right)' + \int_0^1 \frac{3\xi^2}{\sqrt{t}} x \left((\sqrt{t}-1)^2 \xi \right) d\xi = 0,$$

where $t > 1$. We have

$$\limsup_{t \rightarrow \infty} \int_{g(t,\beta)}^t \Theta(s) \int_{g(s,\beta)}^{g(t,\beta)} \frac{1}{r_1(u)} \int_u^{g(t,\beta)} \frac{1}{r_2(v)} dv du ds = \frac{4}{3},$$

and $(1-p) \limsup_{u \rightarrow 0} \frac{u}{f(u)} = \frac{1}{2} < \frac{4}{3}$. Thus, all conditions of Corollary 2.1 are satisfied then all bounded solutions of the above equation are oscillatory.

Remark 2.1 If $\alpha = 0, \beta = 1, q(t, \xi) = q(t), g(t, \xi) = g(t)$ and $\sigma(\xi) = \xi$ then, our results extend the results in [13].

2.2. Oscillation Criteria If $z(t) = x(t) - px(\tau(t))$

In this section, we will present some oscillation criteria for Equation (1) under the case $z(t) = x(t) + p(t)x(\tau(t))$ and the condition

$$\frac{f(u)}{u} \geq k > 0 \text{ for } u \neq 0 \text{ and } t \geq t_0 \tag{16}$$

Lemma 2.2 If $x(t)$ is an eventually positive solution of (1), then for sufficiently large t , there are only two possible cases:

- (i) $z(t) > 0, z'(t) < 0$ and $(r_1(t)z'(t))' > 0$
- (ii) $z(t) > 0, z'(t) > 0$ and $(r_1(t)z'(t))' > 0$.

Proof. The proof of this lemma is similar to the proof Lemma 1 in [9] and we omit the details. \square

Theorem 2.3 Let (16) holds. If

$$\int_{t_0}^{\infty} \frac{1}{r_1(v)} \int_v^{\infty} \frac{1}{r_2(u)} \int_u^{\infty} \Theta(s) ds du dv = \infty, \tag{17}$$

and there exist a positive real function $\rho(t)$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left(k\rho(s)(1-p)\Theta(s) - \frac{\rho^2(s)r_1(g(s,a))}{4\rho(s)R(g(s,a))g'(s,a)} \right) ds = \infty. \tag{18}$$

Then every solution of Equation (1) is either oscillatory or tends to zero.

Proof. Let x be a non-oscillatory solution of Equation (1) on the interval I . Without loss of generality we may assume that $x(t) > 0$. Then there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(g(t, \xi)) > 0$ for $t \geq t_1$. By Lemma 2.2, we have two cases for $z(t)$. In the Case (i), since $z(t) > 0$ and $z'(t) < 0$, we get $\lim_{t \rightarrow \infty} z(t) = \mu \geq 0$. Let $\mu > 0$, then we have $\mu < z(t) < \mu + \varepsilon$ for all $\varepsilon > 0$ and t enough large. Choosing

$\varepsilon < \frac{1-p}{p}\mu$, we obtain

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &\geq \mu - pz(\tau(t)) \geq L(\mu + \varepsilon) \geq Lz(t). \end{aligned}$$

where $L = \frac{\mu - p(\mu + \varepsilon)}{\mu + \varepsilon} > 0$. Hence, from (1), (A₆) and (16), we have

$$\left(r_2(t)(r_1(t)z'(t))' \right) \leq -kL \int_{\alpha}^{\beta} q(t, \xi) z(g(t, \xi)) d\sigma(\xi) < -kL\mu\Theta(t).$$

By integrating two times from t to ∞ , we get

$$-z'(t) > kL\mu \frac{1}{r_1(t)} \int_t^\infty \frac{1}{r_2(u)} \int_u^\infty \Theta(s) ds du.$$

Integrating the last inequality from t_1 to ∞ , we obtain

$$z(t_1) > kL\mu \int_{t_1}^\infty \frac{1}{r_1(v)} \int_v^\infty \frac{1}{r_2(u)} \int_u^\infty \Theta(s) ds du dv.$$

This contradicts to the condition (17), then $\lim_{t \rightarrow \infty} z(t) = 0$, which implies that $\lim_{t \rightarrow \infty} x(t) = 0$. In the Case (ii), since $z(t) > 0$ and $z'(t) > 0$. Then there exist a $t_1 \geq t_0$ such that

$$x(t) \geq (1-p)z(t),$$

for $t \geq t_1$. Thus, from (1), (A₄) and (A₆), we get

$$\left(r_2(t)(r_1(t)z'(t))' \right)' \leq -k(1-p)z(g(t,a))\Theta(t). \tag{19}$$

Also, we have

$$r_1(t)z'(t) = r_1(t_1)z'(t_1) + \int_{t_1}^t \frac{r_2(s)(r_1(s)z'(s))'}{r_2(s)} ds \geq r_2(t)(r_1(t)z'(t))' \int_{t_1}^t \frac{1}{r_2(s)} ds,$$

Since $\left(r_2(t)(r_1(t)z'(t))' \right)' \leq 0$, we obtain

$$z'(g(t,a)) \geq \frac{r_2(t)(r_1(t)z'(t))'}{r_1(g(t,a))} R(g(t,a)). \tag{20}$$

Now, we define

$$\omega(t) = \rho(t) \frac{r_2(t)(r_1(t)z'(t))'}{z(g(t,a))}.$$

By differentiating and using (19) and (20), we get

$$\begin{aligned} \omega'(t) &\leq -k\rho(t)(1-p)\Theta(t) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{R(g(t,a))g'(t,a)}{\rho(t)r_1(g(t,a))}\omega^2(t) \\ &= -k\rho(t)(1-p)\Theta(t) + \frac{\rho'^2(t)r_1(g(t,a))}{4\rho(t)R(g(t,a))g'(t,a)} \\ &\quad - \left(\sqrt{\frac{R(g(t,a))g'(t,a)}{\rho(t)r_1(g(t,a))}}\omega(t) - \rho'(t)\sqrt{\frac{r_1(g(t,a))}{4\rho(t)R(g(t,a))g'(t,a)}} \right)^2. \end{aligned}$$

Hence, we obtain

$$\omega'(t) \leq -k\rho(t)(1-p)\Theta(t) + \frac{\rho'^2(t)r_1(g(t,a))}{4\rho(t)R(g(t,a))g'(t,a)}.$$

By integrating the above inequality from t_1 to t we have

$$\omega(t) \leq \omega(t_1) - \int_{t_1}^t \left(k\rho(s)(1-p)\Theta(s) - \frac{\rho'^2(s)r_1(g(s,a))}{4\rho(s)R(g(s,a))g'(s,a)} \right) ds.$$

Taking the superior limit as $t \rightarrow \infty$ and using (18), we get $\omega(t) \rightarrow -\infty$ which contradicts that $\omega(t) > 0$. This completes the proof of Theorem 2.3. \square

Remark 2.2 We can rewrite the condition (17) in the Theorem 2.3 as following

$$\int_{t_0}^{\infty} \left(\int_{t_0}^r \frac{dv}{r_1(v)} \left(\int_v^r \frac{du}{r_2(u)} \right) \right) \Theta(r) dr = \infty.$$

Remark 2.3 If $\tau(t) = t - \tau$ and $f(x) = x$, then our results extend the results in [3].

Example 2.2 Consider the differential equation

$$\left(\frac{1}{t} \left(\frac{1}{t} z'(t) \right) \right)' + \int_0^1 \frac{t^2 e^{t^2 \xi}}{e^{t^2} - 1} x(\xi(t-1)) (x^2(\xi(t-1)) + 2) d\xi = 0,$$

where $i > 1$ and $z(t) = x(t) + \frac{1}{2}x(t-1)$. Choosing $\rho(t) = 1$ and $k = 2$. Thus, all conditions of Theorem 2 are satisfied then every solutions of this equation is either oscillatory or tends to zero. 3

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