Journal of Applied Mathematics and Physics, 2015, 3, 1270-1281
Published Online October 2015 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2015.310156

Scientific Research Publishing

Existence and Multiplicity of Solutions for Quasilinear $p(x)$-Laplacian Equations in $\mathbb{R}^N$

Honghong Qi, Gao Jia

College of Science, University of Shanghai for Science and Technology, Shanghai, China
Email: qihonghong618@126.com, gaojia89@163.com

Received 10 September 2015; accepted 25 October 2015; published 28 October 2015

Abstract

We establish some results on the existence of multiple nontrivial solutions for a class of $p(x)$-Laplacian elliptic equations without assumptions that the domain is bounded. The main tools used in the proof are the variable exponent theory of generalized Lebesgue-Sobolev spaces, variational methods and a variant of the Mountain Pass Lemma.

Keywords

***p*(*x*)-Laplacian Operator, Generalized Lebesgue-Sobolev Spaces, Variational Method, Multiple Solutions**

1. Introduction

The study of differential and partial differential equations involving variable exponent conditions is a new and interesting topic. The interest in studying such problem was stimulated by their applications in elastic mechanics and fluid dynamics. These physical problems were facilitated by the development of Lebesgue and Sobolev spaces with variable exponent.

The existence and multiplicity of solutions of $p(x)$-Laplacian problems have been studied by several authors (see for example [1] [2], and the references therein).

In [3], A. R. EL Amrouss and F. Kissi proved the existence of multiple solutions of the following problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

How to cite this paper: Qi, H.H. and Jia, G. (2015) Existence and Multiplicity of Solutions for Quasilinear $p(x)$-Laplacian Equations in $\mathbb{R}^N$. *Journal of Applied Mathematics and Physics*, **3**, 1270-1281. http://dx.doi.org/10.4236/jamp.2015.310156

Also Xiaoyan Lin and X. H. Tang in [4] studied the following quasilinear elliptic equation

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+v(x)|u|^{p-2}u=f(x,u), & \text{in } \mathbb{R}^N, \\ u\in W^{1,p}\left(\mathbb{R}^N\right), \end{cases} \tag{2}$$

and they proved the multiplicity of solutions for problem (2) by using the cohomological linking method for cones and a new direct sum decomposition of $W^{1,p}\left(\mathbb{R}^N\right)$.

In this paper, we consider the following problem

$$\begin{cases} -\Delta_{p(x)}u+b(x)|u|^{p(x)-2}u=f(x,u), & \text{in } \mathbb{R}^N, \\ u\in W^{1,p(x)}\left(\mathbb{R}^N\right), \end{cases} \tag{3}$$

where $\Delta_{p(x)}u=\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is the $p(x)$-Laplacian operator; $p(x):\mathbb{R}^N\to\mathbb{R}$ is a Lipschitz continuous function with

$$1<p^-=\inf_{x\in\mathbb{R}^N}p(x)\le p(x)\le p^+=\sup_{x\in\mathbb{R}^N}p(x)<\infty.$$

$b(x)$ is a given continuous function which satisfies

(B_0)

$$b^->0,\ m\left(b^{-1}(0,T]\right)<+\infty,\ \text{for all } T\in\mathbb{R}^+,$$

here m is the Lebesgue measure on $\mathbb{R}^N$.

$f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the subcritical growth condition

(F_0)

$$|f(x,t)|\le c\left(1+|t|^{q(x)-1}\right),\ \forall t\in\mathbb{R},\ \text{a.e. } x\in\mathbb{R}^N,$$

for some $c>0$, where $q(x)\in C\left(\mathbb{R}^N\right)$, $p^+<q^-=\inf_{x\in\mathbb{R}^N}q(x)$, $1<q(x)<p^*(x),\forall x\in\mathbb{R}$, and

$$p^*(x)=\begin{cases} \dfrac{Np(x)}{N-p(x)}, & p(x)<N, \\ \infty, & p(x)\ge N. \end{cases}$$

Define the subspace

$$E=\left\{u\in W^{1,p(x)}\left(\mathbb{R}^N\right)\middle|\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+b(x)|u|^{p(x)}\right)\mathrm{d}x<+\infty\right\}$$

and the functional $\Phi:E\to\mathbb{R}$,

$$\Phi(u)=\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+b(x)|u|^{p(x)}\right)\mathrm{d}x-\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x,\quad \forall u\in E,$$

where $F(x,u)=\int_0^u f(x,t)\mathrm{d}t$.

Clearly, in order to determine the weak solutions of problem (3), we need to find the critical points of functional Φ. It is well known that under (B_0) and (F_0), Φ is well defined and is a C^1 functional. Moreover,

$$\langle\Phi'(u),v\rangle=\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)-2}\nabla u\nabla v+b(x)|u|^{p(x)-2}uv\right)\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)v\mathrm{d}x,$$

for all $u,v\in E$.

If $f(x,0)=0$ for a.e. $x\in\mathbb{R}^N$, the constant function $u=0$ is a trivial solution of problem (3). In the following, the key point is to prove the existence of nontrivial solutions for problem (3).

Set

$$\lambda_* = \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right)\mathrm{d}x}{\int_{\mathbb{R}^N} \frac{1}{p(x)}|u|^{p(x)}\,\mathrm{d}x} > 0. \tag{4}$$

This paper is to show the existence of nontrivial solutions of problem (3) under the following conditions.

(F_1)

$$\lim_{|t|\to\infty}\left(F(x,t) - \frac{\left(p^-\right)^2}{\left(p^+\right)^3}\lambda_*|t|^{p^-}\right) = -\infty, \text{ uniformly for a.e. } x\in\mathbb{R}^N,$$

where λ_* as given in (4).

(F_2) There exist $\mu\in\left[1, p^-\right)$ and $\gamma>0$, such that

$$0<\mu F(x,t)\le tf(x,t), \text{ for a.e. } x\in\mathbb{R}^N, \ 0<|t|\le\gamma.$$

(F_3) There exist $\theta>p^+$ and $K>0$ such that

$$|t|\ge K \Rightarrow 0<\theta F(x,t)\le tf(x,t),$$

for a.e. $x\in\mathbb{R}^N, \forall t\in\mathbb{R}$.

(F_4) $f(x,t) = o\left(|t|^{p^+-1}\right)$ as $t\to 0$ and uniformly for $x\in\mathbb{R}^N$, with $q^->p^+$. Here $q(x)$ is given in the condition (F_0).

We have the following results.

Theorem 1.1. If $b(x)$ satisfies (B_0), $f(x)$ satisfies (F_0), (F_1) and (F_2), then problem (3) possesses at least one nontrivial solution.

Theorem 1.2. Assume $b(x)$ satisfies (B_0), $f(x)$ satisfies (F_0), (F_3) and (F_4), with $f(x,0)=0$ for a.e. $x\in\mathbb{R}^N$, then problem (3) has at least two nontrivial solutions, in which one is non-negative and another is non-positive.

This paper is divided into three sections. In the second section, we state some basic preliminary results and give some lemmas which will be used to prove the main results. The proofs of Theorem 1.1 and Theorem 1.2 are presented in the third section.

2. Preliminaries

In this section, we recall some results on variable exponent Sobolev space $W^{1,p(x)}\left(\mathbb{R}^N\right)$ and basic properties of the variable exponent Lebesgue space $L^{p(x)}\left(\mathbb{R}^N\right)$, we refer to [5]-[8].

Let $p(x)\in L^\infty\left(\mathbb{R}^N\right)$, $p^->1$. Define the variable exponent Lebesgue space:

$$L^{p(x)}\left(\mathbb{R}^N\right) = \left\{u:\mathbb{R}^N\to\mathbb{R}\,\middle|\, u \text{ is a measurable function and } \int_{\mathbb{R}^N}|u|^{p(x)}\,\mathrm{d}x<+\infty\right\}.$$

For $u\in L^{p(x)}\left(\mathbb{R}^N\right)$, we define the following norm

$$|u|_{p(x)} = \inf\left\{\mu>0\,\middle|\,\int_{\mathbb{R}^N}\left|\frac{u(x)}{\mu}\right|^{p(x)}\mathrm{d}x\le 1\right\}.$$

Define the variable exponent Sobolev space:

$$W^{1,p(x)}\left(\mathbb{R}^N\right) = \left\{u:\mathbb{R}^N\to\mathbb{R}\,\middle|\, u\in L^{p(x)}\left(\mathbb{R}^N\right) \text{ and } |\nabla u|\in L^{p(x)}\left(\mathbb{R}^N\right)\right\},$$

which is endowed with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

It can be proved that the spaces $L^{p(x)}\left(\mathbb{R}^N\right)$ and $W^{1,p(x)}\left(\mathbb{R}^N\right)$ are separable and reflexive Banach spaces. See [9] for the details.

Proposition 2.1. [10] [11] Let

$$J(u)=\int_{\mathbb{R}^N}|u|^{p(x)}\mathrm{d}x,\ u\in L^{p(x)}(\mathbb{R}^N).$$

Then we have

1) For $u\neq 0$, $|u|_{p(x)}=\mu\Leftrightarrow J\left(\frac{u}{\mu}\right)=1$;

2) $|u|_{p(x)}>1\Rightarrow |u|_{p(x)}^{p^-}\leq J(u)\leq |u|_{p(x)}^{p^+}$, $|u|_{p(x)}<1\Rightarrow |u|_{p(x)}^{p^+}\leq J(u)\leq |u|_{p(x)}^{p^-}$;

3) $|u|_{p(x)}>1(=1,<1)\Leftrightarrow J(u)>1(=1,<1)$;

4) $|u_n|_{p(x)}\to 0\Leftrightarrow J(u_n)\to 0$, $|u_n|_{p(x)}\to\infty\Leftrightarrow J(u_n)\to\infty$.

For $h(x)\in L^\infty(\mathbb{R}^N)$ with $h^->1$, let $h^*(x):\mathbb{R}^N\to\mathbb{R}$ satisfy

$$\frac{1}{h(x)}+\frac{1}{h^*(x)}=1,\quad \text{a.e. } x\in\mathbb{R}^N.$$

We have the following generalized Hölder type inequality.

Proposition 2.2. [9] [12] For any $u\in L^{h(x)}(\mathbb{R}^N)$ and $v\in L^{h^*(x)}(\mathbb{R}^N)$, we have

$$\left|\int_{\mathbb{R}^N}uv\mathrm{d}x\right|\leq\left(\frac{1}{h^-}+\frac{1}{h^{*-}}\right)|u|_{h(x)}|v|_{h^*(x)}.$$

We consider the case that $b(x)$ satisfies (B_0). Define the norm

$$\|u\|=\inf\left\{\mu>0\,\middle|\,\int_{\mathbb{R}^N}\left(\left|\frac{\nabla u}{\mu}\right|^{p(x)}+b(x)\left|\frac{u}{\mu}\right|^{p(x)}\right)\mathrm{d}x\leq 1\right\}.$$

Then $(E,\|\cdot\|)$ is continuously embedding into $W^{1,p(x)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E,\|\cdot\|)$ is also a separable and reflexive Banach space.

Similar to the Proposition 2.1, we have

Proposition 2.3. [13] The functional $J_1:W^{1,p(x)}(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$J_1(u)=\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+b(x)|u|^{p(x)}\right)\mathrm{d}x$$

has the following properties:

1) $u\neq 0, \|u\|=\mu\Leftrightarrow J_1\left(\frac{u}{\mu}\right)=1$;

2) $\|u\|>1\Rightarrow \|u\|^{p^-}\leq J_1(u)\leq\|u\|^{p^+}$, $\|u\|<1\Rightarrow\|u\|^{p^+}\leq J_1(u)\leq\|u\|^{p^-}$;

3) $\|u_n\|\to 0\Leftrightarrow J_1(u_n)\to 0$.

Lemma 2.4. [13] If $b(x)$ satisfies (B_0), then

1) we have a compact embedding $E\hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$;

2) for any measurable function $q:\mathbb{R}^N\to\mathbb{R}$ with $p(x)\leq q(x)\ll p^*(x)$, we have a compact embedding $E\hookrightarrow\hookrightarrow L^{q(x)}(\mathbb{R}^N)$. Here $u\ll v$ means that $\inf_{x\in\mathbb{R}^N}(v(x)-u(x))>0$.

Now, we consider the eigenvalues of the $p(x)$-Laplacian problem

$$\begin{cases}-\Delta_{p(x)}u+b(x)|u|^{p(x)-2}u=\lambda|u|^{p(x)-2}u, & \text{in } \mathbb{R}^N,\\ u\in W^{1,p(x)}(\mathbb{R}^N).\end{cases}\tag{5}$$

For any $u\in E$, define $G,H:E\to\mathbb{R}$ by

$$G(u)=\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+b(x)|u|^{p(x)}\right)\mathrm{d}x,\ H(u)=\int_{\mathbb{R}^N}\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x.$$

For all $t>0$, set

$$S_t = H^{-1}(t) = \{u \in E : H(u) = t\},$$

then S_t is a C^1 submanifold of *E* since *t* is a regular value of *H*. Put

$$\sum_{t,n} = \{I \subset S_t : I = -I, \gamma(I) \ge n\},$$

where $\gamma(I)$ is the genus of *I*.

Define

$$c_{(n,t)} = \inf_{I \in \sum_{t,n}} \sup_{u \in I} G(u), \quad n = 1, 2, \cdots.$$

We denote by $\{(u_{(n,t)}, \lambda_{(n,t)})\}$ the eigenpair sequences of problem (5) such that

$$H(\pm u_{(n,t)}) = t, \quad G(\pm u_{(n,t)}) = c_{(n,t)},$$

$$\lambda_{(n,t)} = \frac{\int_{\mathbb{R}^N} \left(|\nabla u_{(n,t)}|^{p(x)} + b(x) |u_{(n,t)}|^{p(x)} \right) dx}{\int_{\mathbb{R}^N} |u_{(n,t)}|^{p(x)} dx} \to \infty, \text{ as } n \to \infty.$$

Define

$$\overline{\mu}_* = \inf_{E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + b(x) |u|^{p(x)} \right) dx}{\int_{\mathbb{R}^N} |u|^{p(x)} dx},$$

$$\mu_* = \inf_{E \setminus \{0\}} \frac{G(u)}{H(u)},$$

$\lambda_* = \inf \Lambda$, where $\Lambda = \{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of } (5)\}$.

Lemma 2.5. For all $t > 0$, let $u_{(1,t)}$ be an eigenfunction associated with $\lambda_{(1,t)}$ of the problem (5). Then,

$$G(u_{(1,t)}) = c_{(1,t)} = \inf \{G(u) : u \in S_t\}.$$

Proof. Let $z_t = \inf \{G(u) : u \in S_t\}$. From the definition of $c_{(1,t)}$, it is easy to see that $z_t \le c_{(1,t)}$.

On the other hand, since the functional $G : E \to \mathbb{R}$ is coercive and weakly lower semi-continuous and S_t is weakly closed subset of *E*, there exists $u_0 \in S_t$ such that $G(\pm u_0) = z_t$. Letting $I = \{\pm u_0\}$, then $\gamma(I) = 1$ and $c_{(1,t)} \le z_t$. Thus the lemma follows. □

Lemma 2.6.

$$\left(\frac{p^-}{p^+} \right)^2 \overline{\mu}_* \le \lambda_* \le \left(\frac{p^+}{p^-} \right)^2 \overline{\mu}_*.$$

Proof. From Lemma 2.5, we have

$$\frac{G(u_{(1,t)})}{H(u_{(1,t)})} = \inf \left\{ \frac{G(u)}{H(u)} \middle| u \in S_t \right\}.$$

Since

$$\begin{aligned} \frac{G(u_{(1,t)})}{H(u_{(1,t)})} &= \frac{\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_{(1,t)}|^{p(x)} + b(x) |u_{(1,t)}|^{p(x)} \right) dx}{\int_{\mathbb{R}^N} \frac{1}{p(x)} |u_{(1,t)}|^{p(x)} dx} \\ &\le \frac{p^-}{p^+} \frac{\int_{\mathbb{R}^N} \left(|\nabla u_{(1,t)}|^{p(x)} + b(x) |u_{(1,t)}|^{p(x)} \right) dx}{\int_{\mathbb{R}^N} |u_{(1,t)}|^{p(x)} dx}, \end{aligned}$$

so we have $\frac{p^-}{p^+}\lambda_{(1,t)} \le \inf\left\{\frac{G(u)}{H(u)}\middle| u \in S_t\right\}$. Then,

$$\frac{p^-}{p^+}\lambda_* \le \inf\left\{\frac{G(u)}{H(u)}\middle| u \in S_t\right\}, \quad \text{for all } t > 0.$$

Thus we get $\frac{p^-}{p^+}\lambda_* \le \mu_*$ and $\lambda_* \le \frac{p^+}{p^-}\mu_*$.

Similarly, if $u_{(n,t)}$ is the eigenfunction associated with $\lambda_{(n,t)}$, we get $\lambda_{(n,t)} \ge \frac{p^-}{p^+}\mu_*$ and $\lambda_* \ge \frac{p^-}{p^+}\mu_*$. Finally, we obtain $\frac{p^-}{p^+}\mu_* \le \lambda_* \le \frac{p^+}{p^-}\mu_*$.

On the other hand, it is easy to see that $\frac{p^-}{p^+}\bar{\mu}_* \le \mu_* \le \frac{p^+}{p^-}\bar{\mu}_*$. Thus the lemma follows. □

Now, we consider the truncated problem

$$\begin{cases} -\Delta_{p(x)}u + b(x)|u|^{p(x)-2}u = f_\pm(x,u), \text{ in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{$M_\pm$}$$

where

$$f_\pm(x,t) = \begin{cases} f(x,t), & \text{if } \pm t \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $u^{(+)} = \max(u,0)$ and $u^{(-)} = \max(-u,0)$ the positive and negative parts of u.

Lemma 2.7.

1) If $u \in E$ then $u^{(+)}, u^{(-)} \in E$ and

$$\nabla u^{(+)} = \begin{cases} \nabla u, & u > 0, \\ 0, & u \le 0, \end{cases} \qquad \nabla u^{(-)} = \begin{cases} 0, & u \ge 0, \\ \nabla u, & u < 0. \end{cases}$$

2) The mappings $u \to u^{(\pm)}$ are continuous on E.

Lemma 2.8. All solutions of (M_-) (resp. M_+) are non-positive (resp. non-negative) solutions of problem (3).

Proof. Define $\Phi_\pm(u): E \to \mathbb{R}$,

$$\begin{aligned} \Phi_\pm(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right)\mathrm{d}x - \int_{\mathbb{R}^N} F_\pm(x,u)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right)\mathrm{d}x - \int_{\mathbb{R}^N} F\left(x,u^{(\pm)}\right)\mathrm{d}x, \end{aligned}$$

where $F_\pm(x,s) = \int_0^s f_\pm(x,t)\,\mathrm{d}t$. From Lemma 2.7 and ($F_0$), $\Phi_\pm$ is well defined on E, weakly lower semi-continuous and C^1-functionals.

Let u be a solution of (M_-). Taking $v = u^{(+)}$ in

$$\Phi_-'(u)v = \int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv - f_-(x,u)v\right)\mathrm{d}x = 0,$$

we have

$$J_1\left(u^{(+)}\right) = \int_{\mathbb{R}^N}\left(\left|\nabla u^{(+)}\right|^{p(x)} + b(x)\left|u^{(+)}\right|^{p(x)}\right)\mathrm{d}x = 0.$$

By virtue of Proposition 2.3, we have $\left\|u^{(+)}\right\| = 0$, so $u^{(+)} = 0$ and $u = u^{(-)}$, a.e. $x \in \mathbb{R}^N$, then u is also a critical point of the functional Φ with critical value $\Phi(u)=\Phi_-(u)$.

Similarly, the nontrivial critical points of the functional Φ_+ are non-negative solutions of problem (M_+). □

3. Proof of Main Results

3.1. Proof of Theorem 1.1

To derive the Theorem 1.1, we need the following results.

Proposition 3.1. Φ is coercive on E.

Proof. Put

$$L(x,t)=F(x,t)-\frac{(p^-)^2}{(p^+)^3}\lambda_*|t|^{p^-}.$$

From (F_1) we have, for any $R>0$, there is $M_R>0$ such that

$$L(x,t)\le -R,\quad \forall |t|\ge M_R,\ \text{a.e. } x\in\mathbb{R}^N.$$

By contradiction, let $A\in\mathbb{R}$ and $\{u_n\}\subset E$ such that

$$\|u_n\|\to\infty \ \text{ and } \ \Phi(u_n)\le A. \tag{6}$$

Putting $v_n=\frac{u_n}{\|u_n\|}$, one has $\|v_n\|=1$. For a subsequence, we may assume that for some $v_0\in E$, we have $v_n\rightharpoonup v_0$ weakly in E and $v_n\to v_0$ strongly in $L^{p(x)}(\mathbb{R}^N)$.

Consequently, $v_0\not\equiv 0$. Let $\Omega=\{x\in\mathbb{R}^N: v_0\not\equiv 0\}$, via the result above we have $|\Omega|\not\equiv 0$ and

$$|u_n|\to+\infty,\quad \text{a.e. } x\in\Omega.$$

Set

$$\bar{u}_n=\begin{cases} u_n(x), & x\in\Omega,\\ 0, & x\in\mathbb{R}^N\setminus\Omega,\end{cases}$$

then,

$$|\bar{u}_n|\to+\infty,\quad \text{as } n\to\infty,\ \text{a.e. } x\in\Omega.$$

From (6), (F_1) and Lemma 2.6, we deduce that

$$\begin{aligned} A\ge\Phi(\bar{u}_n)&=\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla\bar{u}_n|^{p(x)}+b(x)|\bar{u}_n|^{p(x)}\right)\mathrm{d}x-\int_{\mathbb{R}^N}F(x,\bar{u}_n)\mathrm{d}x\\ &\ge\frac{1}{p^+}\int_\Omega\left(|\nabla u_n|^{p(x)}+b(x)|u_n|^{p(x)}-\left(\frac{p^-}{p^+}\right)^2\lambda_*|u_n|^{p(x)}\right)\mathrm{d}x-\int_\Omega L(x,u_n)\mathrm{d}x\\ &\ge-\int_\Omega L(x,u_n)\mathrm{d}x\to+\infty.\end{aligned}$$

This is a contradiction. Therefore, Φ is coercive on E. □

Proposition 3.2. Assume $f(x)$ satisfies (F_0) and (F_2), then zero is local maximum for the functional $\Phi(su)$, $u\ne 0$, $s\in\mathbb{R}$.

Proof. From (F_2), there is a constant $c_1>0$ such that

$$F(x,t)\ge c_1|t|^\mu,\quad \text{for } x\in\mathbb{R}^N,\ 0<|t|\le\gamma,\ 1\le\mu<p^-. \tag{7}$$

From (F_0) and $|t|\ge\gamma>1$, there exists $c_2>0$ such that

$$|F(x,t)|\le c_2|t|^{q(x)},\quad x\in\mathbb{R}^N,\ |t|>\gamma. \tag{8}$$

By (7) and (8), we get

$$F(x,t)\ge c_1|t|^\mu-c_2|t|^{q(x)},\quad x\in\mathbb{R}^N,\ t\in\mathbb{R}. \tag{9}$$

For $u \in E, u \neq 0, 0 < s < 1,$ we have

$$\Phi(su) \leq \frac{s^{p^-}}{p^-}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right)\mathrm{d}x - \int_{\mathbb{R}^N}\left(c_1 s^{\mu}|u|^{\mu} - c_2 s^{q^-}|u|^{q(x)}\right)\mathrm{d}x$$
$$\leq \frac{s^{p^-}}{p^-}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right)\mathrm{d}x - c_1 s^{\mu}\|u\|_{L^{\mu}}^{\mu} + c_2 s^{q^-}\int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x.$$

Since $1 \leq \mu < p^- < q^-$, there is a $s_0 = s_0(u) > 0$ such that

$$\Phi(su) < 0, \text{ for all } 0 < s < s_0 < 1. \tag{10}$$

Thus the proposition follows. □

Proof of Theorem 1.1. From Proposition 3.1, we know Φ is coercive on *E*. Since Φ has a global minimizer u_0 on *E*, Φ is weakly lower semi-continuous and $\Phi(0) = 0$, then, in order to prove $u_0 \neq 0$, we need to prove $\Phi(u_0) < 0$. So we have the Theorem 1.1 following from Proposition 3.2. □

3.2. Proof of Theorem 1.2

To find the properties of the $p(x)$-Laplacian operators, we need the following inequalities (see [10]).

Lemma 3.3. For α and β in $\mathbb{R}^N$, then there are the following inequalities

$$\left[\left(|\alpha|^{n-2}\alpha - |\beta|^{n-2}\beta\right)(\alpha-\beta)\right]^{\frac{n}{2}}\left(|\alpha|^{n} + |\beta|^{n}\right)^{\frac{2-n}{2}} \geq (n-1)|\alpha-\beta|^{n}, \quad \text{for } 1 < n < 2;$$
$$\left(|\alpha|^{n-2}\alpha - |\beta|^{n-2}\beta\right)(\alpha-\beta) \geq 2^{-n}|\alpha-\beta|^{n}, \qquad \text{for } n > 2.$$

Proposition 3.4. Assume (F_0), and let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ in *E* and $\Phi'_-(u_n)v = o_n(1)$ for all $v \in E$ as $n \to \infty$, then, for some subsequences, $\nabla u_n(x) \to \nabla u(x)$, a.e. in $\mathbb{R}^N$, as $n \to \infty$ and $\Phi'_-(u)v = 0$ for all $v \in E$.

Proof. Let $R > 0$ and $\eta \in C_0^{\infty}(\mathbb{R}^N)$ such that

$$\eta = 0 \text{ if } |x| \geq 2R, \ \eta = 1 \text{ if } |x| \leq R, \ 0 \leq \eta(x) \leq 1 \text{ for all } x \in \mathbb{R}^N \text{ and } |\nabla\eta| < \frac{2}{R}.$$

Let us denote by $\{P_n\}$ the following sequence

$$P_n = \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u).$$

From Lemma 3.3, we have $P_n \geq 0$ and

$$\int_{B_R(0)} P_n \mathrm{d}x \leq \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)}\cdot\eta\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla u\cdot\eta\mathrm{d}x$$
$$-\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla(u_n - u)\cdot\eta\mathrm{d}x.$$

Recalling that $u_n \rightharpoonup u$ in *E*, we get

$$\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\eta\nabla u\nabla(u_n - u)\mathrm{d}x \to 0, \text{ as } n \to \infty,$$

and so,

$$\begin{aligned}\int_{B_R(0)} P_n \mathrm{d}x &\leq \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)}\eta\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\eta\nabla u_n\nabla u\mathrm{d}x + o_n(1)\\ &= \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\eta\nabla u_n\nabla(u_n - u)\mathrm{d}x + o_n(1).\end{aligned} \tag{11}$$

On the other hand, by (11) and $\Phi'_-(u_n)[\eta(u_n - u)] = o_n(1),$ we obtain

$$\int_{B_R(0)} P_n \mathrm{d}x \leq o_n(1) - \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}(u_n - u)\nabla u_n\nabla\eta\mathrm{d}x$$
$$-\int_{\mathbb{R}^N} b(x)|u_n|^{p(x)-2}u_n(u_n - u)\eta\mathrm{d}x + \int_{\mathbb{R}^N} f\left(x, u_n^{(-)}\right)(u_n - u)\eta\mathrm{d}x$$
$$\leq o_n(1) - \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}(u_n - u)\nabla u_n\nabla\eta\mathrm{d}x$$
$$-\int_{\mathbb{R}^N} b(x)|u_n|^{p(x)-2}u_n(u_n - u)\eta\mathrm{d}x + \int_{\mathbb{R}^N} f(x, u_n)(u_n - u)\eta\mathrm{d}x.$$

Thus,

$$\int_{B_R(0)} P_n \mathrm{d}x \le o_n(1) + c_1 \int_{\sup\eta} |\nabla u_n|^{p(x)-1} |u_n - u| \mathrm{d}x$$
$$+ c_1 \int_{\sup\eta} b(x) |u_n|^{p(x)-1} |u_n - u| \mathrm{d}x + c_2 \int_{\sup\eta} |u_n|^{q(x)-1} |u_n - u| \mathrm{d}x.$$

Combining Hölder's inequality and Sobolev embedding, we deduce that

$$\int_{B_R(0)} P_n \mathrm{d}x \to 0, \text{ as } \to \infty. \tag{12}$$

Let us consider the sets

$$B_1 = \{x \in B_R(0) | 1 < p(x) < 2\} \text{ and } B_2 = \{x \in B_R(0) | p(x) \ge 2\}.$$

From Lemma 3.3, we get

$$P_n \ge (p^- - 1)^2 \frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p(x)}}, \text{ if } x \in B_1, \tag{13}$$

$$P_n \ge 2^{-p^+} |\nabla u_n - \nabla u|^{p(x)}, \text{ if } x \in B_2. \tag{14}$$

Applying again Hölder's inequality,

$$\int_{B_1} |\nabla u_n - \nabla u|^{p(x)} \mathrm{d}x \le C |g_n|_{L^{\frac{2}{p(x)}}(B_1)} |h_n|_{L^{\frac{2}{2-p(x)}}(B_1)}, \tag{15}$$

where

$$g_n(x) = \frac{|\nabla u_n - \nabla u|^{p(x)}}{(|\nabla u_n| + |\nabla u|)^{\frac{p(x)(2-p(x))}{2}}},$$

and

$$h_n(x) = (|\nabla u_n| + |\nabla u|)^{\frac{p(x)(2-p(x))}{2}}.$$

Then,

$$\int_{B_1} |h_n|^{\frac{2}{2-p(x)}} \mathrm{d}x = \int_{B_1} (|\nabla u_n| + |\nabla u|)^{p(x)} \mathrm{d}x < +\infty. \tag{16}$$

From (12) and (13), we have

$$\int_{B_1} |g_n|^{\frac{2}{p(x)}} \mathrm{d}x \le C \int_{B_1} P_n \mathrm{d}x \to 0, \text{ as } n \to \infty. \tag{17}$$

By (15)-(17), we obtain

$$\int_{B_1} |\nabla u_n - \nabla u|^{p(x)} \mathrm{d}x \to 0, \text{ as } n \to \infty. \tag{18}$$

(12) and (14) imply that

$$\int_{B_2} |\nabla u_n - \nabla u|^{p(x)} \mathrm{d}x \to 0, \text{ as } n \to \infty. \tag{19}$$

From (18) and (19), $\nabla u_n \to \nabla u$ a.e. in $B_R(0)$. Because R is arbitrary, it follows that for some subsequence

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N.$$

Combined with Lebesgue's dominated convergence theorem, we get

$$|\nabla u_n|^{p(x)-2} \nabla u_n \rightharpoonup |\nabla u|^{p(x)-2} \nabla u \quad \text{in} \left(L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N) \right)^N. \tag{20}$$

By (20) and $\Phi'_-(u_n)v = o_n(1)$, we derive that $\Phi'_-(u)v = 0$ for all $v \in E$. □

Proposition 3.5. Assume (F_0), and let $d \in \mathbb{R}$ and $\{u_n\}$ be a $(PS)_d$ sequence in E for Φ_-, then $\{u_n\}$ is bounded in E.

Proof. From (F_0), we have $|F(x,t)| \le c_1 |t|^{q(x)}$, $\forall t \in \mathbb{R}$, a.e. $x \in \mathbb{R}^N$. It is clear that

$$\begin{aligned}\Phi_-(u_n) - \frac{1}{\theta}\Phi'_-(u_n)u_n &= \int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}\right)dx - \int_{\mathbb{R}^N} F\left(x, u_n^{(-)}\right)dx \\ &\quad - \frac{1}{\theta}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}\right)dx + \frac{1}{\theta}\int_{\mathbb{R}^N} f\left(x, u_n^{(-)}\right)u_n dx \\ &\ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}\right)dx - \int_{\mathbb{R}^N} F\left(x, u_n^{(-)}\right)dx \\ &\ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right)J_1(u_n) - c_1\int_{\mathbb{R}^N}\left|u_n^{(-)}\right|^{q(x)} dx \\ &\ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right)J_1(u_n) - c_1\int_{\mathbb{R}^N}|u_n|^{q(x)} dx.\end{aligned}$$

Assume that $\|u_n\| > 1$ for some $n \in N$, then, by Proposition 2.3, Hölder's inequality and Sobolev embedding, we have

$$d + 1 + \|u_n\| \ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|^{p^-} - c_1\|u_n\|^{q^+}. \tag{21}$$

Since $\theta > p^+$ and $1 < p^- < q^+$, (21) implies that $\{u_n\}$ is bounded in E. □

Proposition 3.6. Assume $b(x)$ satisfies (B_0), $f(x)$ satisfies (F_0) and (F_4), and let $\{u_n\}$ be a $(PS)_d$ sequence in E, then Φ_- satisfies the (PS) condition.

Proof. From Proposition 3.4, we have

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)dx\right) = \lim_{n\to\infty}\int_{\mathbb{R}^N} P_n dx = 0. \tag{22}$$

By Lemma 2.4, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)|u_n|^{p(x)} dx = \int_{\mathbb{R}^N} b(x)|u|^{p(x)} dx. \tag{23}$$

On the other hand, Lebesgue's dominated convergence theorem and the weak convergence of $\{u_n\}$ to u in E show

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)|u|^{p(x)-2} u u_n dx = \int_{\mathbb{R}^N} b(x)|u|^{p(x)} dx. \tag{24}$$

Moreover, since $b(x)^{\frac{p(x)-1}{p(x)}}|u_n|^{p(x)-2}u_n$ are bounded in $L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)$, then we have

$$b(x)^{\frac{p(x)-1}{p(x)}}|u_n|^{p(x)-2}u_n \rightharpoonup b(x)^{\frac{p(x)-1}{p(x)}}|u|^{p(x)-2}u \quad \text{in } L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N).$$

Therefore, by virtue of the definition of weak convergence, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)|u_n|^{p(x)-2} u_n u dx = \int_{\mathbb{R}^N} b(x)|u|^{p(x)} dx. \tag{25}$$

By (23)-(25), we have

$$\begin{aligned}&\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} b(x)\left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u)dx\right) \\ &= \lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\left(b(x)|u_n|^{p(x)} + b(x)|u|^{p(x)} - b(x)|u_n|^{p(x)-2}u_n u - b(x)|u|^{p(x)-2}u u_n\right)dx\right) \\ &= 0.\end{aligned} \tag{26}$$

By (22) and (26), we get

$$\lim_{n\to\infty}\Big(\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n-\nabla u)\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}b(x)\left(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right)(u_n-u)\,\mathrm{d}x\Big)=0.$$

Then combining Lemma 3.3, we obtain

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n-\nabla u|^{p(x)}\,\mathrm{d}x+\int_{\mathbb{R}^N}b(x)|u_n-u|^{p(x)}\,\mathrm{d}x\right)=0,$$

which imply that $u_n\to u$ in E. □

Proposition 3.7. There exist $r>0$ and $l>0$ such that $\Phi_-(u)\ge l$, for all $u\in E$ with $\|u\|=r$.

Proof. The conditions (F_0) and (F_4) imply that

$$|F(x,t)|\le\varepsilon|t|^{p^+}+C(\varepsilon)|t|^{q(x)},\ \text{for all }(x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

For $\|u\|$ small enough, combined with Proposition 2.3, we have

$$\begin{aligned}\Phi_-(u)&\ge\frac{1}{p^+}J_1(u)-\int_{\mathbb{R}^N}F\left(x,u^{(-)}\right)\mathrm{d}x\\&\ge\frac{1}{p^+}\|u\|^{p^+}-\varepsilon\int_{\mathbb{R}^N}\left|u^{(-)}\right|^{p^+}\mathrm{d}x-C(\varepsilon)\int_{\mathbb{R}^N}\left|u^{(-)}\right|^{q(x)}\mathrm{d}x\\&\ge\frac{1}{p^+}\|u\|^{p^+}-\varepsilon\int_{\mathbb{R}^N}|u|^{p^+}\mathrm{d}x-C(\varepsilon)\int_{\mathbb{R}^N}|u|^{q(x)}\mathrm{d}x.\end{aligned}\tag{27}$$

By the condition (F_0), it follows

$$p^-\le p(x)\le p^+<q^-\le q(x)<p^*(x),$$

from Lemma 2.4, which implies the existence of $C_4,C_5>0$ such that

$$\|u\|_{L^{p^+}}\le C_4\|u\|\ \text{ and }\ |u|_{q(x)}\le C_5\|u\|,\quad\text{for all }u\in E.\tag{28}$$

Using (28) and Proposition 2.1, we deduce

$$\int_{\mathbb{R}^N}\left(|u|^{q(x)}\right)\mathrm{d}x\le|u|_{q(x)}^{q^-}\le C_6\|u\|^{q^-}.$$

Combining (27), it results in that

$$\Phi_-(u)\ge\frac{1}{p^+}\|u\|^{p^+}-\varepsilon C_4^{p^+}\|u\|^{p^+}-C_7\|u\|^{q^-},$$

here C_i are positives constants. Taking $\varepsilon>0$ such that $\varepsilon C_4^{p^+}\le\frac{1}{2p^+}$, we obtain

$$\Phi_-(u)\ge\frac{1}{2p^+}\|u\|^{p^+}-C_7\|u\|^{q^-}\ge\|u\|^{p^+}\left(\frac{1}{2p^+}-C_7\|u\|^{q^--p^+}\right).$$

Since $p^+<q^-$, the function $t\to\left(\frac{1}{2P^+}-C_7t^{q^--p^+}\right)$ is strictly positive in a neighborhood of zero. It follows that there exist $r>0$ and $l>0$ such that

$$\Phi_-(u)\ge l,\ \forall u\in E:\|u\|=r.$$

□

Proposition 3.8. If $u\in E$ and $s>1$, we have $\Phi_-(su)\to-\infty$, as $s\to+\infty$, for a certain $u\in E$.

Proof. From the condition (F_3), we get

$$F(x,t) \geq c|t|^{\theta}, \ |t| \geq K, \text{ for all } x \in \mathbb{R}^N.$$

For $u \in E$ and $s > 1$, we have

$$\begin{aligned}\Phi_-(su) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|s\nabla u|^{p(x)} + b(x)|su|^{p(x)}\right)\mathrm{d}x - \int_{\mathbb{R}^N} F\left(x,(su)^{(-)}\right)\mathrm{d}x \\ &\leq s^{p^+}\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right)\mathrm{d}x - cs^{\theta}\int_{\mathbb{R}^N} \left|u^{(-)}\right|^{\theta}\mathrm{d}x.\end{aligned}$$

Since $\theta > p^+$, we obtain

$$\Phi_-(su) \to -\infty, \text{ as } s \to +\infty.$$

□

Proof of Theorem 1.2. According to the Mountain Pass Lemma, the functional Φ_- has a critical point $u^{(-)}$ with $\Phi_-\left(u^{(-)}\right) \geq l$. But, $\Phi_-(0) = 0$, that is, $u^{(-)} \neq 0$, a.e. $x \in \mathbb{R}^N$. Therefore, the problem (M_-) has a non-trivial solution which, by Lemma 2.8, is a non-positive solution of the problem (3).

Similarly, for functional Φ_+, we still can show that there exists furthermore a non-negative solution. The proof of Theorem 1.2 is now complete. □

Acknowledgements

This work has been supported by the Natural Science Foundation of China (No. 11171220) and Shanghai Leading Academic Discipline Project (XTKX2012) and Hujiang Foundation of China (B14005).

References

[1] EI Hamidi, A. (2004) Existence Results to Elliptic Systems with Nonstandard Growth Conditions. *Journal of Mathematical Analysis and Applications*, **300**, 30-42. http://dx.doi.org/10.1016/j.jmaa.2004.05.041

[2] Mihăilescu, M. (2007) Existence and Multiplicity of Solutions for a Neumann Problem Involving the $p(x)$-Laplace Operator. *Nonlinear Analysis*, **67**, 1419-1425. http://dx.doi.org/10.1016/j.na.2006.07.027

[3] EI Amrouss, A.R. and Kissi, F. (2013) Multiplicity of Solutions for a General $p(x)$-Laplacian Dirichlet Problem. *Arab Journal of Mathematics and Mathematical Sciences*, **19**, 205-216. http://dx.doi.org/10.1016/j.ajmsc.2012.09.006

[4] Lin, X.Y. and Tang, X.H. (2013) Existence of Infinitely Many Solutions for p-Laplacian Equations in $\mathbb{R}^N$. *Nonlinear Analysis*, **92**, 72-81. http://dx.doi.org/10.1016/j.na.2013.06.011

[5] Diening, L. (2004) Riesz Potential and Sobolev Embeddings on Generalized Lebesgue and Sobolev Spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. *Mathematische Nachrichten*, **268**, 31-43. http://dx.doi.org/10.1002/mana.200310157

[6] Edmunds, D.E. and Rákosnik, J. (2002) Sobolev Embeddings with Variable Exponent II. *Mathematische Nachrichten*, **246-247**, 53-67. http://dx.doi.org/10.1002/1522-2616(200212)246:1<53::AID-MANA53>3.0.CO;2-T

[7] Fu, Y.Q. (2007) Existence of Solutions for $p(x)$-Laplacian Problem on an Unbounded Domain. *Topological Methods in Nonlinear Analysis*, **30**, 235-249.

[8] Canino, A. (1995) Multiplicity of Solutions for Quasilinear Elliptic Equations. *Topological Methods in Nonlinear Analysis*, **6**, 357-370.

[9] Fan, X.L. and Zhao, D. (2001) On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *Journal of Mathematical Analysis and Applications*, **263**, 424-446. http://dx.doi.org/10.1006/jmaa.2000.7617

[10] Fan, X.L. and Han, X.Y. (2004) Existence and Multiplicity of Solutions for $p(x)$-Laplacian Equations in $\mathbb{R}^N$. *Nonlinear Analysis*, **59**, 173-188. http://dx.doi.org/10.1016/j.na.2004.07.009

[11] Musielak, J. (1983) Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, Springer-Verlag, Berlin.

[12] Kováčik, O., Rákosnik, J. (1991) On Spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. *Czechoslovak Mathematical Journal*, **41**, 592-618.

[13] Alves, C.O. and Liu, S.B. (2010) On Superlinear $p(x)$-Laplacian Equations in $\mathbb{R}^N$. *Nonlinear Analysis*, **73**, 2566-2579. http://dx.doi.org/10.1016/j.na.2010.06.033